Pointfree Pointwise Suprema in Unital Archimedean \( \ell \)-Groups

R. N. Ball, A. W. Hager, and J. Walters-Wayland

University of Denver, Wesleyan University, Chapman University

8 January 2015
1 Preliminaries
   Motivation, context, definitions, and notation
   The Madden representation for $W$. 

2 Pointwise joins
   In $R_L$
   Nice properties
   In $W$
   Limitations of the $W$ definition

3 Nakano-Stone Theorems
   The classical Nakano-Stone Theorems
   The Nakano-Stone Theorems for pointwise joins
   Is $\beta G$ a pointwise $\sigma$-completion?

4 Unconditional pointwise completeness
   Truncate sequences
   Cuts and mobile downsets
   Is $\epsilon G$ an unconditional pointwise completion?
Outline

1 Preliminaries
   Motivation, context, definitions, and notation
   The Madden representation for $\textbf{W}$.

2 Pointwise joins
   In $\mathcal{RL}$
   Nice properties
   In $\textbf{W}$
   Limitations of the $\textbf{W}$ definition
Outline

1 Preliminaries
   Motivation, context, definitions, and notation
   The Madden representation for $\mathcal{W}$.

2 Pointwise joins
   In $\mathcal{RL}$
   Nice properties
   In $\mathcal{W}$
   Limitations of the $\mathcal{W}$ definition

3 Nakano-Stone Theorems
   The classical Nakano-Stone Theorems
   The Nakano-Stone Theorems for pointwise joins
   Is $\beta G$ a pointwise $\sigma$-completion?
1 Preliminaries
   Motivation, context, definitions, and notation
   The Madden representation for $\mathbf{W}$.

2 Pointwise joins
   In $\mathcal{RL}$
   Nice properties
   In $\mathbf{W}$
   Limitations of the $\mathbf{W}$ definition

3 Nakano-Stone Theorems
   The classical Nakano-Stone Theorems
   The Nakano-Stone Theorems for pointwise joins
   Is $\beta G$ a pointwise $\sigma$-completion?

4 Unconditional pointwise completeness
   Truncate sequences
   Cuts and mobile downsets
   Is $\epsilon G$ an unconditional pointwise completion?
Table of contents

1 Preliminaries
   Motivation, context, definitions, and notation
   The Madden representation for $\mathcal{W}$.

2 Pointwise joins
   In $\mathcal{RL}$
   Nice properties
   In $\mathcal{W}$
   Limitations of the $\mathcal{W}$ definition

3 Nakano-Stone Theorems
   The classical Nakano-Stone Theorems
   The Nakano-Stone Theorems for pointwise joins
   Is $\beta G$ a pointwise $\sigma$-completion?

4 Unconditional pointwise completeness
   Truncate sequences
   Cuts and mobile downsets
   Is $\epsilon G$ an unconditional pointwise completion?
Join versus pointwise join

• In any ℓ-group, \( g_0 = \bigvee_n g_n \) means

\[ \exists f \, \forall n \, (g_n \leq f < g_0). \]

• In \( \mathbb{C}X \), \( g_0 = \bigvee^\bullet g_n \) means

\[ \forall x \, (g_0(x) = \bigvee g_n(x)). \]

• In \( \mathbb{C}X \), \( g_0 = \bigvee^\bullet g_n \) iff

\[ \forall r \in \mathbb{R} \, (g_0^{-1}(r, \infty) = \bigcup g_n^{-1}(r, \infty)) \]

• Every pointwise join is a join, but not conversely.
• Example: take \( g_0 = 1 \) and \( g_n \equiv |nx| \land 1 \) all \( n \). Draw a picture.
Join versus pointwise join

• In any $\ell$-group, $g_0 = \bigvee_n g_n$ means

$$\not\exists \ f \ \forall \ n \ (g_n \leq f < g_0).$$

• In $\mathbb{C}X$, $g_0 = \bigvee^* g_n$ means

$$\forall \ x \ (g_0(x) = \bigvee g_n(x)).$$

• In $\mathbb{C}X$, $g_0 = \bigvee^* g_n$ iff

$$\forall \ r \in \mathbb{R} \ (g_0^{-1}(r, \infty) = \bigcup g_n^{-1}(r, \infty))$$

• Every pointwise join is a join, but not conversely.

• Example: take $g_0 = 1$ and $g_n \equiv |nx| \wedge 1$ all $n$. Draw a picture.
Join versus pointwise join

- In any $\ell$-group, $g_0 = \bigvee_n g_n$ means
  \[ \nexists f \ \forall n \ (g_n \leq f < g_0). \]

- In $\mathcal{C}X$, $g_0 = \bigvee^\bullet g_n$ means
  \[ \forall x \ (g_0(x) = \bigvee g_n(x)). \]

- In $\mathcal{C}X$, $g_0 = \bigvee^\bullet g_n$ iff
  \[ \forall r \in \mathbb{R} \ (g_0^{-1}(r, \infty) = \bigcup g_n^{-1}(r, \infty)). \]

- Every pointwise join is a join, but not conversely.

- Example: take $g_0 = 1$ and $g_n \equiv |nx| \wedge 1$ all $n$. Draw a picture.
Join versus pointwise join

• In any ℓ-group, $g_0 = \bigvee_n g_n$ means
  \[
  \not\exists f \forall n \ (g_n \leq f < g_0).
  \]

• In $\mathcal{C}X$, $g_0 = \bigvee^\bullet g_n$ means
  \[
  \forall x \ (g_0(x) = \bigvee g_n(x)).
  \]

• In $\mathcal{C}X$, $g_0 = \bigvee^\bullet g_n$ iff
  \[
  \forall r \in \mathbb{R} \ (-1)^{-1}(r, \infty) = \bigcup g_n^{-1}(r, \infty))
  \]

• Every pointwise join is a join, but not conversely.

• Example: take $g_0 = 1$ and $g_n \equiv |nx| \land 1$ all $n$. Draw a picture.
Join versus pointwise join

• In any $\ell$-group, $g_0 = \bigvee_n g_n$ means

$$\nexists f \forall n \ (g_n \leq f < g_0).$$

• In $\mathbb{C}X$, $g_0 = \bigvee^\cdot g_n$ means

$$\forall x \ (g_0(x) = \bigvee g_n(x)).$$

• In $\mathbb{C}X$, $g_0 = \bigvee^\cdot g_n$ iff

$$\forall r \in \mathbb{R} \ (g_0^{-1}(r, \infty) = \bigcup g_n^{-1}(r, \infty)).$$

• Every pointwise join is a join, but not conversely.

• Example: take $g_0 = 1$ and $g_n \equiv |nx| \wedge 1$ all $n$. Draw a picture.
Frames in place of spaces

• Definition of frame
  A frame is a complete bounded distributive lattice $L$ such that

  $$b \land \bigvee A = \bigvee_{A} (b \land a)$$

  for all $A \subseteq L$ and $b \in L$. A frame map is a bounded lattice homomorphism which preserves finite meets and arbitrary joins.

• Definition of regularity
  A frame $L$ is regular if $a = \bigvee_{b \prec a} b$, where $b \prec a$ means that $b^{*} \lor a = \top$. A frame $L$ is completely regular if $L$ is regular and the $\prec$ relation interpolates. We assume all frames to be completely regular.

• Definition of $\mathcal{R}L$
  For a frame $L$, $\mathcal{R}L$ designates the family of frame maps $\mathcal{O} \mathcal{R} \rightarrow L$. The category is $\mathcal{W}$. 

Frames in place of spaces

• Definition of frame
  A frame is a complete bounded distributive lattice $L$ such that

  $$b \land \bigvee A = \bigvee_{A} (b \land a)$$

  for all $A \subseteq L$ and $b \in L$. A frame map is a bounded lattice homomorphism which preserves finite meets and arbitrary joins.

• Definition of regularity
  A frame $L$ is regular if $a = \bigvee_{b \prec a} b$, where $b \prec a$ means that $b^* \lor a = \top$. A frame $L$ is completely regular if $L$ is regular and the $\prec$ relation interpolates. We assume all frames to be completely regular.

• Definition of $RL$
  For a frame $L$, $RL$ designates the family of frame maps $\emptyset \to L$. The category is $\mathcal{W}$. 
Frames in place of spaces

• **Definition of frame**
  A *frame* is a complete bounded distributive lattice $L$ such that
  \[ b \wedge \bigvee A = \bigvee_A (b \wedge a) \]
  for all $A \subseteq L$ and $b \in L$. A *frame map* is a bounded lattice homomorphism which preserves finite meets and arbitrary joins.

• **Definition of regularity**
  A frame $L$ is *regular* if $a = \bigvee_{b \prec a} b$, where $b \prec a$ means that $b^* \lor a = \top$. A frame $L$ is *completely regular* if $L$ is regular and the $\prec$ relation interpolates. We assume all frames to be completely regular.

• **Definition of $RL$**
  For a frame $L$, $RL$ designates the family of frame maps $\mathcal{O} \mathcal{R} \rightarrow L$. The category is $\mathcal{W}$. 
Frames in place of spaces

• **Definition of frame**
  A *frame* is a complete bounded distributive lattice \( L \) such that
  \[
  b \wedge \bigvee A = \bigvee_{A} (b \wedge a)
  \]
  for all \( A \subseteq L \) and \( b \in L \). A *frame map* is a bounded lattice homomorphism which preserves finite meets and arbitrary joins.

• **Definition of regularity**
  A frame \( L \) is *regular* if \( a = \bigvee_{b \prec a} b \), where \( b \prec a \) means that \( b^{*} \lor a = \top \). A frame \( L \) is *completely regular* if \( L \) is regular and the \( \prec \) relation interpolates. We assume all frames to be completely regular.

• **Definition of \( RL \)**
  For a frame \( L \), \( RL \) designates the family of frame maps \( \Theta : \mathcal{R} \rightarrow L \). The category is \( \mathcal{W} \).
Frames in place of spaces

• Definition of frame
  A *frame* is a complete bounded distributive lattice $L$ such that
  
  \[ b \land \bigvee A = \bigvee_{A} (b \land a) \]

  for all $A \subseteq L$ and $b \in L$. A *frame map* is a bounded lattice homomorphism which preserves finite meets and arbitrary joins.

• Definition of regularity
  A frame $L$ is *regular* if $a = \bigvee_{b \prec a} b$, where $b \prec a$ means that $b^{*} \lor a = \top$. A frame $L$ is *completely regular* if $L$ is regular and the $\prec$ relation interpolates. We assume all frames to be completely regular.

• Definition of $RL$
  For a frame $L$, $RL$ designates the family of frame maps $\Omega R \rightarrow L$. The category is $W$. 
Frames in place of spaces

- **Definition of frame**
  A *frame* is a complete bounded distributive lattice $L$ such that

  $$b \land \bigvee A = \bigvee_{A} (b \land a)$$

  for all $A \subseteq L$ and $b \in L$. A *frame map* is a bounded lattice homomorphism which preserves finite meets and arbitrary joins.

- **Definition of regularity**
  A frame $L$ is *regular* if $a = \bigvee_{b \prec a} b$, where $b \prec a$ means that $b^* \lor a = \top$. A frame $L$ is *completely regular* if $L$ is regular and the $\prec$ relation interpolates. We assume all frames to be completely regular.

- **Definition of $RL$**
  For a frame $L$, $RL$ designates the family of frame maps $OIR \to L$. The category is $W$. 
What is $\mathbf{W}$?

- $\mathbf{W}$ is the categorical context for $\mathcal{RL}$.
What is \( \mathbb{W} \)?

- \( \mathbb{W} \) is the categorical context for \( \mathcal{RL} \)
What is $\mathcal{W}$?

- $\mathcal{W}$ is the categorical context for $\mathcal{RL}$

Definition of $\mathcal{W}$

An object is an archimedean lattice-ordered group $G$ equipped with a weak order unit, i.e., an element $1 \in G^+$ such that $1 \land g = 0$ implies $g = 0$. (This is not Mundici's unit.) A morphism is a group and lattice homomorphism which takes units to units. The kernels of the morphisms are called $\mathcal{W}$-kernels.

- What is a $\mathcal{W}$-kernel?
- A $\mathcal{W}$-kernel in a $\mathcal{W}$-object $G$ is a subset $K \subseteq G$ with the following properties.
  - $K$ is a subgroup,
  - $K$ is a convex sublattice,
  - if there is some $r \in G^+$ such that $(ng - r)^+ = (ng - r) \lor 0 \in K$ then $g \in K$ (the element $r$ is called a regulator for $g$), and
  - $g \land 1 \in K$ implies $g \in K$. 
What is $\mathcal{W}$?

- $\mathcal{W}$ is the categorical context for $\mathcal{RL}$

- **Definition of $\mathcal{W}$**
  An object is an archimedean lattice-ordered group $G$ equipped with a weak order unit, i.e., an element $1 \in G^+$ such that $1 \wedge g = 0$ implies $g = 0$. (This is not Mundici’s unit.) A morphism is a group and lattice homomorphism which takes units to units. The kernels of the morphisms are called $\mathcal{W}$-kernels.

- What is a $\mathcal{W}$-kernel?
- A $\mathcal{W}$-kernel in a $\mathcal{W}$-object $G$ is a subset $K \subseteq G$ with the following properties.
  - $K$ is a subgroup,
  - $K$ is a convex sublattice,
  - if there is some $r \in G^+$ such that $(ng - r)^+ = (ng - r) \vee 0 \in K$ then $g \in K$ (the element $r$ is called a regulator for $g$), and
  - $g \wedge 1 \in K$ implies $g \in K$. 
What is $\mathbf{W}$?

- $\mathbf{W}$ is the categorical context for $\mathcal{RL}$

- **Definition of $\mathbf{W}$**
  An object is an archimedean lattice-ordered group $G$ equipped with a weak order unit, i.e., an element $1 \in G^+$ such that $1 \land g = 0$ implies $g = 0$. (This is not Mundici’s unit.) A morphism is a group and lattice homomorphism which takes units to units. The kernels of the morphisms are called $\mathbf{W}$-kernels.

  - **What is a $\mathbf{W}$-kernel?**
  - A $\mathbf{W}$-kernel in a $\mathbf{W}$-object $G$ is a subset $K \subseteq G$ with the following properties.
    - $K$ is a subgroup,
    - $K$ is a convex sublattice,
    - if there is some $r \in G^+$ such that $(ng - r)^+ = (ng - r) \lor 0 \in K$ then $g \in K$ (the element $r$ is called a regulator for $g$), and
    - $g \land 1 \in K$ implies $g \in K$. 
What is $\mathbf{W}$?

- $\mathbf{W}$ is the categorical context for $\mathcal{RL}$

**Definition of $\mathbf{W}$**

An object is an archimedean lattice-ordered group $G$ equipped with a weak order unit, i.e., an element $1 \in G^+$ such that $1 \land g = 0$ implies $g = 0$. (This is not Mundici’s unit.) A morphism is a group and lattice homomorphism which takes units to units. The kernels of the morphisms are called $\mathbf{W}$-kernels.

- What is a $\mathbf{W}$-kernel?
- A $\mathbf{W}$-kernel in a $\mathbf{W}$-object $G$ is a subset $K \subseteq G$ with the following properties.
  - $K$ is a subgroup,
  - $K$ is a convex sublattice,
  - if there is some $r \in G^+$ such that $(ng - r)^+ = (ng - r) \lor 0 \in K$ then $g \in K$ (the element $r$ is called a regulator for $g$), and
  - $g \land 1 \in K$ implies $g \in K$. 
What is $\mathbf{W}$?

- $\mathbf{W}$ is the categorical context for $\mathcal{RL}$

**Definition of $\mathbf{W}$**

An object is an archimedean lattice-ordered group $G$ equipped with a weak order unit, i.e., an element $1 \in G^+$ such that $1 \land g = 0$ implies $g = 0$. (This is not Mundici’s unit.) A morphism is a group and lattice homomorphism which takes units to units. The kernels of the morphisms are called $\mathbf{W}$-kernels.

- What is a $\mathbf{W}$-kernel?
- A $\mathbf{W}$-kernel in a $\mathbf{W}$-object $G$ is a subset $K \subseteq G$ with the following properties.
  - $K$ is a subgroup,
  - $K$ is a convex sublattice,
  - if there is some $r \in G^+$ such that $(ng - r)^+ = (ng - r) \lor 0 \in K$ then $g \in K$ (the element $r$ is called a regulator for $g$), and
  - $g \land 1 \in K$ implies $g \in K$. 
What is \( \mathbb{W} \)?

- \( \mathbb{W} \) is the categorical context for \( \mathbb{RL} \).

**Definition of \( \mathbb{W} \)**

An object is an archimedean lattice-ordered group \( G \) equipped with a weak order unit, i.e., an element \( 1 \in G^+ \) such that \( 1 \land g = 0 \) implies \( g = 0 \). (This is not Mundici’s unit.) A morphism is a group and lattice homomorphism which takes units to units. The kernels of the morphisms are called \( \mathbb{W} \)-kernels.

- What is a \( \mathbb{W} \)-kernel?
- A \( \mathbb{W} \)-kernel in a \( \mathbb{W} \)-object \( G \) is a subset \( K \subseteq G \) with the following properties.
  - \( K \) is a subgroup,
  - \( K \) is a convex sublattice,
  - if there is some \( r \in G^+ \) such that \((ng - r)^+ = (ng - r) \lor 0 \in K\) then \( g \in K \) (the element \( r \) is called a regulator for \( g \)), and
  - \( g \land 1 \in K \) implies \( g \in K \).
What is $\mathcal{W}$?

• $\mathcal{W}$ is the categorical context for $\mathcal{RL}$

• Definition of $\mathcal{W}$
An object is an archimedean lattice-ordered group $G$ equipped with a weak order unit, i.e., an element $1 \in G^+$ such that $1 \land g = 0$ implies $g = 0$. (This is not Mundici’s unit.) A morphism is a group and lattice homomorphism which takes units to units. The kernels of the morphisms are called $\mathcal{W}$-kernels.

• What is a $\mathcal{W}$-kernel?
• A $\mathcal{W}$-kernel in a $\mathcal{W}$-object $G$ is a subset $K \subseteq G$ with the following properties.
  • $K$ is a subgroup,
  • $K$ is a convex sublattice,
  • if there is some $r \in G^+$ such that $(ng - r)^+ = (ng - r) \lor 0 \in K$ then $g \in K$ (the element $r$ is called a regulator for $g$), and
  • $g \land 1 \in K$ implies $g \in K$. 
What is $\mathbf{W}$?

- $\mathbf{W}$ is the categorical context for $\mathcal{RL}$

- **Definition of $\mathbf{W}$**
  
  An object is an archimedean lattice-ordered group $G$ equipped with a weak order unit, i.e., an element $1 \in G^+$ such that $1 \land g = 0$ implies $g = 0$. (This is not Mundici’s unit.) A morphism is a group and lattice homomorphism which takes units to units. The kernels of the morphisms are called $\mathbf{W}$-kernels.

- What is a $\mathbf{W}$-kernel?

- A $\mathbf{W}$-kernel in a $\mathbf{W}$-object $G$ is a subset $K \subseteq G$ with the following properties.
  
  - $K$ is a subgroup,
  - $K$ is a convex sublattice,
  - if there is some $r \in G^+$ such that $(ng - r)^+ = (ng - r) \lor 0 \in K$ then $g \in K$ (the element $r$ is called a regulator for $g$), and
  - $g \land 1 \in K$ implies $g \in K$.  

An example of a regulator

\[ G \equiv \{ g + \frac{u}{|x|} : g \in C[-1, 1], u \in \mathbb{R} \} \]
\[ K = \{ g : \exists \epsilon > 0 \ (g|(-\epsilon, \epsilon) = 0) \} \]

Claim
\[ [K] = G \]

Proof
\[ r = \frac{1}{|x|} \] serves as a regulator for 1.
Draw a picture.
An example of a regulator

\[ G \equiv \{ g + \frac{u}{|x|} : g \in \mathbb{C}[\mathbb{C}[0, 1], u \in \mathbb{R} \} \]

\[ K = \{ g : \exists \epsilon > 0 (|g|(-\epsilon, \epsilon) = 0) \} \]

Claim

\[ [K] = G \]

Proof

\[ r = \frac{1}{|x|} \text{ serves as a regulator for } 1. \]

Draw a picture.

Lemma

For every \( K \subseteq G \) such that \([K] = G\) there is a countable subset \( K_0 \subseteq K \) such that \([K_0] = G\).
The Madden representation for $\mathbb{W}$

- $\mathcal{RL}$ is a $\mathbb{W}$-object. In fact, it is universal among $\mathbb{W}$-objects.

- **Theorem (Madden)**
  
  For every $\mathbb{W}$-object $G$ there exists a regular Lindelöf frame $L$ and a $\mathbb{W}$-isomorphism $G \rightarrow \hat{G} \subseteq \mathcal{RL}$ such that $\hat{G}$ is cozero dense in $\mathcal{RL}$, i.e., the subframe of $L$ generated by $\{\text{coz } \hat{g} : g \in G\}$ is all of $L$. The frame and the injection are unique with respect to their properties.

  - Note that $L$ is completely regular.
  - Banaschewski has suggested that the $L$ be called the Madden frame of $G$, and I will follow his suggestion. The notation is $L \equiv \mathcal{M}G$.

- $\mathcal{M}G$ is the frame of $\mathbb{W}$-kernels of $G$

- $\mathcal{M}$ is a functor, i.e., respects morphisms.
The Madden representation for $\mathcal{W}$

- $RL$ is a $\mathcal{W}$-object. In fact, it is universal among $\mathcal{W}$-objects.

**Theorem (Madden)**

*For every $\mathcal{W}$-object $G$ there exists a regular Lindelöf frame $L$ and a $\mathcal{W}$-isomorphism $G \to \hat{G} \leq RL$ such that $\hat{G}$ is cozero dense in $RL$, i.e., the subframe of $L$ generated by \{coz $\hat{g} : g \in G$\} is all of $L$. The frame and the injection are unique with respect to their properties.*

- Note that $L$ is completely regular.
- Banaschewski has suggested that the $L$ be called the Madden frame of $G$, and I will follow his suggestion. The notation is $L \equiv M G$.
- $M G$ is the frame of $\mathcal{W}$-kernels of $G$
- $M$ is a functor, i.e., respects morphisms.
The Madden representation for $\mathcal{W}$

- $\mathcal{R}L$ is a $\mathcal{W}$-object. In fact, it is universal among $\mathcal{W}$-objects.

- **Theorem (Madden)**

  For every $\mathcal{W}$-object $G$ there exists a regular Lindelöf frame $L$ and a $\mathcal{W}$-isomorphism $G \to \hat{G} \leq \mathcal{R}L$ such that $\hat{G}$ is cozero dense in $\mathcal{R}L$, i.e., the subframe of $L$ generated by $\{\text{coz} \, \hat{g} : g \in G\}$ is all of $L$. The frame and the injection are unique with respect to their properties.

- Note that $L$ is completely regular.

- Banaschewski has suggested that the $L$ be called the Madden frame of $G$, and I will follow his suggestion. The notation is $L \equiv \mathcal{M}G$.

- $\mathcal{M}G$ is the frame of $\mathcal{W}$-kernels of $G$

- $\mathcal{M}$ is a functor, i.e., respects morphisms.
The Madden representation for $\mathsf{W}$

- $\mathcal{RL}$ is a $\mathsf{W}$-object. In fact, it is universal among $\mathsf{W}$-objects.

- **Theorem (Madden)**

  For every $\mathsf{W}$-object $G$ there exists a regular Lindelöf frame $L$ and a $\mathsf{W}$-isomorphism $G \to \hat{G} \subseteq \mathcal{RL}$ such that $\hat{G}$ is cozero dense in $\mathcal{RL}$, i.e., the subframe of $L$ generated by $\{\text{coz } \hat{g} : g \in G\}$ is all of $L$. The frame and the injection are unique with respect to their properties.

  - Note that $L$ is completely regular.
  - Banaschewski has suggested that the $L$ be called the Madden frame of $G$, and I will follow his suggestion. The notation is $L \equiv \mathcal{M}G$.
  - $\mathcal{M}G$ is the frame of $\mathsf{W}$-kernels of $G$
  - $\mathcal{M}$ is a functor, i.e., respects morphisms.
The Madden representation for $\mathbb{W}$

- $RL$ is a $\mathbb{W}$-object. In fact, it is universal among $\mathbb{W}$-objects.

- **Theorem (Madden)**
  For every $\mathbb{W}$-object $G$ there exists a regular Lindelöf frame $L$ and a $\mathbb{W}$-isomorphism $G \to \hat{G} \leq RL$ such that $\hat{G}$ is cozero dense in $RL$, i.e., the subframe of $L$ generated by $\{\text{coz} \hat{g} : g \in G\}$ is all of $L$. The frame and the injection are unique with respect to their properties.

- Note that $L$ is completely regular.

- Banaschewskii has suggested that the $L$ be called the Madden frame of $G$, and I will follow his suggestion. The notation is $L \equiv \mathcal{M}G$.

- $\mathcal{M}G$ is the frame of $\mathbb{W}$-kernels of $G$

- $\mathcal{M}$ is a functor, i.e., respects morphisms.
The Madden representation for $\mathbb{W}$

- $\mathbb{RL}$ is a $\mathbb{W}$-object. In fact, it is universal among $\mathbb{W}$-objects.

- **Theorem (Madden)**

  For every $\mathbb{W}$-object $G$ there exists a regular Lindelöf frame $L$ and a $\mathbb{W}$-isomorphism $G \rightarrow \hat{G} \subseteq \mathbb{RL}$ such that $\hat{G}$ is cozero dense in $\mathbb{RL}$, i.e., the subframe of $L$ generated by $\{\text{coz} \hat{g} : g \in G\}$ is all of $L$. The frame and the injection are unique with respect to their properties.

  - Note that $L$ is completely regular.
  - Banaschewski has suggested that the $L$ be called the Madden frame of $G$, and I will follow his suggestion. The notation is $L \equiv \mathcal{M} G$.

- $\mathcal{M} G$ is the frame of $\mathbb{W}$-kernels of $G$

- $\mathcal{M}$ is a functor, i.e., respects morphisms.
The Madden representation for \( \mathcal{W} \)

- \( \mathcal{RL} \) is a \( \mathcal{W} \)-object. In fact, it is universal among \( \mathcal{W} \)-objects.

**Theorem (Madden)**

For every \( \mathcal{W} \)-object \( G \) there exists a regular Lindelöf frame \( L \) and a \( \mathcal{W} \)-isomorphism \( G \rightarrow \hat{G} \leq \mathcal{RL} \) such that \( \hat{G} \) is cozero dense in \( \mathcal{RL} \), i.e., the subframe of \( L \) generated by \( \{ \text{coz} \hat{g} : g \in G \} \) is all of \( L \). The frame and the injection are unique with respect to their properties.

- Note that \( L \) is completely regular.
- Banaschewski has suggested that the \( L \) be called the Madden frame of \( G \), and I will follow his suggestion. The notation is \( L \equiv \mathcal{M}G \).

- \( \mathcal{M}G \) is the frame of \( \mathcal{W} \)-kernels of \( G \)
- \( \mathcal{M} \) is a functor, i.e., respects morphisms.
Table of contents

1 Preliminaries
   Motivation, context, definitions, and notation
   The Madden representation for $W$

2 Pointwise joins
   In $RL$
   Nice properties
   In $W$
   Limitations of the $W$ definition

3 Nakano-Stone Theorems
   The classical Nakano-Stone Theorems
   The Nakano-Stone Theorems for pointwise joins
   Is $\beta G$ a pointwise $\sigma$-completion?

4 Unconditional pointwise completeness
   Truncate sequences
   Cuts and mobile downsets
   Is $\epsilon G$ an unconditional pointwise completion?
What should we mean by a pointwise join in $\mathcal{RL}$?

**Definition of pointwise joins in $\mathcal{RL}$**

We say that $g_0$ is the *pointwise join of* $\{g_n\}$, and write $g_0 = \vee g_n$, provided that $g_0 (r, \infty) = \vee g_n (r, \infty)$ holds in $L$ for all $r \in \mathbb{R}$.

Dually, $g_0 = \wedge g_n$ if $g_0 (-\infty, r) = \vee g_n (-\infty, r)$ for all $r \in \mathbb{R}$.
What should we mean by a pointwise join in $RL$?

**Definition of pointwise joins in $RL$**

We say that $g_0$ is the *pointwise join of* $\{g_n\}$, and write $g_0 = \bigvee \bullet g_n$, provided that $g_0(r, \infty) = \bigvee g_n(r, \infty)$ holds in $L$ for all $r \in \mathbb{R}$. Dually, $g_0 = \bigwedge \bullet g_n$ if $g_0(-\infty, r) = \bigvee g_n(-\infty, r)$ for all $r \in \mathbb{R}$.

- Observe that the frame definition coincides with the spatial definition in case $L$ is the topology of a Tychonoff space.
- A frame map is completely determined by its values on the right rays alone, or on the left rays alone. This makes the appearance of only these rays in the definition less mysterious.
- Recall that $f \succeq g$ in $RL$ iff $f(r, \infty) \succeq g(r, \infty)$ for all $r \in \mathbb{R}$.
- It follows that $g_0 = \bigvee \bullet g_n$ implies $g_0 = \bigvee g_n$, and dually.
- It follows that $g_0 = \bigvee \bullet g_n = f_0$ implies $g_0 = f_0$. 
What should we mean by a pointwise join in $\mathcal{RL}$?

**Definition of pointwise joins in $\mathcal{RL}$**

We say that $g_0$ is the pointwise join of $\{g_n\}$, and write $g_0 = \bigvee^\bullet g_n$, provided that $g_0 (r, \infty) = \bigvee g_n (r, \infty)$ holds in $L$ for all $r \in \mathbb{R}$. Dually, $g_0 = \bigwedge^\bullet g_n$ if $g_0 (-\infty, r) = \bigvee g_n (-\infty, r)$ for all $r \in \mathbb{R}$.

- Observe that the frame definition coincides with the spatial definition in case $L$ is the topology of a Tychonoff space.
- A frame map is completely determined by its values on the right rays alone, or on the left rays alone. This makes the appearance of only these rays in the definition less mysterious.
- Recall that $f \succeq g$ in $\mathcal{RL}$ iff $f(r, \infty) \succeq g(r, \infty)$ for all $r \in \mathbb{R}$.
- It follows that $g_0 = \bigvee^\bullet g_n$ implies $g_0 = \bigvee g_n$, and dually.
- It follows that $g_0 = \bigvee^\bullet g_n = f_0$ implies $g_0 = f_0$. 
What should we mean by a pointwise join in $RL$?

Definition of pointwise joins in $RL$

We say that $g_0$ is the pointwise join of $\{g_n\}$, and write $g_0 = \bigvee\cdot g_n$, provided that $g_0(r, \infty) = \bigvee g_n(r, \infty)$ holds in $L$ for all $r \in \mathbb{R}$. Dually, $g_0 = \bigwedge\cdot g_n$ if $g_0(-\infty, r) = \bigvee g_n(-\infty, r)$ for all $r \in \mathbb{R}$.

- Observe that the frame definition coincides with the spatial definition in case $L$ is the topology of a Tychonoff space.
- A frame map is completely determined by its values on the right rays alone, or on the left rays alone. This makes the appearance of only these rays in the definition less mysterious.
- Recall that $f \succeq g$ in $RL$ iff $f(r, \infty) \succeq g(r, \infty)$ for all $r \in \mathbb{R}$.
- It follows that $g_0 = \bigvee\cdot g_n$ implies $g_0 = \bigvee g_n$, and dually.
- It follows that $g_0 = \bigvee\cdot g_n = f_0$ implies $g_0 = f_0$. 
What should we mean by a pointwise join in \( RL \)?

**Definition of pointwise joins in \( RL \)**

We say that \( g_0 \) is the *pointwise join of* \( \{g_n\} \), and write \( g_0 = \vee^\bullet g_n \), provided that \( g_0 (r, \infty) = \bigvee g_n (r, \infty) \) holds in \( L \) for all \( r \in \mathbb{R} \). Dually, \( g_0 = \bigwedge^\bullet g_n \) if \( g_0 (-\infty, r) = \bigvee g_n (-\infty, r) \) for all \( r \in \mathbb{R} \).

- Observe that the frame definition coincides with the spatial definition in case \( L \) is the topology of a Tychonoff space.
- A frame map is completely determined by its values on the right rays alone, or on the left rays alone. This makes the appearance of only these rays in the definition less mysterious.
- Recall that \( f \geq g \) in \( RL \) iff \( f(r, \infty) \geq g(r, \infty) \) for all \( r \in \mathbb{R} \).
- It follows that \( g_0 = \vee^\bullet g_n \) implies \( g_0 = \bigvee g_n \), and dually.
- It follows that \( g_0 = \bigvee^\bullet g_n = f_0 \) implies \( g_0 = f_0 \).
What should we mean by a pointwise join in $\mathcal{RL}$?

Definition of pointwise joins in $\mathcal{RL}$

We say that $g_0$ is the *pointwise join* of $\{g_n\}$, and write $g_0 = \bigvee^\bullet g_n$, provided that $g_0(r, \infty) = \bigvee g_n(r, \infty)$ holds in $L$ for all $r \in \mathbb{R}$. Dually, $g_0 = \bigwedge^\bullet g_n$ if $g_0(-\infty, r) = \bigvee g_n(-\infty, r)$ for all $r \in \mathbb{R}$.

- Observe that the frame definition coincides with the spatial definition in case $L$ is the topology of a Tychonoff space.
- A frame map is completely determined by its values on the right rays alone, or on the left rays alone. This makes the appearance of only these rays in the definition less mysterious.
- Recall that $f \succeq g$ in $\mathcal{RL}$ iff $f(r, \infty) \succeq g(r, \infty)$ for all $r \in \mathbb{R}$.
- It follows that $g_0 = \bigvee^\bullet g_n$ implies $g_0 = \bigvee g_n$, and dually.
- It follows that $g_0 = \bigvee^\bullet g_n = f_0$ implies $g_0 = f_0$. 
Pointwise joins behave nicely with respect to the $\ell$-group operations

Let $\{f_n\}, \{g_n\} \subseteq \mathcal{RL}$ and $f_0, g_0 \in \mathcal{RL}$.

- $f_0 = \bigvee \{f_n\}$ iff $-f_0 = \bigwedge \{-f_n\}$, and dually.
- $f_0 = \bigvee \{f_0\} = \bigwedge \{f_0\}$.
- If $f_0 = \bigvee f_n$ and $g_0 = \bigvee g_n$ then $f_0 \boxplus g_0 = \bigvee (f_0 \boxplus g_0)$, where $\boxplus$ stands for one of the $\ell$-group operations $+, -, \lor, \land$.
- If $f_0 = \bigvee f_n$ and $0 \leq r \in \mathbb{R}$ then $rf_0 = \bigvee rf_n$. 
Pointwise joins behave nicely with respect to the $\ell$-group operations

Let $\{f_n\}, \{g_n\} \subseteq \mathcal{RL}$ and $f_0, g_0 \in \mathcal{RL}$.

- $f_0 = \bigvee f_n$ iff $-f_0 = \bigwedge (-f_n)$, and dually.
- $f_0 = \bigvee \{f_0\} = \bigwedge \{f_0\}$.
- If $f_0 = \bigvee f_n$ and $g_0 = \bigvee g_n$ then $f_0 \sqcap g_0 = \bigvee (f_0 \sqcap g_0)$, where $\sqcap$ stands for one of the $\ell$-group operations $+$, $-$, $\vee$, or $\wedge$.
- If $f_0 = \bigvee f_n$ and $0 \leq r \in \mathbb{R}$ then $rf_0 = \bigvee rf_n$. 
Pointwise joins behave nicely with respect to the $\ell$-group operations

Let $\{f_n\}, \{g_n\} \subseteq RL$ and $f_0, g_0 \in RL$.

- $f_0 = \bigvee f_n$ iff $-f_0 = \bigwedge (-f_n)$, and dually.
- $f_0 = \bigvee \{f_0\} = \bigwedge \{f_0\}$.
- If $f_0 = \bigvee f_n$ and $g_0 = \bigvee g_n$ then $f_0 \boxdot g_0 = \bigvee (f_0 \boxdot g_0)$, where $\boxdot$ stands for one of the $\ell$-group operations $+, -, \lor$, or $\land$.
- If $f_0 = \bigvee f_n$ and $0 \leq r \in \mathbb{R}$ then $rf_0 = \bigvee rf_n$. 
Pointwise joins behave nicely with respect to the $\ell$-group operations

Let $\{f_n\}, \{g_n\} \subseteq \mathcal{RL}$ and $f_0, g_0 \in \mathcal{RL}$.

- $f_0 = \bigvee^\bullet f_n$ iff $-f_0 = \bigwedge^\bullet (-f_n)$, and dually.
- $f_0 = \bigvee^\bullet \{f_0\} = \bigwedge^\bullet \{f_0\}$.
- If $f_0 = \bigvee^\bullet f_n$ and $g_0 = \bigvee^\bullet g_n$ then $f_0 \boxdot g_0 = \bigvee^\bullet (f_0 \boxdot g_0)$, where $\boxdot$ stands for one of the $\ell$-group operations $+$, $-$, $\lor$, or $\land$.
- If $f_0 = \bigvee^\bullet f_n$ and $0 \leq r \in \mathbb{R}$ then $rf_0 = \bigvee^\bullet rf_n$. 
**Theorem**

Let $G \leq \mathcal{R}L$ and $H \leq \mathcal{R}M$. Then for any $\mathcal{W}$-morphism $\theta : G \to H$, if $g_0 = \bigvee^\bullet g_n$ in $G$ then $\theta g_0 = \bigvee^\bullet \theta(g_n)$ in $H$. 

**Proof.** Since $M$ is a functor, there is a unique frame map $M\theta \equiv f : L \to M$ which realizes $\theta$ in the sense that $\theta(g)(U) = f \circ g(U)$ for all $U \in \mathcal{OR}$ and $g \in G$. Therefore we have, for $r \in \mathcal{R}$, 

$$\bigvee I \theta(g_i)(r, \infty) = \bigvee I f \circ g_i(r, \infty) = f(\bigvee I g_i(r, \infty)) = f \circ g_0(r, \infty) = \theta(g_0)(r, \infty).$$
**W-morphisms preserve pointwise joins**

**Theorem**

Let $G \leq R L$ and $H \leq R M$. Then for any $W$-morphism $\theta : G \to H$, if $g_0 = \bigvee^W g_n$ in $G$ then $\theta g_0 = \bigvee^W \theta(g_n)$ in $H$.

**Proof.**

- Since $M$ is a functor, there is a unique frame map $M \theta \equiv f : L \to M$ which realizes $\theta$ in the sense that $\theta(g)(U) = f \circ g(U)$ for all $U \in O R$ and $g \in G$.

- Therefore we have, for $r \in R$,

$$\bigvee_l \theta(g_i)(r, \infty) = \bigvee_l f \circ g_i(r, \infty) = f \left( \bigvee_l g_i(r, \infty) \right) = f \circ g_0(r, \infty) = \theta(g_0)(r, \infty).$$
\(\mathbf{W}\)-morphisms preserve pointwise joins

**Theorem**

Let \( G \leq \mathcal{RL} \) and \( H \leq \mathcal{RM} \). Then for any \( \mathbf{W}\)-morphism \( \theta : G \to H \), if \( g_0 = \bigvee^\bullet g_n \) in \( G \) then \( \theta g_0 = \bigvee^\bullet \theta(g_n) \) in \( H \).

**Proof.**

- Since \( \mathcal{M} \) is a functor, there is a unique frame map \( \mathcal{M}\theta \equiv f : L \to M \) which realizes \( \theta \) in the sense that \( \theta(g)(U) = f \circ g(U) \) for all \( U \in \mathcal{O}\mathbb{R} \) and \( g \in G \).

- Therefore we have, for \( r \in \mathbb{R} \),

\[
\bigvee_l \theta(g_i)(r, \infty) = \bigvee_l f \circ g_i(r, \infty) = f \left( \bigvee_l g_i(r, \infty) \right) = f \circ g_0(r, \infty) = \theta(g_0)(r, \infty).
\]
The converse is a surprise

- The preceding theorem implies that if $g_0 = \bigvee^\bullet g_n$ in $\mathcal{RL}$ then $\bigvee \theta(g_n) = \theta(g_0)$ for every morphism $\theta$ out of $G$.
- This could be interpreted as saying that pointwise joins are ‘context-free’.
- The surprise is that this feature differentiates the pointwise joins from all others. That is, the converse holds.

**Theorem**

Suppose $\{g_n\}$ is a subset and $g_0$ an element of a $\mathbf{W}$-object $G \subseteq \mathcal{RL}$. If $\bigvee \theta(g_n) = \theta(g_0)$ for all morphisms $\theta$ out of $G$ then $\bigvee^\bullet g_n = g_0$.

- This theorem permits us to recast the definition of pointwise join without reference to any representation.

**Definition of pointwise join in $\mathbf{W}$**

In a $\mathbf{W}$-object $G$, we say that an element $g_0$ is the pointwise join of a family $\{g_n\}$, and write $\bigvee^\bullet g_n = g_0$, if $\bigvee \theta(g_n) = \theta(g_0)$ for every morphism $\theta$ out of $G$. 
The converse is a surprise

- The preceding theorem implies that if $g_0 = \bigvee^\bullet g_n$ in $RL$ then $\bigvee \theta(g_n) = \theta(g_0)$ for every morphism $\theta$ out of $G$.
- This could be interpreted as saying that pointwise joins are ‘context-free’.
- The surprise is that this feature differentiates the pointwise joins from all others. That is, the converse holds.

**Theorem**

*Suppose $\{g_n\}$ is a subset and $g_0$ an element of a $W$-object $G \leq RL$. If $\bigvee \theta(g_n) = \theta(g_0)$ for all morphisms $\theta$ out of $G$ then $\bigvee^\bullet g_n = g_0$.**

- This theorem permits us to recast the definition of pointwise join without reference to any representation.

**Definition of pointwise join in $W$**

In a $W$-object $G$, we say that an element $g_0$ is the **pointwise join** of a family $\{g_n\}$, and write $\bigvee^\bullet g_n = g_0$, if $\bigvee \theta(g_n) = \theta(g_0)$ for every morphism $\theta$ out of $G$. 
The converse is a surprise

- The preceding theorem implies that if \( g_0 = \bigvee^\bullet g_n \) in \( RL \) then \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
- This could be interpreted as saying that pointwise joins are ‘context-free’.
- The surprise is that this feature differentiates the pointwise joins from all others. That is, the converse holds.

- **Theorem**

  *Suppose \( \{g_n\} \) is a subset and \( g_0 \) an element of a \( W \)-object \( G \subseteq RL \). If \( \bigvee \theta(g_n) = \theta(g_0) \) for all morphisms \( \theta \) out of \( G \) then \( \bigvee^\bullet g_n = g_0 \).*

  - This theorem permits us to recast the definition of pointwise join without reference to any representation.

- **Definition of pointwise join in \( W \)**

  In a \( W \)-object \( G \), we say that an element \( g_0 \) is the *pointwise join* of a family \( \{g_n\} \), and write \( \bigvee^\bullet g_n = g_0 \), if \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
The converse is a surprise

- The preceding theorem implies that if \( g_0 = \bigvee^\bullet g_n \) in \( RL \) then \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
- This could be interpreted as saying that pointwise joins are ‘context-free’.
- The surprise is that this feature differentiates the pointwise joins from all others. That is, the converse holds.

- **Theorem**

  Suppose \( \{g_n\} \) is a subset and \( g_0 \) an element of a \( W \)-object \( G \leq RL \). If \( \bigvee \theta(g_n) = \theta(g_0) \) for all morphisms \( \theta \) out of \( G \) then \( \bigvee^\bullet g_n = g_0 \).

  - This theorem permits us to recast the definition of pointwise join without reference to any representation.

- **Definition of pointwise join in \( W \)**

  In a \( W \)-object \( G \), we say that an element \( g_0 \) is the pointwise join of a family \( \{g_n\} \), and write \( \bigvee^\bullet g_n = g_0 \), if \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
The converse is a surprise

- The preceding theorem implies that if \( g_0 = \bigvee g_n \) in \( \mathcal{RL} \) then \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
- This could be interpreted as saying that pointwise joins are ‘context-free’.
- The surprise is that this feature differentiates the pointwise joins from all others. That is, the converse holds.

**Theorem**

Suppose \( \{g_n\} \) is a subset and \( g_0 \) an element of a \( \mathbf{W} \)-object \( G \leq \mathcal{RL} \). If \( \bigvee \theta(g_n) = \theta(g_0) \) for all morphisms \( \theta \) out of \( G \) then \( \bigvee^\bullet g_n = g_0 \).

- This theorem permits us to recast the definition of pointwise join without reference to any representation.

**Definition of pointwise join in \( \mathbf{W} \)**

In a \( \mathbf{W} \)-object \( G \), we say that an element \( g_0 \) is the **pointwise join** of a family \( \{g_n\} \), and write \( \bigvee^\bullet g_n = g_0 \), if \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
The converse is a surprise

- The preceding theorem implies that if $g_0 = \bigvee^\bullet g_n$ in $RL$ then $\bigvee \theta(g_n) = \theta(g_0)$ for every morphism $\theta$ out of $G$.
- This could be interpreted as saying that pointwise joins are ‘context-free’.
- The surprise is that this feature differentiates the pointwise joins from all others. That is, the converse holds.

- **Theorem**

  *Suppose $\{g_n\}$ is a subset and $g_0$ an element of a $W$-object $G \subseteq RL$. If $\bigvee \theta(g_n) = \theta(g_0)$ for all morphisms $\theta$ out of $G$ then $\bigvee^\bullet g_n = g_0$."

  - This theorem permits us to recast the definition of pointwise join without reference to any representation.

- **Definition of pointwise join in $W$**

  *In a $W$-object $G$, we say that an element $g_0$ is the *pointwise join* of a family $\{g_n\}$, and write $\bigvee^\bullet g_n = g_0$, if $\bigvee \theta(g_n) = \theta(g_0)$ for every morphism $\theta$ out of $G$. "*
The converse is a surprise

• The preceding theorem implies that if \( g_0 = \bigvee^\bullet g_n \) in \( RL \) then \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
• This could be interpreted as saying that pointwise joins are ‘context-free’.
• The surprise is that this feature differentiates the pointwise joins from all others. That is, the converse holds.

• Theorem

Suppose \( \{g_n\} \) is a subset and \( g_0 \) an element of a \( W \)-object \( G \subseteq RL \). If \( \bigvee \theta(g_n) = \theta(g_0) \) for all morphisms \( \theta \) out of \( G \) then \( \bigvee^\bullet g_n = g_0 \).

• This theorem permits us to recast the definition of pointwise join without reference to any representation.

• Definition of pointwise join in \( W \)

In a \( W \)-object \( G \), we say that an element \( g_0 \) is the pointwise join of a family \( \{g_n\} \), and write \( \bigvee^\bullet g_n = g_0 \), if \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
The converse is a surprise

- The preceding theorem implies that if \( g_0 = \bigvee \cdot g_n \) in \( RL \) then \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
- This could be interpreted as saying that pointwise joins are ‘context-free’.
- The surprise is that this feature differentiates the pointwise joins from all others. That is, the converse holds.

- **Theorem**

  Suppose \( \{g_n\} \) is a subset and \( g_0 \) an element of a \( W \)-object \( G \leq RL \). If \( \bigvee \theta(g_n) = \theta(g_0) \) for all morphisms \( \theta \) out of \( G \) then \( \bigvee \cdot g_n = g_0 \).

  - This theorem permits us to recast the definition of pointwise join without reference to any representation.

- **Definition of pointwise join in \( W \)**

  In a \( W \)-object \( G \), we say that an element \( g_0 \) is the *pointwise join* of a family \( \{g_n\} \), and write \( \bigvee \cdot g_n = g_0 \), if \( \bigvee \theta(g_n) = \theta(g_0) \) for every morphism \( \theta \) out of \( G \).
The working version of the definition

- **Theorem**
  \[ \bigvee g_n = g_0 \iff [(g_n - r)^+ : n \in \mathbb{N}] = [(g_0 - r)^+] \text{ for all } r \in \mathbb{R}. \]

- **Proof.**
  In the Madden representation, \( g(r, \infty) \) is evaluated in the frame \( MG \) of \( W \)-kernels of \( G \) as \([ (g - r)^+ ] \). \( \square \)

- **Corollary**
  In any \( W \)-object \( G \), if \( g_0 = \bigvee K \) then \( g_0 = \bigvee K_0 \) for some countable subset \( K_0 \subseteq K \).
The working version of the definition

- **Theorem**
  \[ \bigvee^\bullet g_n = g_0 \text{ iff } [(g_n - r)^+ : n \in \mathbb{N}] = [(g_0 - r)^+] \text{ for all } r \in \mathbb{R}. \]

- **Proof.**
  In the Madden representation, \( g(r, \infty) \) is evaluated in the frame \( \mathcal{M}G \) of \( W \)-kernels of \( G \) as \([ (g - r)^+ ] \). \( \square \)

- **Corollary**
  In any \( W \)-object \( G \), if \( g_0 = \bigvee^\bullet K \) then \( g_0 = \bigvee^\bullet K_0 \) for some countable subset \( K_0 \subseteq K \). 
The working version of the definition

- **Theorem**
  \[ \bigvee^\bullet g_n = g_0 \quad \text{iff} \quad [(g_n - r)^+ : n \in \mathbb{N}] = [(g_0 - r)^+] \quad \text{for all} \quad r \in \mathbb{R}. \]

- **Proof.**
  In the Madden representation, \( g(r, \infty) \) is evaluated in the frame \( \mathcal{M}G \) of \( \mathbf{W} \)-kernels of \( G \) as \( [(g - r)^+] \). \( \square \)

- **Corollary**
  In any \( \mathbf{W} \)-object \( G \), if \( g_0 = \bigvee^\bullet K \) then \( g_0 = \bigvee^\bullet K_0 \) for some countable subset \( K_0 \subseteq K \).
The working version of the definition

- **Theorem**
  \[ \bigvee^\bullet g_n = g_0 \text{ iff } [(g_n - r)^+ : n \in \mathbb{N}] = [(g_0 - r)^+] \text{ for all } r \in \mathbb{R}. \]

- **Proof.**
  In the Madden representation, \( g(r, \infty) \) is evaluated in the frame \( M_G \) of \( \mathbb{W} \)-kernels of \( G \) as \( [(g - r)^+] \).

- **Corollary**
  In any \( \mathbb{W} \)-object \( G \), if \( g_0 = \bigvee^\bullet K \) then \( g_0 = \bigvee^\bullet K_0 \) for some countable subset \( K_0 \subseteq K \).
The working version of the definition

• **Theorem**
  \[ \forall^\bullet g_n = g_0 \iff [(g_n - r)^+ : n \in \mathbb{N}] = [(g_0 - r)^+] \text{ for all } r \in \mathbb{R}. \]

• **Proof.**
  In the Madden representation, \( g(r, \infty) \) is evaluated in the frame \( MG \) of \( W \)-kernels of \( G \) as \( [(g - r)^+] \).

• **Corollary**
  In any \( W \)-object \( G \), if \( g_0 = \bigvee^\bullet K \) then \( g_0 = \bigvee^\bullet K_0 \) for some countable subset \( K_0 \subseteq K \).
The working version of the definition

• **Theorem**
  \[ \bigvee^\bullet g_n = g_0 \text{ iff } [(g_n - r)^+ : n \in \mathbb{N}] = [(g_0 - r)^+] \text{ for all } r \in \mathbb{R}. \]

• **Proof.**
  In the Madden representation, \( g(r, \infty) \) is evaluated in the frame \( \mathcal{M}G \) of \( \mathbb{W} \)-kernels of \( G \) as \( [(g - r)^+] \).

• **Corollary**
  In any \( \mathbb{W} \)-object \( G \), if \( g_0 = \bigvee^\bullet K \) then \( g_0 = \bigvee^\bullet K_0 \) for some countable subset \( K_0 \subseteq K \).
The \textbf{W} definition differs from the spatiiale definition if the space is not Lindelöf.

- **Theorem**
  Suppose $L = \emptyset X$ for a Tychonoff space $X$, so that $G \equiv RL \approx CX$. Then pointwise joins in $CX$ are \textbf{W}-pointwise in $G$ iff $X$ is Lindelöf.

- **Proof outline**
  Begin with a cozero cover $\{\text{coz} \ g : g \in C\}$ of $X$ having no countable subcover. Clearly $\bigvee_C (ng \land 1) = 1$ in $CX$. But this join cannot be \textbf{W}-pointwise because there is no countable subset $C_0 \subseteq C$ such that $\bigvee_{C_0} (ng \land 1) = 1$.

  - What’s going on?
  - $L$ has a regular Lindelöf reflection $\lambda L$. Draw a picture.
  - $\Omega L = X$ and $RL \approx R\lambda L$.
  - The Madden frame of any \textbf{W}-object $G$ is Lindelöf.
  - In $RL$, the \textbf{W} definition is actually capturing pointwise joins in $R\lambda L$. 

\[\[\]\]
The \textbf{W} definition differs from the spatiiale definition if the space is not Lindelöf.

- **Theorem**
  Suppose \( L = \emptyset X \) for a Tychonoff space \( X \), so that \( G \equiv RL \approx CX \). Then pointwise joins in \( CX \) are \textbf{W}-pointwise in \( G \) iff \( X \) is Lindelöf.

- **Proof outline**
  Begin with a cozero cover \( \{ \text{coz} \ g : g \in C \} \) of \( X \) having no countable subcover. Clearly \( \bigvee_C (ng \land 1) = 1 \) in \( CX \). But this join cannot be \textbf{W}-pointwise because there is no countable subset \( C_0 \subseteq C \) such that \( \bigvee_{C_0} (ng \land 1) = 1 \).

  - What’s going on?
  - \( L \) has a regular Lindelöf reflection \( \lambda L \). Draw a picture.
  - \( \Omega\lambda L = X \) and \( RL \approx RL \).
  - The Madden frame of any \textbf{W}-object \( G \) is Lindelöf.
  - In \( RL \), the \textbf{W} definition is actually capturing pointwise joins in \( RL \).
The **W** definition differs from the spatiiale definition if the space is not Lindelöf.

- **Theorem**
  Suppose \( L = \emptyset X \) for a Tychonoff space \( X \), so that \( G \equiv RL \approx CX \). Then pointwise joins in \( CX \) are **W**-pointwise in \( G \) iff \( X \) is Lindelöf.

- **Proof outline**
  Begin with a cozero cover \( \{ \text{coz} \ g : g \in C \} \) of \( X \) having no countable subcover. Clearly \( \bigvee_{C} (ng \land 1) = 1 \) in \( CX \). But this join cannot be **W**-pointwise because there is no countable subset \( C_0 \subseteq C \) such that \( \bigvee_{C_0} (ng \land 1) = 1 \).

- **What’s going on?**
  - \( L \) has a regular Lindelöf reflection \( \lambda L \). Draw a picture.
  - \( \Omega\lambda L = X \) and \( RL \approx R\lambda L \).
  - The Madden frame of any **W**-object \( G \) is Lindelöf.
  - In \( RL \), the **W** definition is actually capturing pointwise joins in \( R\lambda L \).
The $\mathbf{W}$ definition differs from the spatial definition if the space is not Lindelöf.

- **Theorem**
  Suppose $L = \emptyset X$ for a Tychonoff space $X$, so that $G \equiv RL \approx CX$. Then pointwise joins in $CX$ are $\mathbf{W}$-pointwise in $G$ iff $X$ is Lindelöf.

- **Proof outline**
  Begin with a cozero cover $\{\text{coz } g : g \in C\}$ of $X$ having no countable subcover. Clearly $\bigvee_{C}(ng \land 1) = 1$ in $CX$. But this join cannot be $\mathbf{W}$-pointwise because there is no countable subset $C_0 \subseteq C$ such that $\bigvee_{C_0}(ng \land 1) = 1$.

  - What’s going on?
  - $L$ has a regular Lindelöf reflection $\lambda L$. Draw a picture.
  - $\Omega \lambda L = X$ and $RL \approx R\lambda L$.
  - The Madden frame of any $\mathbf{W}$-object $G$ is Lindelöf.
  - In $RL$, the $\mathbf{W}$ definition is actually capturing pointwise joins in $R\lambda L$.
The \( \mathbf{W} \) definition differs from the spatiale definition if the space is not Lindelöf.

- **Theorem**
  Suppose \( L = \emptyset X \) for a Tychonoff space \( X \), so that \( G \equiv RL \approx CX \). Then pointwise joins in \( CX \) are \( \mathbf{W} \)-pointwise in \( G \) iff \( X \) is Lindelöf.

- **Proof outline**
  Begin with a cozero cover \( \{ \text{coz } g : g \in C \} \) of \( X \) having no countable subcover. Clearly \( \bigvee_C (ng \wedge 1) = 1 \) in \( CX \). But this join cannot be \( \mathbf{W} \)-pointwise because there is no countable subset \( C_0 \subseteq C \) such that \( \bigvee_{C_0} (ng \wedge 1) = 1 \).

- **What’s going on?**
  - \( L \) has a regular Lindelöf reflection \( \lambda L \). Draw a picture.
  - \( \Omega \lambda L = X \) and \( RL \approx R\lambda L \).
  - The Madden frame of any \( \mathbf{W} \)-object \( G \) is Lindelöf.
  - In \( RL \), the \( \mathbf{W} \) definition is actually capturing pointwise joins in \( R\lambda L \).
The $W$ definition differs from the spatiale definition if the space is not Lindelőf.

• Theorem
Suppose $L = \emptyset X$ for a Tychonoff space $X$, so that $G \equiv RL \approx CX$. Then pointwise joins in $CX$ are $W$-pointwise in $G$ iff $X$ is Lindelőf.

• Proof outline
Begin with a cozero cover $\{coz g : g \in C\}$ of $X$ having no countable subcover. Clearly $\bigvee_C (ng \wedge 1) = 1$ in $CX$. But this join cannot be $W$-pointwise because there is no countable subset $C_0 \subseteq C$ such that $\bigvee_{C_0} (ng \wedge 1) = 1$.

• What’s going on?
• $L$ has a regular Lindelőf reflection $\lambda L$. Draw a picture.
• $\Omega \lambda L = X$ and $RL \approx R\lambda L$.
• The Madden frame of any $W$-object $G$ is Lindelőf.
• In $RL$, the $W$ definition is actually capturing pointwise joins in $R\lambda L$. 
The $\mathbf{W}$ definition differs from the spatiale definition if the space is not Lindelöf.

- **Theorem**
  Suppose $L = \emptyset X$ for a Tychonoff space $X$, so that $G \equiv RL \approx CX$. Then pointwise joins in $CX$ are $\mathbf{W}$-pointwise in $G$ iff $X$ is Lindelöf.

- **Proof outline**
  Begin with a cozero cover \{coz $g : g \in C$\} of $X$ having no countable subcover. Clearly $\bigvee^\bullet_C (ng \land 1) = 1$ in $CX$. But this join cannot be $\mathbf{W}$-pointwise because there is no countable subset $C_0 \subseteq C$ such that $\bigvee_{C_0} (ng \land 1) = 1$.

  - What’s going on?
  - $L$ has a regular Lindelöf reflection $\lambda L$. Draw a picture.
  - $\Omega \lambda L = X$ and $RL \approx R\lambda L$.
  - The Madden frame of any $\mathbf{W}$-object $G$ is Lindelöf.
  - In $RL$, the $\mathbf{W}$ definition is actually capturing pointwise joins in $R\lambda L$. 
The **W** definition differs from the spatiole definition if the space is not Lindelöf.

- **Theorem**
  Suppose $L = \emptyset X$ for a Tychonoff space $X$, so that $G \equiv RL \approx CX$. Then pointwise joins in $CX$ are **W**-pointwise in $G$ iff $X$ is Lindelöf.

- **Proof outline**
  Begin with a cozero cover $\{\text{coz } g : g \in C\}$ of $X$ having no countable subcover. Clearly $\bigvee_C (ng \land 1) = 1$ in $CX$. But this join cannot be **W**-pointwise because there is no countable subset $C_0 \subseteq C$ such that $\bigvee_{C_0} (ng \land 1) = 1$.
  - What’s going on?
  - $L$ has a regular Lindelöf reflection $\lambda L$. Draw a picture.
  - $\Omega \lambda L = X$ and $RL \approx R\lambda L$.
  - The Madden frame of any **W**-object $G$ is Lindelöf.
  - In $RL$, the **W** definition is actually capturing pointwise joins in $R\lambda L$. 
The \( \mathbf{W} \) definition differs from the spatial definition if the space is not Lindelöf.

- **Theorem**
  Suppose \( L = \emptyset X \) for a Tychonoff space \( X \), so that \( G \equiv RL \approx CX \). Then pointwise joins in \( CX \) are \( \mathbf{W} \)-pointwise in \( G \) iff \( X \) is Lindelöf.

- **Proof outline**
  Begin with a cozero cover \( \{ \text{coz} \, g : g \in C \} \) of \( X \) having no countable subcover. Clearly \( \bigvee_{C} (ng \wedge 1) = 1 \) in \( CX \). But this join cannot be \( \mathbf{W} \)-pointwise because there is no countable subset \( C_{0} \subseteq C \) such that \( \bigvee_{C_{0}} (ng \wedge 1) = 1 \).

  - What’s going on?
  - \( L \) has a regular Lindelöf reflection \( \lambda L \). Draw a picture.
  - \( \Omega \lambda L = X \) and \( RL \approx R\lambda L \).
  - The Madden frame of any \( \mathbf{W} \)-object \( G \) is Lindelöf.
  - In \( RL \), the \( \mathbf{W} \) definition is actually capturing pointwise joins in \( R\lambda L \).
Table of contents

1 Preliminaries
   Motivation, context, definitions, and notation
   The Madden representation for W.

2 Pointwise joins
   In RL
   Nice properties
   In W
   Limitations of the W definition

3 Nakano-Stone Theorems
   The classical Nakano-Stone Theorems
   The Nakano-Stone Theorems for pointwise joins
   Is βG a pointwise σ-completion?

4 Unconditional pointwise completeness
   Truncate sequences
   Cuts and mobile downsets
   Is εG an unconditional pointwise completion?
Conditional pointwise completeness of $\mathcal{RL}$

- There is a relationship between the completeness properties of $\mathcal{RL}$ and the disconnectivity of $L$.
- A good example is the classical Nakano-Stone Theorem. We give the pointfree formulation of this famous theorem below.
- A $\mathcal{W}$-object $G$ is called conditionally complete (conditionally $\sigma$-complete) if every bounded (countable) subset of $G$ has a join in $G$.
- A frame $L$ is called extremally (basically) disconnected if $a^* \vee a^{**} = \top$, $a \in L$ ($a \in \text{coz } L$).

**Theorem (Banaschewski, Hong)**

$\mathcal{RL}$ is conditionally complete ($\sigma$-complete) iff $L$ is extremally (basically) disconnected.
Conditional pointwise completeness of $\mathcal{R}L$

- There is a relationship between the completeness properties of $\mathcal{R}L$ and the disconnectivity of $L$.
- A good example is the classical Nakano-Stone Theorem. We give the pointfree formulation of this famous theorem below.
- A $\mathcal{W}$-object $G$ is called conditionally complete (conditionally $\sigma$-complete) if every bounded (countable) subset of $G$ has a join in $G$.
- A frame $L$ is called extremally (basically) disconnected if $a^* \lor a^{**} = \top$, $a \in L$ ($a \in \text{coz } L$).

Theorem (Banaschewski, Hong)

$\mathcal{R}L$ is conditionally complete ($\sigma$-complete) iff $L$ is extremally (basically) disconnected.
Conditional pointwise completeness of $\mathcal{RL}$

- There is a relationship between the completeness properties of $\mathcal{RL}$ and the disconnectivity of $L$.
- A good example is the classical Nakano-Stone Theorem. We give the pointfree formulation of this famous theorem below.
- A $\mathcal{W}$-object $G$ is called *conditionally complete* (*conditionally $\sigma$-complete*) if every bounded (countable) subset of $G$ has a join in $G$.
- A frame $L$ is called extremally (basically) disconnected if $a^* \vee a^{**} = \top$, $a \in L$ ($a \in \text{coz } L$).

**Theorem (Banaschewski, Hong)**

$\mathcal{RL}$ is conditionally complete ($\sigma$-complete) iff $L$ is extremally (basically) disconnected.
Conditional pointwise completeness of $\mathcal{RL}$

- There is a relationship between the completeness properties of $\mathcal{RL}$ and the disconnectivity of $L$.

- A good example is the classical Nakano-Stone Theorem. We give the pointfree formulation of this famous theorem below.

- A $\mathcal{W}$-object $G$ is called *conditionally complete (conditionally $\sigma$-complete)* if every bounded (countable) subset of $G$ has a join in $G$.

- A frame $L$ is called extremally (basically) disconnected if $a^* \vee a^{**} = \top$, $a \in L$ ($a \in \text{coz } L$).

- **Theorem (Banaschewski, Hong)**

  $\mathcal{RL}$ is conditionally complete ($\sigma$-complete) iff $L$ is extremally (basically) disconnected.
Conditional pointwise completeness of $\mathcal{R}L$

- There is a relationship between the completeness properties of $\mathcal{R}L$ and the disconnectivity of $L$.
- A good example is the classical Nakano-Stone Theorem. We give the pointfree formulation of this famous theorem below.
- A $\mathcal{W}$-object $G$ is called *conditionally complete* (*conditionally $\sigma$-complete*) if every bounded (countable) subset of $G$ has a join in $G$.
- A frame $L$ is called extremally (basically) disconnected if $a^* \lor a^{**} = \top$, $a \in L$ ($a \in \text{coz} L$).

- **Theorem (Banaschewski, Hong)**
  
  $\mathcal{R}L$ is *conditionally complete* ($\sigma$-complete) iff $L$ is extremally (basically) disconnected.
Conditional pointwise completeness of $\mathcal{RL}$

- There is a relationship between the completeness properties of $\mathcal{RL}$ and the disconnectivity of $L$.
- A good example is the classical Nakano-Stone Theorem. We give the pointfree formulation of this famous theorem below.
- A $\mathcal{W}$-object $G$ is called conditionally complete (conditionally $\sigma$-complete) if every bounded (countable) subset of $G$ has a join in $G$.
- A frame $L$ is called extremally (basically) disconnected if $a^* \lor a^{**} = \top$, $a \in L$ ($a \in \text{coz } L$).

- **Theorem (Banaschewski, Hong)**

  $\mathcal{RL}$ is conditionally complete ($\sigma$-complete) iff $L$ is extremally (basically) disconnected.
Conditional pointwise completeness

- **Definition of conditional pointwise completeness**
  A \( W \)-object \( G \) is said to be *conditionally pointwise complete* (\( \sigma \)-complete) if every bounded (countable) subset of \( G^+ \) has a pointwise join in \( G \).

- **The pointwise Nakano-Stone Theorem**
  [B., Hager, Walters Wayland] \( RL \) is conditionally pointwise complete (\( \sigma \)-complete) iff \( L \) is a Boolean frame (a \( P \)-frame).
  - The theorem tells us that, in order for bounded (countable) subsets to have pointwise joins in \( RL \), we need \( L \) to be Boolean (a \( P \)-frame).
  - Often in analysis one calculates joins in \( R^I \), where \( I \) is some discrete index set. That is because pointwise limits always exist in this setting.
  - But now we have a deeper understanding of the situation. The joins are the right ones, not because \( I \) is discrete, but because its topology is Boolean.
Conditional pointwise completeness

• Definition of conditional pointwise completeness
  A $W$-object $G$ is said to be *conditionally pointwise complete* ($\sigma$-complete) if every bounded (countable) subset of $G^+$ has a pointwise join in $G$.

• The pointwise Nakano-Stone Theorem
  [B., Hager, Walters Wayland] $RL$ is conditionally pointwise complete ($\sigma$-complete) iff $L$ is a Boolean frame (a $P$-frame).
  • The theorem tells us that, in order for bounded (countable) subsets to have pointwise joins in $RL$, we need $L$ to be Boolean (a $P$-frame).
  • Often in analysis one calculates joins in $R^I$, where $I$ is some discrete index set. That is because pointwise limits always exist in this setting.
  • But now we have a deeper understanding of the situation. The joins are the right ones, not because $I$ is discrete, but because its topology is Boolean.
Conditional pointwise completeness

- **Definition of conditional pointwise completeness**
  A $\mathcal{W}$-object $G$ is said to be *conditionally pointwise complete* ($\sigma$-complete) if every bounded (countable) subset of $G^+$ has a pointwise join in $G$.

- **The pointwise Nakano-Stone Theorem**
  [B., Hager, Walters Wayland] $\mathcal{RL}$ is conditionally pointwise complete ($\sigma$-complete) iff $L$ is a Boolean frame (a $P$-frame).
  - The theorem tells us that, in order for bounded (countable) subsets to have pointwise joins in $\mathcal{RL}$, we need $L$ to be Boolean (a $P$-frame).
  - Often in analysis one calculates joins in $\mathbb{R}^I$, where $I$ is some discrete index set. That is because pointwise limits always exist in this setting.
  - But now we have a deeper understanding of the situation. The joins are the right ones, not because $I$ is discrete, but because its topology is Boolean.
Conditional pointwise completeness

• Definition of conditional pointwise completeness
  A $\mathfrak{W}$-object $G$ is said to be conditionally pointwise complete ($\sigma$-complete) if every bounded (countable) subset of $G^+$ has a pointwise join in $G$.

• The pointwise Nakano-Stone Theorem
  [B., Hager, Walters Wayland] $\mathcal{RL}$ is conditionally pointwise complete ($\sigma$-complete) iff $L$ is a Boolean frame (a $P$-frame).
  • The theorem tells us that, in order for bounded (countable) subsets to have pointwise joins in $\mathcal{RL}$, we need $L$ to be Boolean (a $P$-frame).
  • Often in analysis one calculates joins in $\mathbb{R}^I$, where $I$ is some discrete index set. That is because pointwise limits always exist in this setting.
  • But now we have a deeper understanding of the situation. The joins are the right ones, not because $I$ is discrete, but because its topology is Boolean.
Conditional pointwise completeness

• Definition of conditional pointwise completeness
  A \( W \)-object \( G \) is said to be conditionally pointwise complete (\( \sigma \)-complete) if every bounded (countable) subset of \( G^+ \) has a pointwise join in \( G \).

• The pointwise Nakano-Stone Theorem
  [B., Hager, Walters Wayland] \( RL \) is conditionally pointwise complete (\( \sigma \)-complete) iff \( L \) is a Boolean frame (a \( P \)-frame).
  • The theorem tells us that, in order for bounded (countable) subsets to have pointwise joins in \( RL \), we need \( L \) to be Boolean (a \( P \)-frame).
  • Often in analysis one calculates joins in \( R^I \), where \( I \) is some discrete index set. That is because pointwise limits always exist in this setting.
  • But now we have a deeper understanding of the situation. The joins are the right ones, not because \( I \) is discrete, but because its topology is Boolean.
Conditional pointwise completeness

- Definition of conditional pointwise completeness
  A \( W \)-object \( G \) is said to be \textit{conditionally pointwise complete} (\( \sigma \)-complete) if every bounded (countable) subset of \( G^+ \) has a pointwise join in \( G \).

- The pointwise Nakano-Stone Theorem
  [B., Hager, Walters Wayland] \( RL \) is conditionally pointwise complete (\( \sigma \)-complete) iff \( L \) is a Boolean frame (a \( P \)-frame).
  - The theorem tells us that, in order for bounded (countable) subsets to have pointwise joins in \( RL \), we need \( L \) to be Boolean (a \( P \)-frame).
  - Often in analysis one calculates joins in \( R^I \), where \( I \) is some discrete index set. That is because pointwise limits always exist in this setting.
  - But now we have a deeper understanding of the situation. The joins are the right ones, not because \( I \) is discrete, but because its topology is Boolean.
Conditional pointwise completeness

- **Definition of conditional pointwise completeness**
  A $\mathcal{W}$-object $G$ is said to be *conditionally pointwise complete* ($\sigma$-complete) if every bounded (countable) subset of $G^+$ has a pointwise join in $G$.

- **The pointwise Nakano-Stone Theorem**
  [B., Hager, Walters Wayland] $RL$ is conditionally pointwise complete ($\sigma$-complete) iff $L$ is a Boolean frame (a $P$-frame).
  - The theorem tells us that, in order for bounded (countable) subsets to have pointwise joins in $RL$, we need $L$ to be Boolean (a $P$-frame).
  - Often in analysis one calculates joins in $\mathbb{R}^I$, where $I$ is some discrete index set. That is because pointwise limits always exist in this setting.
  - But now we have a deeper understanding of the situation. The joins are the right ones, not because $I$ is discrete, but because its topology is Boolean.
When all existing joins are pointwise

- One might ask when every existing (countable) join in $G$ is pointwise.

- **Theorem**

  A $\mathcal{W}$-object $G$ has the feature that every existing (countable) join is pointwise iff $G$ is cozero dense in $\mathbb{R}L$ for some Boolean frame ($P$-frame) $L$, i.e., iff $\mathcal{M}L$ is Boolean (a $P$-frame).
When all existing joins are pointwise

• One might ask when every existing (countable) join in $G$ is pointwise.

• Theorem
  A $\mathbf{W}$-object $G$ has the feature that every existing (countable) join is pointwise iff $G$ is cozero dense in $\mathcal{RL}$ for some Boolean frame ($P$-frame) $L$, i.e., iff $\mathcal{M}L$ is Boolean (a $P$-frame).
When all existing joins are pointwise

- One might ask when every existing (countable) join in $G$ is pointwise.

- **Theorem**
  A $\mathcal{W}$-object $G$ has the feature that every existing (countable) join is pointwise iff $G$ is cozero dense in $\mathcal{RL}$ for some Boolean frame ($P$-frame) $L$, i.e., iff $\mathcal{ML}$ is Boolean (a $P$-frame).
The $P$-frame reflection of $L$ and the epicompletion of $G$

- Not every frame is a $P$-frame, of course, but, as luck would have it, every frame sits inside a canonically associated $P$-frame.

- **Theorem (B, Walters-Wayland, Zenk)**

  $P$-frames are bireflective in frames. That is, every frame $L$ has an extension $\pi_L : L \to \mathcal{P}L$, $\mathcal{P}L$ a $P$-frame, such that every morphism from $L$ into a $P$-frame factors through $\pi_L$.

  - The embedding $\pi_L : L \to \mathcal{P}L$ gives an embedding $\mathcal{R}L \to \mathcal{R}\mathcal{P}L$.
  - When concatenated with the Madden representation $G \to \mathcal{R}\mathcal{M}G$ of an arbitrary $\mathbf{W}$-object $G$, this gives an embedding $G \to \mathcal{R}\mathcal{P}\mathcal{M}G \equiv \beta G$.
  - This is evidently a reflector from $\mathbf{W}$ into the full subcategory of conditionally pointwise $\sigma$ complete objects.
  - This is the very well studied class of epicomplete objects in $\mathbf{W}$.  
The $P$-frame reflection of $L$ and the epicompletion of $G$

- Not every frame is a $P$-frame, of course, but, as luck would have it, every frame sits inside a canonically associated $P$-frame.

- **Theorem (B, Walters-Wayland, Zenk)**

  *$P$-frames are bireflective in frames. That is, every frame $L$ has an extension $\pi_L : L \to PL$, $PL$ a $P$-frame, such that every morphism from $L$ into a $P$-frame factors through $\pi_L$.  
    - The embedding $\pi_L : L \to PL$ gives an embedding $RL \to RPL$. 
    - When concatenated with the Madden representation $G \to RMG$ of an arbitrary $W$-object $G$, this gives an embedding $G \to RPMG \equiv \beta G$. 
    - This is evidently a reflector from $W$ into the full subcategory of conditionally pointwise $\sigma$ complete objects. 
    - This is the very well studied class of epicomplete objects in $W$.***
The $P$-frame reflection of $L$ and the epicompletion of $G$

- Not every frame is a $P$-frame, of course, but, as luck would have it, every frame sits inside a canonically associated $P$-frame.

**Theorem (B, Walters-Wayland, Zenk)**

$P$-frames are bireflective in frames. That is, every frame $L$ has an extension $\pi_L : L \rightarrow PL$, $PL$ a $P$-frame, such that every morphism from $L$ into a $P$-frame factors through $\pi_L$.

- The embedding $\pi_L : L \rightarrow PL$ gives an embedding $RL \rightarrow RPL$.
- When concatenated with the Madden representation $G \rightarrow RMG$ of an arbitrary $W$-object $G$, this gives an embedding $G \rightarrow RPMG \equiv \beta G$.
- This is evidently a reflector from $W$ into the full subcategory of conditionally pointwise $\sigma$ complete objects.
- This is the very well studied class of epicomplete objects in $W$. 
The $P$-frame reflection of $L$ and the epicompletion of $G$

- Not every frame is a $P$-frame, of course, but, as luck would have it, every frame sits inside a canonically associated $P$-frame.

- **Theorem (B, Walters-Wayland, Zenk)**
  
  $P$-frames are bireflective in frames. That is, every frame $L$ has an extension $\pi_L : L \to PL$, $PL$ a $P$-frame, such that every morphism from $L$ into a $P$-frame factors through $\pi_L$.

  - The embedding $\pi_L : L \to PL$ gives an embedding $RL \to RPL$.
  - When concatenated with the Madden representation $G \to R\mathcal{M}G$ of an arbitrary $\mathcal{W}$-object $G$, this gives an embedding $G \to R\mathcal{P}\mathcal{M}G \equiv \beta G$.
  - This is evidently a reflector from $\mathcal{W}$ into the full subcategory of conditionally pointwise $\sigma$ complete objects.
  - This is the very well studied class of epicomplete objects in $\mathcal{W}$. 
The $P$-frame reflection of $L$ and the epicompletion of $G$

- Not every frame is a $P$-frame, of course, but, as luck would have it, every frame sits inside a canonically associated $P$-frame.

- **Theorem (B, Walters-Wayland, Zenk)**

  $P$-frames are bireflective in frames. That is, every frame $L$ has an extension $\pi_L : L \to PL$, $PL$ a $P$-frame, such that every morphism from $L$ into a $P$-frame factors through $\pi_L$.

  - The embedding $\pi_L : L \to PL$ gives an embedding $RL \to RPL$.
  - When concatenated with the Madden representation $G \to RMG$ of an arbitrary $W$-object $G$, this gives an embedding $G \to RPMG \equiv \beta G$.
  - This is evidently a reflector from $W$ into the full subcategory of conditionally pointwise $\sigma$ complete objects.
  - This is the very well studied class of epicomplete objects in $W$.
The \( P \)-frame reflection of \( L \) and the epicompletion of \( G \)

- Not every frame is a \( P \)-frame, of course, but, as luck would have it, every frame sits inside a canonically associated \( P \)-frame.

- **Theorem (B, Walters-Wayland, Zenk)**

  \( P \)-frames are bireflective in frames. That is, every frame \( L \) has an extension \( \pi_L : L \to PL \), \( PL \) a \( P \)-frame, such that every morphism from \( L \) into a \( P \)-frame factors through \( \pi_L \).

  - The embedding \( \pi_L : L \to PL \) gives an embedding \( RL \to RPL \).
  - When concatenated with the Madden representation \( G \to RMG \) of an arbitrary \( W \)-object \( G \), this gives an embedding \( G \to RPMG \equiv \beta G \).
  - This is evidently a reflector from \( W \) into the full subcategory of conditionally pointwise \( \sigma \) complete objects.
  - This is the very well studied class of epicomplete objects in \( W \).
The $P$-frame reflection of $L$ and the epicompletion of $G$

- Not every frame is a $P$-frame, of course, but, as luck would have it, every frame sits inside a canonically associated $P$-frame.

- **Theorem (B, Walters-Wayland, Zenk)**

  $P$-frames are bireflective in frames. That is, every frame $L$ has an extension $\pi_L : L \to PL$, $PL$ a $P$-frame, such that every morphism from $L$ into a $P$-frame factors through $\pi_L$.

  - The embedding $\pi_L : L \to PL$ gives an embedding $RL \to RPL$.
  - When concatenated with the Madden representation $G \to RMG$ of an arbitrary $W$-object $G$, this gives an embedding $G \to RPMG \equiv \beta G$.
  - This is evidently a reflector from $W$ into the full subcategory of conditionally pointwise $\sigma$ complete objects.
  - This is the very well studied class of epicomplete objects in $W$. 
Is $\beta G$ a pointwise $\sigma$-completion?

- The reflector $G \to \text{RPM}_G$ maps each $\mathcal{W}$-object canonically into a pointwise $\sigma$-complete object. Can this extension be characterized in terms of pointwise joins?
- In particular, is $G$ pointwise dense in $\text{RPM}_G$ in some sense?
- We strongly suspect that this is the case.
- But we don’t know that yet.
- In any case we have an appealing characterization of epicompleteness.

**Theorem**

A $\mathcal{W}$-object $G$ is epicomplete iff every countable mobile subset of $G^+$ has a pointwise join in $G$. 
Is $\beta G$ a pointwise $\sigma$-completion?

• The reflector $G \to \text{RPM} G$ maps each $\mathbf{W}$-object canonically into a pointwise $\sigma$-complete object. Can this extension be characterized in terms of pointwise joins?

• In particular, is $G$ pointwise dense in $\text{RPM} G$ in some sense?

• We strongly suspect that this is the case.

• But we don’t know that yet.

• In any case we have an appealing characterization of epicompleteness.

**Theorem**

A $\mathbf{W}$-object $G$ is epicomplete iff every countable mobile subset of $G^+$ has a pointwise join in $G$. 
Is $\beta G$ a pointwise $\sigma$-completion?

- The reflector $G \rightarrow \text{RPM}_G$ maps each $\mathcal{W}$-object canonically into a pointwise $\sigma$-complete object. Can this extension be characterized in terms of pointwise joins?
- In particular, is $G$ pointwise dense in $\text{RPM}_G$ in some sense?
- We strongly suspect that this is the case.
- But we don’t know that yet.
- In any case we have an appealing characterization of epicompleteness.

**Theorem**

A $\mathcal{W}$-object $G$ is epicomplete iff every countable mobile subset of $G^+$ has a pointwise join in $G$. 
Is $\beta G$ a pointwise $\sigma$-completion?

- The reflector $G \to \text{RPM}_G$ maps each $\mathcal{W}$-object canonically into a pointwise $\sigma$-complete object. Can this extension be characterized in terms of pointwise joins?
- In particular, is $G$ pointwise dense in $\text{RPM}_G$ in some sense?
- We strongly suspect that this is the case.
- But we don’t know that yet.
- In any case we have an appealing characterization of epicompleteness.

**Theorem**
A $\mathcal{W}$-object $G$ is epicomplete iff every countable mobile subset of $G^+$ has a pointwise join in $G$. 
Is $\beta G$ a pointwise $\sigma$-completion?

- The reflector $\mathcal{G} \rightarrow \text{RPM} \mathcal{G}$ maps each $\mathcal{W}$-object canonically into a pointwise $\sigma$-complete object. Can this extension be characterized in terms of pointwise joins?
- In particular, is $\mathcal{G}$ pointwise dense in $\text{RPM} \mathcal{G}$ in some sense?
- We strongly suspect that this is the case.
- But we don’t know that yet.
- In any case we have an appealing characterization of epicompleteness.

**Theorem**

A $\mathcal{W}$-object $\mathcal{G}$ is epicomplete iff every countable mobile subset of $\mathcal{G}^+$ has a pointwise join in $\mathcal{G}$.
Is $\beta G$ a pointwise $\sigma$-completion?

- The reflector $G \to \text{RPM} G$ maps each $\mathcal{W}$-object canonically into a pointwise $\sigma$-complete object. Can this extension be characterized in terms of pointwise joins?
- In particular, is $G$ pointwise dense in $\text{RPM} G$ in some sense?
- We strongly suspect that this is the case.
- But we don’t know that yet.
- In any case we have an appealing characterization of epicompleteness.

**Theorem**

A $\mathcal{W}$-object $G$ is epicomplete iff every countable mobile subset of $G^+$ has a pointwise join in $G$. 
Table of contents

1 Preliminaries
   Motivation, context, definitions, and notation
   The Madden representation for $\mathbb{W}$.

2 Pointwise joins
   In $\mathcal{RL}$
   Nice properties
   In $\mathbb{W}$
   Limitations of the $\mathbb{W}$ definition

3 Nakano-Stone Theorems
   The classical Nakano-Stone Theorems
   The Nakano-Stone Theorems for pointwise joins
   Is $\beta G$ a pointwise $\sigma$-completion?

4 Unconditional pointwise completeness
   Truncate sequences
   Cuts and mobile downsets
   Is $\epsilon G$ an unconditional pointwise completion?
Truncate sequences

• Proposition
Any member of $\mathcal{R}L^+$ is the pointwise join of its truncates, i.e., for any $g \in \mathcal{R}L^+$, \( \bigvee_n (g \land n) = g \).

• This raises an important question: which sequences in $\mathcal{R}L$ are sequences of truncates of members of $\mathcal{R}L$, so called truncate sequences?

• Proposition
Let $\{g_n\}$ be the sequence of truncates of $g_0 \in \mathcal{R}^+L$, i.e., $g_n = g_0 \land n$ for all $n$. Then
  - $g_{n+1} \land n = g_n$, and
  - \( \bigvee_n g_n(-\infty, n) = \top \).

Conversely, any sequence in $\mathcal{R}^+L$ having these two properties is the sequence of truncates of some member of $\mathcal{R}L$. 
Truncate sequences

- **Proposition**
  Any member of $\mathcal{RL}^+$ is the pointwise join of its truncates, i.e., for any $g \in \mathcal{RL}^+$, $\bigvee_n (g \wedge n) = g$.

  - This raises an important question: which sequences in $\mathcal{RL}$ are sequences of truncates of members of $\mathcal{RL}$, so called *truncate sequences*?

- **Proposition**
  Let $\{g_n\}$ be the sequence of truncates of $g_0 \in \mathcal{RL}^+$, i.e., $g_n = g_0 \wedge n$ for all $n$. Then
  - $g_{n+1} \wedge n = g_n$, and
  - $\bigvee_n g_n (-\infty, n) = \top$.

  Conversely, any sequence in $\mathcal{R}^+L$ having these two properties is the sequence of truncates of some member of $\mathcal{RL}$. 
Truncate sequences

• Proposition
Any member of $\mathcal{RL}^+$ is the pointwise join of its truncates, i.e., for any $g \in \mathcal{RL}^+$, $\bigvee_n (g \land n) = g$.

• This raises an important question: which sequences in $\mathcal{RL}$ are sequences of truncates of members of $\mathcal{RL}$, so called truncate sequences?

• Proposition
Let $\{g_n\}$ be the sequence of truncates of $g_0 \in \mathcal{RL}^+$, i.e., $g_n = g_0 \land n$ for all $n$. Then
  - $g_{n+1} \land n = g_n$, and
  - $\bigvee_n g_n (-\infty, n) = \top$.

Conversely, any sequence in $\mathcal{RL}^+$ having these two properties is the sequence of truncates of some member of $\mathcal{RL}$.
Truncate sequences

• Proposition
Any member of $\mathcal{RL}^+$ is the pointwise join of its truncates, i.e., for any $g \in \mathcal{RL}^+$, $\bigvee_n (g \land n) = g$.

• This raises an important question: which sequences in $\mathcal{RL}$ are sequences of truncates of members of $\mathcal{RL}$, so called truncate sequences?

• Proposition
Let $\{g_n\}$ be the sequence of truncates of $g_0 \in \mathcal{RL}^+$, i.e., $g_n = g_0 \land n$ for all $n$. Then
- $g_{n+1} \land n = g_n$, and
- $\bigvee_n g_n(\neg\infty, n) = \top$.

Conversely, any sequence in $\mathcal{RL}^+$ having these two properties is the sequence of truncates of some member of $\mathcal{RL}$. 
Truncate sequences

• Proposition

Any member of $\mathcal{RL}^+$ is the pointwise join of its truncates, i.e., for any $g \in \mathcal{RL}^+$, $\bigvee_n (g \wedge n) = g$.

• This raises an important question: which sequences in $\mathcal{RL}$ are sequences of truncates of members of $\mathcal{RL}$, so called truncate sequences?

• Proposition

Let $\{g_n\}$ be the sequence of truncates of $g_0 \in \mathcal{RL}^+$, i.e., $g_n = g_0 \wedge n$ for all $n$. Then

• $g_{n+1} \wedge n = g_n$, and
• $\bigvee_n g_n (−\infty, n) = T$.

Conversely, any sequence in $\mathcal{RL}^+$ having these two properties is the sequence of truncates of some member of $\mathcal{RL}$. 
Truncate sequences

• Proposition
  Any member of $RL^+$ is the pointwise join of its truncates, i.e., for any $g \in RL^+$, $\bigvee_n (g \land n) = g$.

  • This raises an important question: which sequences in $RL$ are sequences of truncates of members of $RL$, so called truncate sequences?

• Proposition
  Let $\{g_n\}$ be the sequence of truncates of $g_0 \in R^+L$, i.e., $g_n = g_0 \land n$ for all $n$. Then
  
  • $g_{n+1} \land n = g_n$, and
  • $\bigvee_n g_n (-\infty, n) = \top$.

  Conversely, any sequence in $R^+L$ having these two properties is the sequence of truncates of some member of $RL$. 
Truncate sequences

• Proposition
Any member of $\mathcal{RL}^+$ is the pointwise join of its truncates, i.e., for any $g \in \mathcal{RL}^+$, $\bigvee_n (g \wedge n) = g$.

• This raises an important question: which sequences in $\mathcal{RL}$ are sequences of truncates of members of $\mathcal{RL}$, so called truncate sequences?

• Proposition
Let $\{g_n\}$ be the sequence of truncates of $g_0 \in \mathcal{RL}^+$, i.e., $g_n = g_0 \wedge n$ for all $n$. Then
- $g_{n+1} \wedge n = g_n$, and
- $\bigvee_n g_n(-\infty, n) = \top$.

Conversely, any sequence in $\mathcal{RL}^+$ having these two properties is the sequence of truncates of some member of $\mathcal{RL}$.
Cuts and mobile downsets

• **Definition of a cut**
  A downset \( Z \subseteq G \) is called a *cut* if it has a join in some extension of \( G \).

• **Definition of a mobile downset**
  A downset \( Z \subseteq G \) is said to be *mobile* if \( Z + g \nsubseteq Z \) for all \( 0 < g \in G \).

• **Theorem**
  A downset \( Z \subseteq G \) is a cut iff it is mobile.
  
  • **Draw a picture**
Cuts and mobile downsets

• **Definition of a cut**
  A downset $Z \subseteq G$ is called a *cut* if it has a join in some extension of $G$.

• **Definition of a mobile downset**
  A downset $Z \subseteq G$ is said to be *mobile* if $Z + g \not\subseteq Z$ for all $0 < g \in G$.

• **Theorem**
  A downset $Z \subseteq G$ is a cut iff it is mobile.
    • Draw a picture
Cuts and mobile downsets

• **Definition of a cut**
  A downset $Z \subseteq G$ is called a *cut* if it has a join in some extension of $G$.

• **Definition of a mobile downset**
  A downset $Z \subseteq G$ is said to be *mobile* if $Z + g \not\subseteq Z$ for all $0 < g \in G$.

• **Theorem**
  A downset $Z \subseteq G$ is a cut iff it is mobile.
    • Draw a picture
Cuts and mobile downsets

- **Definition of a cut**
  A downset $Z \subseteq G$ is called a *cut* if it has a join in some extension of $G$.

- **Definition of a mobile downset**
  A downset $Z \subseteq G$ is said to be *mobile* if $Z + g \not\subseteq Z$ for all $0 < g \in G$.

- **Theorem**
  A downset $Z \subseteq G$ is a cut iff it is mobile.
  
  - Draw a picture
Cuts and mobile downsets

• **Definition of a cut**
  A downset \( Z \subseteq G \) is called a *cut* if it has a join in some extension of \( G \).

• **Definition of a mobile downset**
  A downset \( Z \subseteq G \) is said to be *mobile* if \( Z + g \not\subseteq Z \) for all \( 0 < g \in G \).

• **Theorem**
  A downset \( Z \subseteq G \) is a cut iff it is mobile.
  - **Draw a picture**
Cuts and mobile downsets

• **Definition of a cut**
  A downset $Z \subseteq G$ is called a *cut* if it has a join in some extension of $G$.

• **Definition of a mobile downset**
  A downset $Z \subseteq G$ is said to be *mobile* if $Z + g \not\in Z$ for all $0 < g \in G$.

• **Theorem**
  A downset $Z \subseteq G$ is a cut iff it is mobile.
  
  • **Draw a picture**
Cuts and mobile downsets

• Definition of a cut
  A downset $Z \subseteq G$ is called a cut if it has a join in some extension of $G$.

• Definition of a mobile downset
  A downset $Z \subseteq G$ is said to be mobile if $Z + g \not\subseteq Z$ for all $0 < g \in G$.

• Theorem
  A downset $Z \subseteq G$ is a cut iff it is mobile.
    • Draw a picture
Pointwise completeness

• **Definition**
  A \( W \)-object is *pointwise complete* if every cut in \( G \) has a pointwise join in \( G \).

• **Theorem**
  The following are equivalent for a \( W \)-object \( G \).
  
  - \( G \) is pointwise complete.
  - Every mobile downset of \( G \) has a pointwise join in \( G \).
  - \( G \) is conditionally pointwise complete and every truncate sequence in \( G^+ \) has a join in \( G \).
  - \( G \) is of the form \( RL \) for a Boolean frame \( L \).
Pointwise completeness

• Definition
  A \textit{W}-object is \textit{pointwise complete} if every cut in \( G \) has a pointwise join in \( G \).

• Theorem
  The following are equivalent for a \textit{W}-object \( G \).
  
  • \( G \) is pointwise complete.
  • Every mobile downset of \( G \) has a pointwise join in \( G \).
  • \( G \) is conditionally pointwise complete and every truncate sequence in \( G^+ \) has a join in \( G \).
  • \( G \) is of the form \( RL \) for a Boolean frame \( L \).
Pointwise completeness

• Definition
  A $W$-object is pointwise complete if every cut in $G$ has a pointwise join in $G$.

• Theorem
  The following are equivalent for a $W$-object $G$.
  
  • $G$ is pointwise complete.
  • Every mobile downset of $G$ has a pointwise join in $G$.
  • $G$ is conditionally pointwise complete and every truncate sequence in $G^+$ has a join in $G$.
  • $G$ is of the form $RL$ for a Boolean frame $L$. 
Pointwise completeness

• Definition
A \( W \)-object is pointwise complete if every cut in \( G \) has a pointwise join in \( G \).

• Theorem
The following are equivalent for a \( W \)-object \( G \).

  • \( G \) is pointwise complete.
  • Every mobile downset of \( G \) has a pointwise join in \( G \).
  • \( G \) is conditionally pointwise complete and every truncate sequence in \( G^+ \) has a join in \( G \).
  • \( G \) is of the form \( RL \) for a Boolean frame \( L \).
Pointwise completeness

• Definition
  A \( W \)-object is \textit{pointwise complete} if every cut in \( G \) has a pointwise join in \( G \).

• Theorem
  The following are equivalent for a \( W \)-object \( G \).
  
  • \( G \) is pointwise complete.
  
  • Every mobile downset of \( G \) has a pointwise join in \( G \).
  
  • \( G \) is conditionally pointwise complete and every truncate sequence in \( G^+ \) has a join in \( G \).
  
  • \( G \) is of the form \( RL \) for a Boolean frame \( L \).
Pointwise completeness

• **Definition**
  A \( W \)-object is *pointwise complete* if every cut in \( G \) has a pointwise join in \( G \).

• **Theorem**
  The following are equivalent for a \( W \)-object \( G \).
  
  • \( G \) is pointwise complete.
  
  • Every mobile downset of \( G \) has a pointwise join in \( G \).
  
  • \( G \) is conditionally pointwise complete and every truncate sequence in \( G^+ \) has a join in \( G \).
  
  • \( G \) is of the form \( RL \) for a Boolean frame \( L \).
Pointwise completeness

• **Definition**
  A **W**-object is *pointwise complete* if every cut in **G** has a pointwise join in **G**.

• **Theorem**
  The following are equivalent for a **W**-object **G**.
  
  • **G** is pointwise complete.
  
  • Every mobile downset of **G** has a pointwise join in **G**.
  
  • **G** is conditionally pointwise complete and every truncate sequence in **G**\(^{+}\) has a join in **G**.
  
  • **G** is of the form **R**\( \mathcal{L} \) for a Boolean frame **L**.
Pointwise completeness

• **Definition**
  A \( W \)-object is *pointwise complete* if every cut in \( G \) has a pointwise join in \( G \).

• **Theorem**
  The following are equivalent for a \( W \)-object \( G \).
  
  • \( G \) is pointwise complete.
  • Every mobile downset of \( G \) has a pointwise join in \( G \).
  • \( G \) is conditionally pointwise complete and every truncate sequence in \( G^+ \) has a join in \( G \).
  • \( G \) is of the form \( RL \) for a Boolean frame \( L \).
Essential extensions

**Theorem**
The following are equivalent for an extension $G \leq H$ in $\mathbf{W}$.

- The embedding $G \to H$ is essential, i.e., any morphism out of $H$ whose restriction to $G$ is 1–1 is itself 1–1.
- Every nontrivial $\mathbf{W}$-kernel of $H$ meets $G$ nontrivially.
- If $H$ is divisible then these conditions are equivalent to $G$ being order dense in $H$, i.e., for every $0 < h \in H$ there exists some $0 < g \in G$ such that $g \leq h$.
- The frame map $f : L \equiv M_G \to M_H \equiv M$ which realizes the extension $G \leq H$ ‘drops’ to an isomorphism of the Booleanizations.
Essential extensions

• Theorem

The following are equivalent for an extension $G \leq H$ in $\mathbf{W}$.

- The embedding $G \to H$ is essential, i.e., any morphism out of $H$ whose restriction to $G$ is 1–1 is itself 1–1.
- Every nontrivial $\mathbf{W}$-kernel of $H$ meets $G$ nontrivially.
- If $H$ is divisible then these conditions are equivalent to $G$ being order dense in $H$, i.e., for every $0 < h \in H$ there exists some $0 < g \in G$ such that $g \leq h$.
- The frame map $f : L \equiv \mathcal{M}G \to \mathcal{M}H \equiv M$ which realizes the extension $G \leq H$ ‘drops’ to an isomorphism of the Booleanizations.

\[
\begin{array}{c}
L \overset{f}{\longrightarrow} M \\
\downarrow b_L \quad \downarrow b_M \\
L^{**} \quad \longrightarrow \quad M^{**}
\end{array}
\]
Essential extensions

Theorem
The following are equivalent for an extension $G \leq H$ in $\mathcal{W}$.

- The embedding $G \to H$ is essential, i.e., any morphism out of $H$ whose restriction to $G$ is 1–1 is itself 1–1.
- Every nontrivial $\mathcal{W}$-kernel of $H$ meets $G$ nontrivially.
- If $H$ is divisible then these conditions are equivalent to $G$ being order dense in $H$, i.e., for every $0 < h \in H$ there exists some $0 < g \in G$ such that $g \leq h$.
- The frame map $f : L \equiv \mathcal{M}G \to \mathcal{M}H \equiv M$ which realizes the extension $G \leq H$ ‘drops’ to an isomorphism of the Booleanizations.

\[
\begin{array}{ccc}
L & \xrightarrow{f} & M \\
\downarrow b_L & & \downarrow b_M \\
L^{**} & \xrightarrow{b_M} & M^{**}
\end{array}
\]
Essential extensions

- **Theorem**
  The following are equivalent for an extension $G \leq H$ in $\mathbf{W}$.

  - The embedding $G \to H$ is essential, i.e., any morphism out of $H$ whose restriction to $G$ is 1–1 is itself 1–1.
  - Every nontrivial $\mathbf{W}$-kernel of $H$ meets $G$ nontrivially.
  - If $H$ is divisible then these conditions are equivalent to $G$ being order dense in $H$, i.e., for every $0 < h \in H$ there exists some $0 < g \in G$ such that $g \leq h$.
  - The frame map $f : L \equiv \mathcal{M}G \to \mathcal{M}H \equiv M$ which realizes the extension $G \leq H$ ‘drops’ to an isomorphism of the Booleanizations.

\[
\begin{array}{ccc}
L & \xrightarrow{f} & M \\
b_L & & b_M \\
\downarrow & & \downarrow \\
L^{**} & \to & M^{**}
\end{array}
\]
**Essential extensions**

- **Theorem**
  The following are equivalent for an extension $G \leq H$ in $\mathcal{W}$.
  
  - The embedding $G \rightarrow H$ is essential, i.e., any morphism out of $H$ whose restriction to $G$ is 1–1 is itself 1–1.
  - Every nontrivial $\mathcal{W}$-kernel of $H$ meets $G$ nontrivially.
  - If $H$ is divisible then these conditions are equivalent to $G$ being order dense in $H$, i.e., for every $0 < h \in H$ there exists some $0 < g \in G$ such that $g \leq h$.
  - The frame map $f : L \equiv \mathcal{M}G \rightarrow \mathcal{M}H \equiv M$ which realizes the extension $G \leq H$ ‘drops’ to an isomorphism of the Booleanizations.

\[
\begin{array}{ccc}
L & \xrightarrow{f} & M \\
\downarrow b_L & & \downarrow b_M \\
L^{**} & \rightarrow & M^{**}
\end{array}
\]
Essential extensions

• Theorem
  The following are equivalent for an extension $G \leq H$ in $\mathbf{W}$.
  
  • The embedding $G \to H$ is essential, i.e., any morphism out of $H$ whose restriction to $G$ is 1–1 is itself 1–1.
  • Every nontrivial $\mathbf{W}$-kernel of $H$ meets $G$ nontrivially.
  • If $H$ is divisible then these conditions are equivalent to $G$ being order dense in $H$, i.e., for every $0 < h \in H$ there exists some $0 < g \in G$ such that $g \leq h$.
  • The frame map $f : L \equiv \mathcal{M}G \to \mathcal{M}H \equiv M$ which realizes the extension $G \leq H$ ‘drops’ to an isomorphism of the Booleanizations.

\[
\begin{array}{ccc}
L & \xrightarrow{f} & M \\
| & b_L & | \\
L^{**} & \xrightarrow{b_M} & M^{**}
\end{array}
\]
The maximal essential extension of $G$

- **Theorem**
  [Bernau, Conrad] Every $\mathcal{W}$-object $G$ has a maximal essential extension $G \leq \epsilon G$. That is, $G \leq \epsilon G$ is an essential extension such that an arbitrary extension $G \leq K$ is essential iff there is a $\mathcal{W}$-injection $K \rightarrow \epsilon G$ over $G$.

  - The maximal essential extension $\epsilon G$ is of the form $\mathcal{R}L$ for $L$ Boolean, hence it is pointwise complete.
  - One would like to have a pointwise completion, and the extension $G \leq \epsilon G$ is a likely candidate.
  - But this would require $G$ to be pointwise dense in $\epsilon G$, and we do not know that this is the case.
The maximal essential extension of $G$

- **Theorem**
  [Bernau, Conrad] Every $\mathbf{W}$-object $G$ has a maximal essential extension $G \subseteq \varepsilon G$. That is, $G \subseteq \varepsilon G$ is an essential extension such that an arbitrary extension $G \subseteq K$ is essential iff there is a $\mathbf{W}$-injection $K \to \varepsilon G$ over $G$.

  - The maximal essential extension $\varepsilon G$ is of the form $\mathcal{R}L$ for $L$ Boolean, hence it is pointwise complete.
  - One would like to have a pointwise completion, and the extension $G \subseteq \varepsilon G$ is a likely candidate.
  - But this would require $G$ to be pointwise dense in $\varepsilon G$, and we do not know that this is the case.
The maximal essential extension of $G$

- **Theorem**
  
  [Bernau, Conrad] Every $W$-object $G$ has a maximal essential extension $G \leq \epsilon G$. That is, $G \leq \epsilon G$ is an essential extension such that an arbitrary extension $G \leq K$ is essential iff there is a $W$-injection $K \to \epsilon G$ over $G$.

  - The maximal essential extension $\epsilon G$ is of the form $RL$ for $L$ Boolean, hence it is pointwise complete.
  - One would like to have a pointwise completion, and the extension $G \leq \epsilon G$ is a likely candidate.
  - But this would require $G$ to be pointwise dense in $\epsilon G$, and we do not know that this is the case.
The maximal essential extension of $G$

- **Theorem**
  [Bernau, Conrad] Every $\mathbf{W}$-object $G$ has a maximal essential extension $G \leq \epsilon G$. That is, $G \leq \epsilon G$ is an essential extension such that an arbitrary extension $G \leq K$ is essential iff there is a $\mathbf{W}$-injection $K \to \epsilon G$ over $G$.

  - The maximal essential extension $\epsilon G$ is of the form $RL$ for $L$ Boolean, hence it is pointwise complete.
  - One would like to have a pointwise completion, and the extension $G \leq \epsilon G$ is a likely candidate.
  - But this would require $G$ to be pointwise dense in $\epsilon G$, and we do not know that this is the case.
The maximal essential extension of $G$

- **Theorem**
  [Bernau, Conrad] Every $W$-object $G$ has a maximal essential extension $G \leq \varepsilon G$. That is, $G \leq \varepsilon G$ is an essential extension such that an arbitrary extension $G \leq K$ is essential iff there is a $W$-injection $K \to \varepsilon G$ over $G$.

  - The maximal essential extension $\varepsilon G$ is of the form $RL$ for $L$ Boolean, hence it is pointwise complete.
  - One would like to have a pointwise completion, and the extension $G \leq \varepsilon G$ is a likely candidate.
  - But this would require $G$ to be pointwise dense in $\varepsilon G$, and we do not know that this is the case.
Thank you!