An inductive description of congruence primal arithmetical algebras

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This talk is an application of relational structure theory (RST) for describing congruence primal arithmetical algebras (CPA).

Table of Contents

1 Introduction

2 Relational Structure Theory

3 Description of congruence primal arithmetical algebras
This talk is an application of relational structure theory (RST) for describing congruence primal arithmetical algebras (CPA).

**Relational structure theory**

A sub field of universal algebra, which

- study algebras via decomposition into smaller algebras,
- is related to the structure of the category (of the class generated by an algebra).

The structure of category has full information about “independent of underlying set” as the following sense.

**Theorem**

Let $A, B$ be algebras and assume there is a categorical equivalence $\varphi : \mathcal{V}(A) \to \mathcal{V}(B)$ so that $\varphi(A) = B$. Then $A$ and $B$ are isomorphic iff $\varphi$ commute with the forgetful functors to $\text{Set}$.

**Remark**

The following are properties “independent of underlying set”.
- Structure of subalgebra lattice,
- Structure of congruence lattice,
- Construction of products in a variety.
This talk is an application of relational structure theory (RST) for describing **congruence primal arithmetical algebras** (CPA).

### Congruence primality

Congruence primality is a condition a such as “having simple representation” (represented by congruence relations).

### Arithmecity

Arithmetical = congruence permutable and congruence distributive. Algebraic properties which are generalization of traditional algebras:
- Congruence permutability: groups, rings, etc.
- Congruence distributivity: lattices, Heyting algebras etc.

CPA is a simpleness condition for the categorical structure of the generated class.
2. Relational Structure Theory

1. Introduction

2. Relational Structure Theory

3. Description of congruence primal arithmetical algebras
Summary of the 2nd part.

Definitions

- Term retract. (Compression.)
- Matrix product. (Constructed algebra from family of term retracts.)
- Cover relation. (No lack of information with respect to category structure.)
- Smallness notion of cover.
- Essential part
  - = matrix product of the minimum family of term retracts that has no lack of information.

Facts

- Uniqueness of minimal covers.
- Any algebra is a term retract of an essential algebra.
Definition (Kearnes 2001 [3])

Let $A$ be an algebra.

- $e \in \text{Clo}_1(A)$ is said to be **idempotent** if $e^2 = e$.

  \[
  \text{E}(A) := \text{the set of all idempotent term operations.}
  \]

- If $e \in \text{E}(A)$, an **term retract** of $A$ by $e$, denoted by $A|_{e(A)}$ or simply $e(A)$, is an algebra such that
  - The underlying set is $e(A)$.
  - $\text{Clo}_m(A|_{e(A)}) := \{ e \circ f \mid f \in \text{Clo}_m(A) \}$.

The structure of $A|_{e(A)}$ is determined by the underlying set $e(A) = \{ x \in A \mid e(x) = x \}$ (independent of $e$ itself).
Definition ([3])

Let $A$ be an algebra. $e_1, \ldots, e_l \in E(A)$. $e_1(A) \Join \cdots \Join e_l(A)$ (which is called matrix product in $A$) is an algebra such that
- the underlying set is $e_1(A) \times \cdots \times e_l(A)$.
- $\text{Clo}_m(e_1(A) \Join \cdots \Join e_l(A))$ is a set of tuples $(e_1 t_1, \ldots, e_l t_l)$ where $t_i \in \text{Clo}_{lm}(A)$.

Remark

$X \mapsto e_1(X) \Join \cdots \Join e_l(X)$ is a functor from $\mathcal{V}(A)$ to $\mathcal{V}(e_1(A) \Join \cdots \Join e_l(A))$. 
Definition

Let \( e_1, \ldots, e_n, e \in E(A) \).
\( \{e_1(A), \ldots, e_n(A)\} \) covers \( e(A) \) if
\[
\exists t, f_1, \ldots, f_l \in \text{Clo}(A) ; \ t(e_{i_1} f_1(x), \ldots, e_{i_l} f_l(x)) = e(x).
\]

This case \( e(A) \) is “embedded” in \( e_{i_1}(A) \times \cdots \times e_{i_l}(A) \) by term operations. That is, the term retract

\[
(e_{i_1} f_1, \ldots, e_{i_l} f_l) \circ t(e_{i_1}(A) \boxtimes \cdots \boxtimes e_{i_l}(A))
\]

is isomorphic to \( e(A) \).

“A” “⊂” \( e_{i_1}(A) \boxtimes \cdots \boxtimes e_{i_l}(A) \)

“Embedding”:

\[
x \longmapsto (e_{i_1} f_1(x), \ldots, e_{i_l} f_l(x))
\]

“Retraction”:

\[
t(x_1, \ldots, x_l) \longmapsto (x_1, \ldots, x_l)
\]
Definition ([3])

A cover $\mathcal{U} = \{e_1(A), \ldots, e_n(A)\}$ of $A$ is minimal if

- $\mathcal{U}$ has no proper refinements,
- $\mathcal{U} \setminus \{e_{i_0}(A)\}$ does not cover $A$ for $i_0 = 1, \ldots, n$.

Definition ([3])

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be covers of $A$. $\mathcal{U}_1$ is a refinement of $\mathcal{U}_2$ if for all $U_1 \in \mathcal{U}_1$ there exists $U_2 \in \mathcal{U}_2$ such that $U_1 \subseteq U_2$.

An refinement $\mathcal{U}_1$ of $\mathcal{U}_2$ is said proper if $\mathcal{U}_2$ is not refinement of $\mathcal{U}_1$. 
Definition/Theorem ([3], Behrisch 2009 [4], I. 2013 [5])

Let $A$ be a finite algebra.

- $A$ has a unique “minimal” cover “up to isomorphism”.
- $\text{Ess}(A)$ denotes the matrix product of the minimal cover of $A$ and is said essential part of $A$.
- $A$ has unique essential part “up to isomorphism”.
- An algebra $E$ is said to be essential if $\text{Ess}(E) \simeq E$.

Proposition

Let $A$ be a finite algebra. Then there exist an essential algebra $E$ and a positive integer $n$ s.t. $A$ is isomorphic to a term retract of $E \boxtimes n$.

Remark ([5])

$A$ and $B$ are categorically equivalent iff $\text{Ess}(A) \simeq \text{Ess}(B)$.
3. Description of CPA

1. Introduction

2. Relational Structure Theory

3. Description of congruence primal arithmetical algebras
Definition (Congruence primality)

A finite algebra $A$ is said to be **congruence primal** if an operation $f : A^m \rightarrow A$ preserving $\text{Con}(A)$ is a term operation of $A$.

Definition (Arithmecity)

An algebra $A$ is said to be **arithmetical** if

1. (Congruence permutability) For $\alpha, \beta \in \text{Con}(A)$,

   $$\alpha \circ \beta := \{(x, y) \in A^2 \mid \exists z \in A; (x, z) \in \alpha, (z, y) \in \beta\}$$

   equals to $\alpha \lor \beta$.

2. (Congruence distributivity) The lattice $\text{Con}(A)$ is a distributive.
RST characterization of CPA

**Theorem (cf. [1])**

A finite algebra $A$ is congruence primal and arithmetical iff $A$ has a cover that consists of (two-element) primal algebras.

- Primal = all operations are represented by terms.

**Remark**

A minimal cover of a primal algebra consists a two-element primal algebra. Particularly, a primal algebra $P$ is irreducible iff $|P| = 2$. 
Theorem

- There exists one to one correspondence between matrix products of $n$ primal algebras $\{P_1, \ldots, P_n\}$ and preorders $\leq$ on $\{1, \ldots, n\}$.

- The set (of embedded images of) $\{P_1, \ldots, P_n\}$ is a minimal cover of $\bigotimes_{i=1}^{n} P_i$ iff the corresponding preorder $\leq$ is a partial order.
Notation:
n, m: fixed positive integers,  
I = \{1, \ldots, n\}, \leq),  
P_i: two-element primal algebras,  
E_I = P_1 \boxtimes \cdots \boxtimes P_n: the algebra corresponding to a poset I,  
Q_i = P_i^{\boxtimes m}(= P_i^m := \{0, \ldots, 2^m - 1\} as sets),  
A = E^{\boxtimes m} = (P_1 \boxtimes \cdots \boxtimes P_n)^{\boxtimes m} = Q_1 \boxtimes \cdots \boxtimes Q_n.

Proposition (Definition of E_I)

f = (f_1, \ldots, f_n) (f_i : A^l \to Q_i) is a term operation of A iff f_i depends only on \{j \in I \mid j \leq i\}.

\[ A^l \cong \prod_{j \in I} Q_j^l \xrightarrow{f_i \pi_{\leq i}} Q_i \]

(S shohei Izawa)
**Proposition**

$U \subset A$ is a term retract of $A$ iff

$$\forall a \in A \ [ (\forall i \in I ; \pi_{\leq i}(a) \in \pi_{\leq i}(U)) \Rightarrow a \in U].$$

The case $I$ is a partial ordered set, the following hold.

**Theorem**

For a term retract $U$ of $A$, $\text{Ess}(U) = E$ iff

$$\forall i \in I \exists a \in \pi_{< i}(U); \{|x \in Q_i \mid (a, x) \in \pi_{\leq i}(U)\}| \geq 2.$$ 

**Theorem**

Retracts $U_1 \subset A_1$, $U_2 \subset A_2$ of matrix powers of the essential algebra $E_I$ are isomorphic

$$\iff \exists f : U_1 \to U_2 \text{ bijection so that}$$

$$\forall i \in I \forall a, b \in U_1 [\pi_{\leq i}(a) = \pi_{\leq i}(b) \Rightarrow \pi_{\leq i}(f(a)) = \pi_{\leq i}(f(b))].$$
Example

\[ I = \{1, 2, 3\} \]
\[ 1 < 2, \ 1 < 3 \]
\[ 2 \perp 3 \]
Example

\[ I = \{1, 2, 3\} \]
\[ 1 < 2, \ 1 < 3 \]
\[ 2 \perp 3 \]

\{ (0), (1) \}
Example

\[ I = \{1, 2, 3\} \]

1 < 2, 1 < 3

2 ⊥ 3

\{(0, 0), (0, 1), (1, 0)\}
Example

\[ I = \{1, 2, 3\} \]
\[ 1 < 2, \ 1 < 3 \]
\[ 2 \perp 3 \]

\[ \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0)\} \]
Example

$I = \{1, 2, 3\} \quad \text{common} <3\text{ subtrees}
1 < 2, \ 1 < 3 \quad \text{same branching}
2 \perp 3$

\begin{align*}
\{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0)\}
\end{align*}
Example of a minimal $\text{Ess} = E_I$ algebra that has a ternary branch
(= not isomorphic to any term retract of their essential part)

$I = \{1, 2, 3, 4\}$

$1 < 2 < 3 < 4$

$\text{Ess} = E_I \iff$ Branchings appear in each level

(Shohei Izawa)


Shohei Izawa, Composition of matrix products and categorical equivalence, Algebra universalis 69(2013) 327-356.