Uniform Deductive Interpolation

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A set of equations over the real numbers

\[ \{ x + z = 2y, \ 3x = y + z \} \]

has (equational) consequences

\[ 2x + y = 2z, \ 4x = 3y, \ x = x, \ 4x + w = 3y + w, \ldots \]

It has the same consequences restricted to the variables \( x, y \) as

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Moreover, the same holds restricting to all variables different to \( z \).
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Moreover, the same holds restricting to all variables different to \( z \).
For any formula $\alpha(\bar{x}, \bar{y})$ of intuitionistic propositional logic IPC, there exist **left** and **right uniform interpolants**, 

$$\alpha^L(\bar{y}) \quad \text{and} \quad \alpha^R(\bar{y}),$$

such that for any formula $\beta(\bar{y}, \bar{z})$,

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Pitts’ theorem consists of two parts:

**Variable restriction:** for $\alpha(\bar{x}, \bar{y})$, there exist $\alpha^L(\bar{y})$, $\alpha^R(\bar{y})$ such that

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**Craig interpolation:** for $\alpha(\bar{x}, \bar{y})$, $\gamma(\bar{y}, \bar{z})$ satisfying

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Variable Restriction and Craig Interpolation

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Which **varieties of algebras** admit uniform interpolation?
The **equational consequence relation** for a variety $\mathcal{V}$ is defined by

$$\Sigma \models_{\mathcal{V}} \alpha \approx \beta \iff \text{for all } A \in \mathcal{V} \text{ and } e \in \text{hom}(\text{Fm}(\omega), A),$$

$$\Sigma \subseteq \ker(e) \implies \alpha \approx \beta \in \ker(e).$$

We also write $\Sigma \models_{\mathcal{V}} \Delta$ to denote that $\Sigma \models_{\mathcal{V}} \varepsilon$ for all $\varepsilon \in \Delta$. 
A variety $\mathcal{V}$ admits **deductive interpolation** if whenever

$$\Sigma \subseteq \text{Eq}(X), \quad \varepsilon \in \text{Eq}(Z), \quad Y = X \cap Z \neq \emptyset, \quad \text{and} \quad \Sigma \models_{\mathcal{V}} \varepsilon,$$

there exists $\Delta \subseteq \text{Eq}(Y)$ satisfying

$$\Sigma \models_{\mathcal{V}} \Delta \quad \text{and} \quad \Delta \models_{\mathcal{V}} \varepsilon.$$
The variety $\mathcal{HA}$ of Heyting algebras admits deductive interpolation (equivalently, in this setting, Craig interpolation); e.g., for

$$\{x \land (y \lor z) \approx x, \ (x \land y) \land z \approx z\} \models_{\mathcal{HA}} (x \land w) \lor y \approx y,$$

we obtain

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Theorem (Pigozzi, Bacsich, Maksimova, Czelakowski, …)

A variety with the congruence extension property admits deductive interpolation if and only if it has the amalgamation property.

Further relationships between various forms of interpolation and amalgamation are described in:

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A variety $\mathcal{V}$ has **right variable restriction** if for $X$ and $\Sigma \subseteq \text{Eq}(X)$ finite and $Y \subseteq X$, there is a finite $\Delta \subseteq \text{Eq}(Y)$ such that for all $\varepsilon \in \text{Eq}(Y)$,

$$\Sigma \models_{\mathcal{V}} \varepsilon \iff \Delta \models_{\mathcal{V}} \varepsilon.$$ 

**Note.** The following set always satisfies the above equivalence

$$\Delta = \{ \varepsilon \in \text{Eq}(Y) : \Sigma \models_{\mathcal{V}} \varepsilon \},$$

and will be finite up to equivalence in $\mathcal{V}$ if $\mathcal{V}$ is locally finite.
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Equivalently, for any finite $X$ and $Y \subseteq X$, 

$$\Theta \in \operatorname{Con}^\text{fg}(F_\mathcal{V}(X)) \implies \Theta \cap F_\mathcal{V}(Y)^2 \in \operatorname{Con}^\text{fg}(F_\mathcal{V}(Y)),$$

where $\operatorname{Con}^\text{fg}(A)$ denotes the finitely generated congruences on $A \in \mathcal{V}$.
Equivalently, for any finite \( X \) and \( Y \subseteq X \), the natural embedding

\[ i : \text{Con}^\text{fg}(F^\mathcal{V}(Y)) \rightarrow \text{Con}^\text{fg}(F^\mathcal{V}(X)) \]

has a right adjoint \( j : \text{Con}^\text{fg}(F^\mathcal{V}(X)) \rightarrow \text{Con}^\text{fg}(F^\mathcal{V}(Y)) \) satisfying

\[ i(\Theta) \subseteq \Psi \iff \Theta \subseteq j(\Psi). \]
Restricting Homomorphisms

**Theorem**

The following are equivalent for any variety $\mathcal{V}$:

1. $\mathcal{V}$ admits right variable restriction.

2. For finitely presented $A, B \in \mathcal{V}$ and a homomorphism $f : A \to B$, the lifted map $f^* : \text{Con}^{\text{fg}}(A) \to \text{Con}^{\text{fg}}(B)$ has a right adjoint.
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The following varieties admit right variable restriction:

- any locally finite variety
- Heyting algebras and modal algebras
- abelian groups, abelian $\ell$-groups, and MV-algebras.

Groups and S4-algebras do not have the property; e.g., in S4-algebras,

\[ \top \approx x \land \Box(x \rightarrow \Diamond y) \land \Box(y \rightarrow \Diamond x) \land \Box(x \rightarrow z) \land \Box(y \rightarrow \neg z) \]

does not restrict for the variable $z$ (Ghilardi and Zawadowski 1995).
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**Theorem**

The following are equivalent for any variety $\mathcal{V}$:

(1) $\mathcal{V}$ admits right uniform deductive interpolation.

(2) $\mathcal{V}$ admits deductive interpolation and right variable restriction.
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Examples

The following varieties admit right uniform deductive interpolation:

- the varieties generated by two and three element Heyting chains
- Heyting algebras and modal algebras
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S4-algebras, groups, and varieties generated by \( n \)-element Heyting chains for \( n \geq 4 \) do not have the property.
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\( \forall \) admits **left variable restriction** if for \( X \) and \( \Sigma \subseteq \text{Eq}(X) \) finite and \( Y \subseteq X \), there exists a finite \( \Delta \subseteq \text{Eq}(Y) \) such that for all \( \Pi \subseteq \text{Eq}(Y) \),

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Heyting algebras have left variable restriction, but not the variety $\mathcal{ISL}$ of implicative semilattices, e.g.,

$$\Sigma = \{ \top \approx ((x \to z) \land (y \to z)) \to z \},$$

provides the consequences

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**Theorem**

The following are equivalent for any variety $\mathcal{V}$:

1. $\mathcal{V}$ admits left uniform deductive interpolation.
2. $\mathcal{V}$ admits deductive interpolation and left variable restriction.
A variety $\mathcal{V}$ admits **left uniform deductive interpolation** if for $X$ and $\Sigma \subseteq \text{Eq}(X)$ finite, and $Y \subseteq X$, there exists a finite $\Delta \subseteq \text{Eq}(Y)$ such that whenever $\Pi \subseteq \text{Eq}(Z)$ and $\emptyset \neq X \cap Z \subseteq Y$,

$$\Pi \models_\mathcal{V} \Sigma \iff \Pi \models_\mathcal{V} \Delta.$$

**Theorem**

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Restricting Homomorphisms

Theorem

The following are equivalent for any variety \( \mathcal{V} \):

1. For finitely presented \( A, B \in \mathcal{V} \) and a homomorphism \( f : A \to B \), the lifted map \( f^* : \text{Con}^{fg}(A) \to \text{Con}^{fg}(B) \) has a left adjoint.

2. \( \mathcal{V} \) has the left variable restriction property and for any finite \( X \), the join-semilattice \( \text{Con}^{fg}(F_{\mathcal{V}}(X)) \) is residuated.
Uniform deductive interpolation can be understood as a (weaker) form of quantifier elimination.

In particular, under certain conditions (e.g., for varieties of Heyting and modal algebras), uniform deductive interpolation for \( \forall \) implies the existence of a model completion for the first-order theory of \( \forall \).

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