The Conrad Program: From $\ell$-Groups to Algebras of Logic

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Far West Algebras and Logics
According to Jean Yves Girard (1995), non-commutative linear logic — and by extension, related logics — is Far West.
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Lambek's Calculus is a non-commutative logic. It was introduced in 1958 provide a foundation for the study of linguistics. The algebraic counterparts of this calculus are pointed residuated lattices (to be discussed shortly).
Residuated Lattices
A residuated lattice (or an RL) is an algebra 
\( A = \langle A, \land, \lor, \cdot, \backslash, /, e \rangle \) such that:

(i) \( \langle A, \land, \lor \rangle \) is a lattice;

(ii) \( \langle A, \cdot, e \rangle \) is a monoid; and

(iii) the operation \( \cdot \) is residuated with residuals \( \backslash \) and \( / \). This means that, for all \( x, y, z \in A \),

\[ xy \leq z \iff x \leq z/y \iff y \leq x \backslash z. \]
A residuated lattice (or an RL) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, e \rangle$ such that:

(i) $\langle A, \wedge, \vee \rangle$ is a lattice;
(ii) $\langle A, \cdot, e \rangle$ is a monoid; and
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$$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$ 

An algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, e, 0 \rangle$ is said to be a pointed residuated lattice (pointed RL) or an FL-algebra provided:

(i) $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, e \rangle$ is an RL; and
(ii) $0$ is a distinguished element of $\mathbf{A}$. 
A residuated lattice (or an RL) is an algebra $A = \langle A, \wedge, \vee, \cdot, \backslash, /, e \rangle$ such that:

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An algebra $A = \langle A, \wedge, \vee, \cdot, \backslash, /, e, 0 \rangle$ is said to be a pointed residuated lattice (pointed RL) or an FL-algebra provided:

(i) $A = \langle A, \wedge, \vee, \cdot, \backslash, /, e \rangle$ is an RL; and

(ii) $0$ is a distinguished element of $A$.

It is easy to see that the implications in the definition of an RL can be described by equations. Hence the classes $\mathcal{RL}$ and $\mathcal{PRL}$ of RLs and pointed RLs, respectively, are finitely based varieties.
Let $\mathcal{R}$ be a ring with identity and let $I(\mathcal{R})$ denote the lattice of two-sided ideals of $\mathcal{R}$. Then $I(\mathcal{R}) = \langle I(R), \cap, \cup, \cdot, \setminus, /, R, \{0\} \rangle$ is a (non-necessarily commutative) pointed RL, where, for $I, J \in I(R)$:

$$I \cdot J = \{ \sum_{k=1}^{n} a_k b_k | a_k \in I; b_k \in J; n \in \mathbb{Z}^+ \}$$
Let $\mathbb{R}$ be a ring with identity and let $I(\mathbb{R})$ denote the lattice of two-sided ideals of $\mathbb{R}$. Then $I(\mathbb{R}) = \langle I(R), \cap, \lor, \cdot, \setminus, /, R, \{0\} \rangle$ is a (non-necessarily commutative) pointed RL, where, for $I, J \in I(R)$:

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For a related example, consider an integral domain $\mathbb{R}$ and its field of quotients $\mathbb{K}$. Let $L(\mathbb{K})$ denote the lattice of $\mathbb{R}$-submodules of $\mathbb{K}$. Then $L(\mathbb{K}) = \langle L(K), \cap, \lor, \cdot, \setminus, /, R, \{0\} \rangle$ is a pointed RL, where, for $I, J \in L(K)$:

$$I \cdot J = \{\sum_{k=1}^{n} a_k b_k | a_k \in I; b_k \in J; n \in \mathbb{Z}^+\}$$
A lattice-ordered group (ℓ-group) is an algebra $G = \langle G, \wedge, \vee, \cdot, -1, e \rangle$ such that (i) $\langle G, \wedge, \vee \rangle$ is a lattice; (ii) $\langle G, \cdot, -1, e \rangle$ is a group; and (iii) multiplication preserves finite joins.
A lattice-ordered group ($\ell$-group) is an algebra $G = \langle G, \wedge, \vee, \cdot, -1, e \rangle$ such that (i) $\langle G, \wedge, \vee \rangle$ is a lattice; (ii) $\langle G, \cdot, -1, e \rangle$ is a group; and (iii) multiplication preserves finite joins.

The variety of $\ell$-groups is term equivalent to the subvariety, $\mathcal{L}G$, of $\mathcal{RL}$ defined by the equation $x(x \backslash e) \approx e$; the term equivalence is given by $x^{-1} = e/x$ and $x/y = xy^{-1}$, $y \backslash x = y^{-1}x$. 
A lattice-ordered group \((\ell\text{-group})\) is an algebra \(G = \langle G, \wedge, \vee, \cdot, -^1, e \rangle\) such that (i) \(\langle G, \wedge, \vee \rangle\) is a lattice; (ii) \(\langle G, \cdot, -^1, e \rangle\) is a group; and (iii) multiplication preserves finite joins.

The variety of \(\ell\)-groups is term equivalent to the subvariety, \(\mathcal{L}G\), of \(\mathcal{RL}\) defined by the equation \(x(x\setminus e) \approx e\); the term equivalence is given by \(x^{-1} = e/x\) and \(x/y = xy^{-1}, y\setminus x = y^{-1}x\).

The variety of Boolean algebras is term-equivalent to the subvariety, \(\mathcal{B}A\), of \(\mathcal{PRL}\) axiomatized, relative to \(\mathcal{PRL}\), by identities \(x \lor y \approx (x \rightarrow y) \rightarrow y, xy \approx x \land y\) and \(x \land 0 \approx 0\).
A lattice-ordered group \((\ell\text{-group})\) is an algebra 
\(G = \langle G, \land, \lor, \cdot, -1, e \rangle\) such that (i) \(\langle G, \land, \lor \rangle\) is a lattice; 
(ii) \(\langle G, \cdot, -1, e \rangle\) is a group; and (iii) multiplication preserves finite joins.

The variety of \(\ell\text{-groups}\) is term equivalent to the subvariety, \(\mathcal{L}G\), of \(\mathcal{RL}\) defined by the equation \(x(x\setminus e) \approx e\); the term equivalence is given by 
\(x^{-1} = e/x\) and \(x/y = xy^{-1}, y\setminus x = y^{-1}x\).

The variety of Boolean algebras is term-equivalent to the subvariety, \(\mathcal{BA}\), of \(\mathcal{PRL}\) axiomatized, relative to \(\mathcal{PRL}\), by identities \(x \lor y \approx (x \rightarrow y) \rightarrow y, xy \approx x \land y\) and \(x \land 0 \approx 0\).

The variety of Heyting algebras is term-equivalent to the subvariety, \(\mathcal{HA}\), of \(\mathcal{PRL}\) axiomatized, relative to \(\mathcal{PRL}\), by identities \(xy \approx x \land y\) and \(x \land 0 \approx 0\).
A lattice-ordered group (ℓ-group) is an algebra $G = \langle G, \land, \lor, \cdot, ^{-1}, e \rangle$ such that (i) $\langle G, \land, \lor \rangle$ is a lattice; (ii) $\langle G, \cdot, ^{-1}, e \rangle$ is a group; and (iii) multiplication preserves finite joins.

The variety of ℓ-groups is term equivalent to the subvariety, $\mathcal{LG}$, of $\mathcal{RL}$ defined by the equation $x(x\backslash e) \approx e$; the term equivalence is given by $x^{-1} = e/x$ and $x/y = xy^{-1}$, $y \backslash x = y^{-1}x$.

The variety of Boolean algebras is term-equivalent to the subvariety, $\mathcal{BA}$, of $\mathcal{PRL}$ axiomatized, relative to $\mathcal{PRL}$, by identities $x \lor y \approx (x \to y) \to y$, $xy \approx x \land y$ and $x \land 0 \approx 0$.

The variety of Heyting algebras is term-equivalent to the subvariety, $\mathcal{HA}$, of $\mathcal{PRL}$ axiomatized, relative to $\mathcal{PRL}$, by identities $xy \approx x \land y$ and $x \land 0 \approx 0$.

The variety of MV-algebras is term-equivalent to the subvariety, $\mathcal{MV}$, of $\mathcal{PRL}$ axiomatized, relative to $\mathcal{PRL}$, by identities $x \lor y \approx (x \to y) \to y$, $xy \approx yx$, and $x \land 0 \approx 0$. 
More Examples of Residuated Lattices

Four Important Varieties of RLs

- The variety \( \mathbf{IRL} \) of integral RLs. It is characterized by the law \( x \land e \approx e \).
- The variety \( \mathbf{CRL} \) of commutative RLs. It is characterized by the law \( xy \approx yx \).
- The variety \( \mathbf{SemRL} \) of semilinear RLs. This is the variety of RLs generated by all totally ordered RLs. Its defining laws will be mentioned later.
- The variety \( \mathbf{GMV} \) of GMV-algebras. It consists of all RLs that satisfy the equations:

\[
x/(y \setminus x \land e) \approx x \lor y \approx (x/y \land e) \setminus x.
\]
More Examples of Residuated Lattices

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$$x/(y \backslash x \land e) \approx x \lor y \approx (x/y \land e) \backslash x.$$  

Note that the preceding equations are the non-commutative and non-integral versions of $(x \rightarrow y) \rightarrow y \approx x \lor y$. 

References

- Far West
- RLs
- Definitions
- Examples (1)
- Examples (2)
- Examples (3)
- Algebra & Logic
- Completions
- $e$-Cyclic RLs
- Direct Limits
- Lat. Completion
The Interplay of Algebra and Logic
Let $X$ be a fixed infinite countable set; $Fm(X)$ be the term algebra over $X$ in the signature of $\mathcal{RL}$; $F(X)$ be the free RL over $X$; and $\Phi \cup \{\beta\} \subseteq Fm(X)$. We say that $\beta$ is a consequence of $\Phi$ – in symbols, $\Phi \models \beta$ if, for every $A \in \mathcal{RL}$ and every homomorphism $\varphi: Fm(X) \to A$, $\varphi(\beta) \geq e$ whenever $\varphi(\alpha) \geq e$, for all $\alpha \in \Phi$. 
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The Consequence Relation $\models$

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For $\Phi \cup \{\beta\} \subseteq Fm(X)$, TFAE:

1. $\Phi \models \beta$
2. $(\beta \land \bar{e}, \bar{e}) \in Cg_{F(X)}(\bar{\Phi})$, 
   \[
   \bar{\Phi} = \{(\bar{\alpha} \land \bar{e}, \bar{e}) \mid \alpha \in \Phi \}.
   \]
A **sequent** is an expression of the form $\Gamma \Rightarrow \beta$, where $\Gamma$ is a finite, possibly empty, sequence of formulas $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta$ is a formula: $\alpha_1, \alpha_2, \ldots, \alpha_n \Rightarrow \beta$.

[Algebraic meaning: $\alpha_1 \alpha_2 \ldots \alpha_n \leq \beta$. ]
The Logic of Residuated Lattices

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[Algebraic meaning: $\alpha_1 \alpha_2 \ldots \alpha_n \leq \beta$.]

Initial Sequents

$\alpha \Rightarrow \alpha$ and $\Rightarrow e$

Structural Rules

$e$-Weakening Rule

$$
\frac{\Gamma, \Delta \Rightarrow \beta}{\Gamma, e, \Delta \Rightarrow \beta} \quad (ew)
$$

Cut Rule:

$$
\frac{\Gamma \Rightarrow \alpha \quad \Sigma, \alpha, \Delta \Rightarrow \beta}{\Sigma, \Gamma, \Delta \Rightarrow \beta}
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The Logic of Residuated Lattices

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**Initial Sequent**: $\alpha \Rightarrow \alpha$ and $\Rightarrow e$

**Structural Rules**

**$e$-Weakening Rule**

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\frac{\Gamma, \Delta \Rightarrow \beta}{\Gamma, e, \Delta \Rightarrow \beta} \quad (ew)
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**Cut Rule**: $\Gamma \Rightarrow \alpha$ $\Sigma, \alpha, \Delta \Rightarrow \beta$

$$
\frac{\Sigma, \Gamma, \Delta \Rightarrow \beta}{\Sigma, \Gamma, \Delta \Rightarrow \beta}
$$

**Dropped Structural Rules**: Weakening, Contraction, and Exchange
The following structural rules are not assumed:

**Weakening Rule**

\[
\frac{\Gamma, \Delta \Rightarrow \beta}{\Gamma, \alpha, \Delta \Rightarrow \beta} \quad (w)
\]

**Contraction Rule**

\[
\frac{\Gamma, \alpha, \alpha, \Delta \Rightarrow \beta}{\Gamma, \alpha, \Delta \Rightarrow \beta} \quad (c)
\]

**Exchange Rule**

\[
\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma} \quad (e)
\]
Rules for Logical Connectives

Definitions of Two Rules

\[
\frac{\Sigma, \alpha, \Delta \Rightarrow \gamma \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \alpha \lor \beta, \Delta \Rightarrow \gamma} \quad (\lor \Rightarrow )
\]

\[
\frac{\Sigma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Sigma, \alpha \cdot \beta, \Delta \Rightarrow \gamma} \quad (\cdot \Rightarrow )
\]

\[
\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \setminus \beta} \quad (\Rightarrow \setminus)
\]
**Definitions of Two Rules**

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\frac{\Sigma, \alpha, \Delta \Rightarrow \gamma \quad \Sigma, \beta, \Delta \Rightarrow \gamma}{\Sigma, \alpha \lor \beta, \Delta \Rightarrow \gamma} \quad (\lor \Rightarrow)
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\[
\frac{\Sigma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Sigma, \alpha \cdot \beta, \Delta \Rightarrow \gamma} \quad (\cdot \Rightarrow)
\]

\[
\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \backslash \beta} \quad (\Rightarrow \backslash)
\]

Let \( \Phi \cup \{\beta\} \subseteq Fm(X) \). We say that \( \Phi \) deduces \( \beta \) or that \( \beta \) is provable from \( \Phi \) in \( \text{RL} \) – in symbols, \( \Phi \vdash \beta \), if there is a proof in \( \text{RL} \) of the sequent \( \Rightarrow \beta \) from the set \( \{ \Rightarrow \alpha : \alpha \in \Phi \} \).

A formula \( \beta \) is said to be a theorem of \( \text{RL} \) provided \( \emptyset \vdash \beta \).
Definitions of Two Rules

\[
\Sigma, \alpha, \Delta \Rightarrow \gamma \quad \Sigma, \beta, \Delta \Rightarrow \gamma \\
\hline
\Sigma, \alpha \lor \beta, \Delta \Rightarrow \gamma
\]

\[
\Sigma, \alpha \cdot \beta, \Delta \Rightarrow \gamma
\]

\[
\alpha, \Gamma \Rightarrow \beta \\
\hline
\Gamma \Rightarrow \alpha \setminus \beta
\]

Let \( \Phi \cup \{\beta\} \subseteq Fm(X) \). We say that \( \Phi \) deduces \( \beta \) or that \( \beta \) is provable from \( \Phi \) in \( \mathcal{RL} \) – in symbols, \( \Phi \vdash \beta \), if there is a proof in \( \mathcal{RL} \) of the sequent \( \Rightarrow \beta \) from the set \( \{ \Rightarrow \alpha : \alpha \in \Phi \} \).

A formula \( \beta \) is said to be a theorem of \( \mathcal{RL} \) provided \( \emptyset \vdash \beta \).

Strong Completeness Theorem: \( \Phi \vdash \beta \) iff \( \Phi \models \beta \). In particular, \( \beta \) is a theorem of \( \mathcal{RL} \) iff the equation \( \beta \land e \approx e \) holds in \( \mathcal{RL} \).
Completions: An Overview
J. Harding and G. Bezhanisvili (2004): The only varieties of Heyting algebras that are closed under Dedekind-MacNeille completions are the trivial variety, the variety of Boolean algebras, and the variety of Heyting algebras.
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Embeddability of algebras in a variety $\mathcal{V}$ into complete algebras in $\mathcal{V}$ does not imply closure under Dedekind-MacNeille completions.
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W. Krull (1936), P. Lorenzen (1939), A. H. Clifford (1940), C. J. Everett and S. Ulam (1945): An $\ell$-group can be embedded into a conditionally complete $\ell$-group iff it is Archimedean. (For all elements $a$ and $b$, $a^n \leq b$, $\forall n \in \mathbb{Z}^+$ implies $a \leq e$.)
Completions of Ordered Structures

- J. Harding and G. Bezhanisvili (2004): The only varieties of Heyting algebras that are closed under Dedekind-MacNeille completions are the trivial variety, the variety of Boolean algebras, and the variety of Heyting algebras.

- Embeddability of algebras in a variety $\mathcal{V}$ into complete algebras in $\mathcal{V}$ does not imply closure under Dedekind-MacNeille completions.

- W. Krull (1936), P. Lorenzen (1939), A. H. Clifford (1940), C. J. Everett and S. Ulam (1945): An $\ell$-group can be embedded into a conditionally complete $\ell$-group iff it is Archimedean. (For all elements $a$ and $b$, $a^n \leq b$, $\forall n \in \mathbb{Z}^+$ implies $a \leq e$.)

It was shown by F. Riesz’s (1940) that any conditionally complete $\ell$-group is strongly projectable. (In the terminology of Riesz spaces, it satisfies the strong projection property.)
Another property that has attracted the attention of algebraists and functional analysts is lateral completeness. Two elements $a, b$ of an RL $L$ are said to be orthogonal if $a \lor b = e$. An non-empty subset $X \subseteq L$ is called orthogonal provided any two distinct elements of $X$ are orthogonal. An RL is said to be laterally complete if all orthogonal subsets of it have a greatest lower bound.
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Most of the main embedding theorems for Riesz spaces and $\ell$-groups involve embeddings into laterally complete objects: M.H. Stone (1941), H. Nakano (1950), S. J. Bernau (1965 & 1966), C. W. Holland (1963), P. Conrad, J. Harvey and C. W. Holland (1968), R. Ball (1980), etc.
Another property that has attracted the attention of algebraists and functional analysts is lateral completeness. Two elements \( a, b \) of an RL \( L \) are said to be orthogonal if \( a \lor b = e \). An non-empty subset \( X \subseteq L \) is called orthogonal provided any two distinct elements of \( X \) are orthogonal. An RL is said to be laterally complete if all orthogonal subsets of it have a greatest lower bound.

Most of the main embedding theorems for Riesz spaces and \( \ell \)-groups involve embeddings into laterally complete objects: M.H. Stone (1941), H. Nakano (1950), S. J. Bernau (1965 & 1966), C. W. Holland (1963), P. Conrad, J. Harvey and C. W. Holland (1968), R. Ball (1980), etc.

Does a semilinear RL have a “minimal” semilinear projectable, strongly projectable, lateral (or a combination of these) completion? Under what conditions is it unique?
Main Question

More precisely:

Let $\mathcal{P}, \mathcal{SP},$ and $\mathcal{LC}$ denote the class of projectable, strongly projectable, and laterally complete semilinear RLs, respectively. Let $\mathcal{K} \in \{\mathcal{P}, \mathcal{SP}, \mathcal{LC}, \mathcal{P} \cap \mathcal{LC}, \mathcal{SP} \cap \mathcal{LC}\}$. Given a variety $\mathcal{V}$ of semilinear RLs and $L \in \mathcal{V}$, is there $M \in \mathcal{V} \cap \mathcal{K}$ such that:

1. $L$ is (isomorphic to) a subalgebra of $M$;
2. $L$ is dense in $M$ (For every $a < e$ in $M$, there exists $b \in L$ such that $a \leq b < e$);
3. no proper subalgebra of $M$ containing $L$ is in $\mathcal{V} \cap \mathcal{K}$; and
4. $M$ is unique up to isomorphism.
More precisely:

Let \( \mathcal{P}, \mathcal{SP}, \) and \( \mathcal{LC} \) denote the class of projectable, strongly projectable, and laterally complete semilinear RLs, respectively. Let \( \mathcal{K} \in \{\mathcal{P}, \mathcal{SP}, \mathcal{LC}, \mathcal{P} \cap \mathcal{LC}, \mathcal{SP} \cap \mathcal{LC}\} \). Given a variety \( \mathcal{V} \) of semilinear RLs and \( L \in \mathcal{V} \), is there \( M \in \mathcal{V} \cap \mathcal{K} \) such that:

1. \( L \) is (isomorphic to) a subalgebra of \( M \);
2. \( L \) is dense in \( M \) (For every \( a < e \) in \( M \), there exists \( b \in L \) such that \( a \leq b < e \));
3. no proper subalgebra of \( M \) containing \( L \) is in \( \mathcal{V} \cap \mathcal{K} \); and
4. \( M \) is unique up to isomorphism.

We use the term \( \mathcal{K} \) completion of \( L \) in \( \mathcal{V} \) for any algebra \( M \) that satisfies (1), (2) and (3) above.
e-Cyclic Residuated Lattices
We call a (pointed) RL \( e \)-cyclic if it satisfies the identity \( e/x \approx x\backslash e \). Most well-studied classes of RLs, with a few notable exceptions, are \( e \)-cyclic.
We call a (pointed) RL $e$-cyclic if it satisfies the identity $e/x \approx x\setminus e$. Most well-studied classes of RLs, with a few notable exceptions, are $e$-cyclic.

If $L$ is an $e$-cyclic RL, then closure system $C(L)$ of (order) convex subalgebras of $L$ is a distributive algebraic lattice. The poset $K(C(L))$ of compact elements of $C(L)$ – consisting of the principal convex subalgebras of $L$ – is a sublattice of $C(L)$.
Convex Subalgebras

We call a (pointed) RL \( e \)-cyclic if it satisfies the identity 
\[ e/x \approx x\setminus e. \] Most well-studied classes of RLs, with a few notable exceptions, are \( e \)-cyclic.

If \( L \) is an \( e \)-cyclic RL, then closure system \( C(L) \) of (order) convex subalgebras of \( L \) is a distributive algebraic lattice. The poset \( \mathcal{K}(C(L)) \) of compact elements of \( C(L) \) – consisting of the principal convex subalgebras of \( L \) – is a sublattice of \( C(L) \). In fact, \( C(L) \) can be embedded (as a complete sublattice) into the congruence lattice of the lattice-reduct of \( L \).
We call a (pointed) RL *e-cyclic* if it satisfies the identity $e/x \approx x \setminus e$. Most well-studied classes of RLs, with a few notable exceptions, are *e*-cyclic.

If $L$ is an *e*-cyclic RL, then closure system $C(L)$ of (order) convex subalgebras of $L$ is a distributive algebraic lattice. The poset $\mathcal{K}(C(L))$ of compact elements of $C(L)$ — consisting of the principal convex subalgebras of $L$ — is a sublattice of $C(L)$. In fact, $C(L)$ can be embedded (as a complete sublattice) into the congruence lattice of the lattice-reduct of $L$.

We note that for all $X, Y \in C(L)$,

$$X \rightarrow Y = \{a \in L : |a| \lor |x| \in Y, \text{ for all } x \in X\},$$

and in particular,

$$X^\perp = X \rightarrow \{e\} = \{a \in L : |a| \lor |x| = e, \text{ for all } x \in X\}.$$

Notation: $|x| = x \wedge (e/x) \wedge e$ is the absolute value of $x$. 

References

- Far West
- RLs
- Algebra & Logic
- Completions
- *e*-Cyclic RLs
- Convex Subs
- Polars
- Normality
- Semilinearity
- Direct Limits
- Lat. Completion
Given $X \in \mathcal{C}(L)$, we refer to the pseudo-complement $X^\perp$ of $X$ as the **polar** of $X$.
Given $X \in \mathcal{C}(L)$, we refer to the pseudo-complement $X^\perp$ of $X$ as the polar of $X$.

The map $\perp : \mathcal{C}(L) \rightarrow \mathcal{C}(L)$ is a self-adjoint inclusion-reversing map, while the map sending $H \in \mathcal{C}(L)$ to its double polar $H^{\perp\perp}$ is an intersection-preserving closure operator on $\mathcal{C}(L)$. By a classical result due to V. Glivenko, the image of this closure operator is a (complete) Boolean algebra $\mathcal{B}(L)$ with least element $\{e\}$ and largest element $L$. The complement of $H$ in $\mathcal{B}(L)$ is precisely $H^\perp$, whereas, for any pair of convex subalgebras $H, K \in \mathcal{B}(L)$,

$$H \vee^{\mathcal{B}(L)} K = (H^\perp \cap K^\perp)^\perp = (H \cup K)^{\perp\perp}.$$ 

On the other hand, meets in $\mathcal{B}(L)$ are just intersections.
Given $X \in C(L)$, we refer to the pseudo-complement $X^\perp$ of $X$ as the **polar** of $X$.

The map $\perp : C(L) \to C(L)$ is a self-adjoint inclusion-reversing map, while the map sending $H \in C(L)$ to its double polar $H^\perp\perp$ is an intersection-preserving closure operator on $C(L)$. By a classical result due to V. Glivenko, the image of this closure operator is a (complete) Boolean algebra $B(L)$ with least element $\{e\}$ and largest element $L$. The complement of $H$ in $B(L)$ is precisely $H^\perp$, whereas, for any pair of convex subalgebras $H, K \in B(L)$,

$$H \lor B(L) K = (H^\perp \cap K^\perp)^\perp = (H \cup K)^{\perp\perp}.$$ 

On the other hand, meets in $B(L)$ are just intersections.

- $L$ is said to be **projectable** if $x^\perp \lor C(L) x^{\perp\perp} = L$, for all $x \in L$.
- $L$ is said to be **strongly projectable** if $H^\perp \lor C(L) H^{\perp\perp} = L$, for all $H \in C(L)$. 

**References**

- Far West
- RLS
- Algebra & Logic
- Completions
- e-Cyclic RLSs
- Convex Subs
- Polars
- Normality
- Semilinearity
- Direct Limits
- Lat. Completion
Two identities of particular interest to us are the **left prelinearity law** $LP$ and the **right prelinearity law** $RP$:

$((x \backslash y) \land e) \lor ((y \backslash x) \land e) \approx e$ and $((y/x) \land e) \lor ((x/y) \land e) \approx e$. 
Two identities of particular interest to us are the left prelinearity law $LP$ and the right prelinearity law $RP$: 
\[(x \backslash y) \land e \lor ((y \backslash x) \land e) \approx e \quad \text{and} \quad ((y / x) \land e) \lor ((x / y) \land e) \approx e.\]

Let $L$ be an RL. Given an element $u \in L$, we define 
\[\lambda_u(x) = (u \backslash xu) \land e \quad \text{and} \quad \rho_u(x) = (ux / u) \land e,\]
for all $x \in L$. We refer to $\lambda_u$ and $\rho_u$ as left conjugation and right conjugation by $u$. 
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A convex subalgebra $H$ of $L$ is said to be normal if for all $x \in H$ and $y \in L$, $(y \backslash xy) \land e \in H$ and $(yx/y) \land e \in H$. 
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Given a normal convex subalgebra $H$ of $L$,

$$\Theta_H = \{ \langle x, y \rangle \in L^2 : (x \backslash y) \land (y \backslash x) \land e \in H \}$$

is a congruence of $L$. Conversely, given a congruence $\Theta$, the equivalence class $[e]_\Theta$ is a normal convex subalgebra.
Two identities of particular interest to us are the left prelinearity law $\text{LP}$ and the right prelinearity law $\text{RP}$:

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Let $L$ be an RL. Given an element $u \in L$, we define

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Given a normal convex subalgebra $H$ of $L$, $\Theta_H = \{ \langle x, y \rangle \in L^2 : (x\backslash y) \land (y\backslash x) \land e \in H \}$ is a congruence of $L$. Conversely, given a congruence $\Theta$, the equivalence class $[e]_\Theta$ is a normal convex subalgebra.

The lattice $\mathcal{NC}(L)$ of normal convex subalgebras of a RL $L$ is isomorphic to its congruence lattice $\text{Con}(L)$. The isomorphism is given by the mutually inverse maps $H \mapsto \Theta_H$ and $\Theta \mapsto [e]_\Theta$. 
For a variety $\mathcal{V}$ of $e$-cyclic RLs, the following statements are equivalent:

1. $\mathcal{V}$ is semilinear.
2. $\mathcal{V}$ satisfies either of the equations
   \[
   \lambda_u((x \lor y)\backslash x) \lor \rho_v((x \lor y)\backslash y) \approx e, \\
   \lambda_u(x/(x \lor y)) \lor \rho_v(y/(x \lor y)) \approx e.
   \]
3. $\mathcal{V}$ satisfies either of the prelinearity laws and the quasi-idenitity
   \[
   x \lor y \approx e \quad \Rightarrow \quad \lambda_u(x) \lor \rho_v(y) \approx e.
   \]
4. $\mathcal{V}$ satisfies either of the prelinearity laws and the normality condition on polars.
Direct Limits
Let \((I, \leq)\) be an up-directed poset, \(\mathcal{V}\) a class of similar algebras, and \((A_i \mid i \in I)\) a family of algebras in \(\mathcal{V}\). A family \((f_{ij} : A_i \to A_j \mid i \leq j \text{ in } I)\) of homomorphisms is a directed system for \((A_i \mid i \in I)\) provided for all \(i \in I\), \(f_{ii} = id_{A_i}\), and for all \(i \leq j \leq k\) in \(I\), \(f_{jk}f_{ij} = f_{ik}\).
Let \((I, \leq)\) be an **up-directed** poset, \(\mathcal{V}\) a class of similar algebras, and \((A_i \mid i \in I)\) a family of algebras in \(\mathcal{V}\). A family \((f_{ij} : A_i \to A_j \mid i \leq j \text{ in } I)\) of homomorphisms is a **directed system** for \((A_i \mid i \in I)\) provided for all \(i \in I\), \(f_{ii} = id_{A_i}\), and for all \(i \leq j \leq k\) in \(I\), \(f_{jk}f_{ij} = f_{ik}\).

\[
\begin{array}{cccc}
A_i & \xrightarrow{f_{ik}} & A_k \\
\downarrow{f_{ij}} & & & \downarrow{f_{jk}} \\
A_j & & & A_j
\end{array}
\]

For such a directed system, a family of homomorphisms \((\varphi_i : A_i \to A \mid i \in I)\) is said to be **compatible** with the system provided \(\varphi_j f_{ij} = \varphi_i\) holds, for all \(i \leq j\) in \(I\).
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\((\varphi_i : A_i \to A \mid i \in I)\) is said to be **compatible** with the system provided \(\varphi_jf_{ij} = \varphi_i\) holds, for all \(i \leq j\) in \(I\).

Such a family is called a **direct limit** of the directed system if for any other compatible family 
\((\psi_i : A_i \to B \mid i \in I)\), there exists a unique \(\psi : A \to B\) rendering the red diagram commutative, for all \(i \in I\).
A construction of the Direct Limit

We note that if the homomorphisms of a directed system are embeddings, then so are those of the direct limit.
A construction of the Direct Limit

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Consider a directed system \((f_{ij}: A_i \to A_j \mid i \leq j \text{ in } I)\) and set:

\[
V = \{ a \in \prod_{i \in I} A_i \mid \exists k \forall j \geq k, a_j = f_{kj}(a_k) \}.
\]

(Here and in the sequel, we write \(a_i\) instead of \(a(i)\), for \(a \in \prod_{i \in I} A_i\) and \(i \in I\).) Define \(\Theta \subseteq V \times V\) by:

\[
a \Theta b \iff \exists k \forall j \geq k, a_j = b_j.
\]
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\[ a \Theta b \iff \exists k \forall j \geq k, a_j = b_j. \]

(i) \(V\) is the universe of a subalgebra \(V\) of \(\prod_{i \in I} A_i\).

(ii) \(\Theta\) is a congruence of \(V\).
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a \Theta b \iff \exists k \forall j \geq k, a_j = b_j.
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(i) \(V\) is the universe of a subalgebra \(V\) of \(\prod_{i \in I} A_i\).

(ii) \(\Theta\) is a congruence of \(V\).

Let us fix an arbitrary element \(e \in \prod_{i \in I} A_i\). For each \(i \in I\), let \(\varphi_i : A_i \to V\) be defined as follows for all \(a \in A_i\):

\[
\varphi_i(a)_j = \begin{cases} f_{ij}(a) & \text{if } i \leq j; \\ e_j & \text{otherwise.} \end{cases}
\]
A construction of the Direct Limit

The map $\varphi_i$ induces a map $\bar{\varphi}_i : A_i \rightarrow V/\Theta$ defined, for all $a \in A_i$, by:

$$\bar{\varphi}_i(a) = [\varphi_i(a)]_\Theta.$$
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**Proposition**

The system $(\overline{\varphi}_i : A_i \to V/\Theta | i \in I)$ is the direct limit of the directed system $(f_{ij} : A_i \to A_j | i \leq j$ in $I)$. 

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  - Definition
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  - Construction (2)
- Lat. Completion
A construction of the Direct Limit

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Proposition

The system $(\bar{\varphi}_i : A_i \rightarrow V/\Theta | i \in I)$ is the direct limit of the directed system $(f_{ij} : A_i \rightarrow A_j | i \leq j$ in $I)$.

We call $i \in I$ a witness for $a \in V$, if for every $k \geq i$, $a_k = f_{ik}(a_i)$. Note the following:

- Every element $a \in V$ has a witness.
- The set of witnesses for an element $a \in V$ is closed upwards.
- For each $a \in V$, and each witness $i$ for $a$, $a \Theta \varphi_i(a_i)$ (equivalently, $[a]_\Theta = [\varphi_i(a_i)]_\Theta$).
The Lateral Completion
Let $B = \langle B, \land, \lor, \neg, \bot, \top \rangle$ be a non-trivial complete Boolean algebra. A partition $C$ of $B$ is a maximal set of disjoint elements of $B$. Alternatively, $C \subseteq B$ satisfies:

1. $\bot \notin C$;
2. for every $c, d \in C$, if $c \neq d$ then $c \land d = \bot$; and
3. if $a \neq \bot$ in $B$, there exists $c \in C$ such that $c \land a \neq \bot$.
Partitions of Boolean Algebras

Let $B = \langle B, \land, \lor, \neg, \bot, \top \rangle$ be a non-trivial complete Boolean algebra. A partition $C$ of $B$ is a maximal set of disjoint elements of $B$. Alternatively, $C \subseteq B$ satisfies:

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Any subset of $B$ that satisfies conditions (1) and (2) above can be extended to a partition.
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Any subset of $B$ that satisfies conditions (1) and (2) above can be extended to a partition.

Given two partitions $C$ and $D$, we say that $D$ is a refinement of $C$, and write $C \preceq D$, if for every $d \in D$ there exists a (necessarily unique) $c \in C$ such that $d \leq c$. It is easily seen that $\preceq$ is a join-semilattice order on the set $\Pi$ of partitions of $B$:

$$C \lor D = \{ c \land d \neq \bot : c \in C, \ d \in D \}$$
Partitions of Boolean Algebras

Let $C, D$ be partitions of a complete Boolean algebra $B$. The following are equivalent:

1. $C \not\leq D$;
2. for every $c \in C$, $\neg c = \bigwedge \{ \neg d : d \in D, d \leq c \}$; and
3. $\{ d \in D : d \leq c \}$ is a partition of the Boolean algebra $[\bot, c]$.
Let $\mathcal{C}, \mathcal{D}$ be partitions of a complete Boolean algebra $\mathcal{B}$. The following are equivalent:

1. $\mathcal{C} \precsim \mathcal{D}$;
2. for every $c \in \mathcal{C}$, $\neg c = \bigwedge\{\neg d : d \in \mathcal{D}, d \leq c\}$; and
3. $\{d \in \mathcal{D} : d \leq c\}$ is a partition of the Boolean algebra $[\bot, c]$.

Recall that all polars of any $e$-cyclic semilinear RL $L$ are normal. Hence for every $C^\perp \in B(L)$ we can consider the quotient algebra $L/C^\perp$. 
Partitions of Boolean Algebras

Let $\mathcal{C}, \mathcal{D}$ be partitions of a complete Boolean algebra $B$. The following are equivalent:

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Recall that all polars of any $e$-cyclic semilinear RL $L$ are normal. Hence for every $C^\bot \in B(L)$ we can consider the quotient algebra $L/C^\bot$.

Let $L$ be a semilinear $e$-cyclic RL, and let $\mathcal{C}, \mathcal{D}$ be partitions of the Boolean algebra $B(L)$. The following are equivalent:

1. $\mathcal{C} \preceq \mathcal{D}$.
2. $L/C^\bot$ is a subdirect product of the algebras $\{L/D^\bot : D \in \mathcal{D}, D \subseteq C\}$.
Partitions of Boolean Algebras

Let $C, D$ be partitions of a complete Boolean algebra $B$. The following are equivalent:

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Recall that all polars of any $e$-cyclic semilinear RL $L$ are normal. Hence for every $C^\perp \in B(L)$ we can consider the quotient algebra $L/C^\perp$.

Let $L$ be a semilinear $e$-cyclic RL, and let $C, D$ be partitions of the Boolean algebra $B(L)$. The following are equivalent:

1. $C \not\leq D$.
2. $L/C^\perp$ is a subdirect product of the algebras $\{L/D^\perp : D \in D, D \subseteq C\}$.

Given a partition $C$ of $B(L)$, we let $L_C = \prod_{C \in C} L/C^\perp$.  

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- RLs
- Algebra & Logic
- Completions
- $e$-Cyclic RLs
- Direct Limits
- Lat. Completion
- Partitions (1)
- Partitions (2)
- LC (1)
- LC (2)
- LC (3)
- LC (4)
- LC (5)
Construction of the Lateral Completion

If $C$ and $A$ are two partitions with $C \preceq A$, we define a homomorphism $\varphi_{CA} : L_C \to L_A$ as follows:
If $C$ and $A$ are two partitions with $C \nleq A$, we define a homomorphism \( \varphi_{CA} : L_C \to L_A \) as follows: (i) for every $A \in A$, we choose the unique $C \in C$ such that $A \subseteq C$. Then, $C^\perp \subseteq A^\perp$, whence there exists a homomorphism $f_{CA} : L/C^\perp \to L/A^\perp$. 
If $C$ and $A$ are two partitions with $C \preceq A$, we define a homomorphism $\varphi_{CA} : L_C \to L_A$ as follows: (i) for every $A \in A$, we choose the unique $C \in C$ such that $A \subseteq C$. Then, $C^\perp \subseteq A^\perp$, whence there exists a homomorphism $f_{CA} : L / C^\perp \to L / A^\perp$. 

\[ L_C \xrightarrow{\pi_C} L_C^\perp \xrightarrow{f_{CA}} L_A^\perp \]

\[ L_A \xrightarrow{\pi_A} \]

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- RLS
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Construction of the Lateral Completion

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(ii) Composing with the projection $\pi_C : L_C \rightarrow L/C'$, we obtain a homomorphism $f_{CA} \pi_C : L_C \rightarrow L/A'$. 
Construction of the Lateral Completion

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(ii) Composing with the projection $\pi_C : L_C \to L/C^\perp$, we obtain a homomorphism $f_{CA}\pi_C : L_C \to L/A^\perp$. (iii) Lastly, by the co-universal property of the product $L_A$, there exists a unique homomorphism $\varphi_{CA} : L_C \to L_A$ such that for all $A \in A$, $\pi'_A \varphi_{CA} = f_{CA}\pi_C$, where $\pi'_A : L_A \to L/A^\perp$ is the canonical projection.
Construction of the Lateral Completion

If \( C \) and \( A \) are two partitions with \( C \preceq A \), we define a homomorphism \( \varphi_{CA} : L_C \to L_A \) as follows: (i) for every \( A \in A \), we choose the unique \( C \in C \) such that \( A \subseteq C \). Then, \( C^\perp \subseteq A^\perp \), whence there exists a homomorphism \( f_{CA} : L/C^\perp \to L/A^\perp \).

(ii) Composing with the projection \( \pi_C : L_C \to L/C^\perp \), we obtain a homomorphism \( f_{CA}\pi_C : L_C \to L/A^\perp \). (iii) Lastly, by the co-universal property of the product \( L_A \), there exists a unique homomorphism \( \varphi_{CA} : L_C \to L_A \) such that for all \( A \in A \), \( \pi_A' \varphi_{CA} = f_{CA}\pi_C \), where \( \pi_A' : L_A \to L/A^\perp \) is the canonical projection.

We can describe \( \varphi_{CA} \) as follows: Every element \( x \in L_C \) is the form \( x = ([x_C]_{C^\perp} \mid C \in C) \), with \( x_C \in L \). Then, \( \varphi_{CA}(x) = ([y_A]_A^\perp \mid A \in A) \), where for every \( A \in A \), \( y_A = x_C \), for the unique \( C \in C \) such that \( A \subseteq C \).
Construction of the Lateral Completion

It can be easily shown, by using for instance the previous description, that \((\varphi_{CA} : L_C \to L_A \mid C \preceq A \text{ in } \Pi)\) is a directed system. Let \((\bar{\varphi}_C : A_C \to O(L) \mid C \in \Pi)\) be the direct limit of this system.
Construction of the Lateral Completion

It can be easily shown, by using for instance the previous description, that \((\varphi_{CA} : L_C \rightarrow L_A \mid C \preceq A \text{ in } \Pi)\) is a directed system. Let \((\bar{\varphi}_C : A_C \rightarrow \mathcal{O}(L) \mid C \in \Pi)\) be the direct limit of this system.

Theorem

\(\mathcal{O}(L)\) is laterally complete and projectable, and \(L\) is densely embeddable into it.
Construction of the Lateral Completion

It can be easily shown, by using for instance the previous description, that $(\varphi_{CA} : L_C \to L_A \mid C \preceq A$ in $\Pi)$ is a directed system. Let $(\bar{\varphi}_C : A_C \to O(L) \mid C \in \Pi)$ be the direct limit of this system.

**Theorem**

$O(L)$ is laterally complete and projectable, and $L$ is densely embeddable into it.

**Corollary**

Given any variety $\mathcal{V}$ of semilinear $e$-cyclic RLs, any algebra in $\mathcal{V}$ can be densely embedded into a laterally complete and projectable member of $\mathcal{V}$.
Construction of the Lateral Completion

It can be easily shown, by using for instance the previous description, that \((\varphi_{CA} : \mathbf{L}_C \to \mathbf{L}_A \mid C \preceq A \text{ in } \Pi)\) is a directed system. Let \((\tilde{\varphi}_C : \mathbf{A}_C \to \mathcal{O}(\mathbf{L}) \mid C \in \Pi)\) be the direct limit of this system.

**Theorem**

\(\mathcal{O}(\mathbf{L})\) is laterally complete and projectable, and \(\mathbf{L}\) is densely embeddable into it.

**Corollary**

Given any variety \(\mathcal{V}\) of semilinear \(e\)-cyclic RLs, any algebra in \(\mathcal{V}\) can be densely embedded into a laterally complete and projectable member of \(\mathcal{V}\).

For a partition \(C\) of \(\mathcal{B}(\mathbf{L})\) and \(x = ([x_C]_{C \perp} \mid C \in C) \in \mathbf{L}_C\), the support of \(x\) (in \(\mathbf{L}_C\)) is the set:

\[\text{Spt}(x) = \{C \in C \mid [x_C]_{C \perp} \neq [e]_{C \perp}\}\]

It is clear that, for any \(x \in \mathbf{L}_C\), \(x = e_C\) if and only if \(\text{Spt}(x) = \emptyset\).
Construction of the Lateral Completion

Let $S \subseteq L$ be an orthogonal subset of $L$. Then for $x \neq y$ in $S$, $x \perp \perp \cap y \perp \perp = \emptyset$. Hence the set $\{x \perp \perp \mid x \in S\}$ can be enlarged to a partition $A$ of $B(L)$. If $C$ denotes the trivial partition $\{L\}$ of $B(L)$, then $L \cong L_C$. Further the embedding $\varphi_C : L_C \rightarrow L_A$ is given by $\varphi_C(x) = ([x]_A \perp \mid x \in S)$, for all $x \in L$. Note that $\text{Spt}(\varphi_C(x)) = x \perp \perp$, for all $x \in L$, and so $\text{Spt}(\varphi_C(x)) \cap \text{Spt}(\varphi_C(y)) = \emptyset$, for $x \neq y$. Thus $\wedge \{\varphi_C(x) \mid x \in S\}$ exists in $L_C$. 

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  - LC (5)
Construction of the Lateral Completion

Let \( S \subseteq L \) be an orthogonal subset of \( L \). Then for \( x \neq y \) in \( S \), \( x^{\perp \perp} \cap y^{\perp \perp} = \emptyset \). Hence the set \( \{x^{\perp \perp} \mid x \in S\} \) can be enlarged to a partition \( \mathcal{A} \) of \( \mathcal{B}(L) \). If \( C \) denotes the trivial partition \( \{L\} \) of \( \mathcal{B}(L) \), then \( L \cong L_C \). Further the embedding \( \varphi_{C,A} : L_C \rightarrow L_A \) is given by \( \varphi_{C,A}(x) = ([x]_{\mathcal{A} \perp} \mid x \in S) \), for all \( x \in L \). Note that \( \text{Spt}(\varphi_{C,A}(x)) = x^{\perp \perp} \), for all \( x \in L \), and so \( \text{Spt}(\varphi_{C,A}(x)) \cap \text{Spt}(\varphi_{C,A}(y)) = \emptyset \), for \( x \neq y \). Thus \( \bigwedge\{\varphi_{C,A}(x) \mid x \in S\} \) exists in \( L_C \).

More generally, if \( S \) is a set of orthogonal elements of \( L_C \) that have pairwise disjoint supports, then \( \bigwedge^C S \) exists.
Construction of the Lateral Completion

Let $S \subseteq L$ be an orthogonal subset of $L$. Then for $x \neq y$ in $S$, $x \bot \cap y \bot = \emptyset$. Hence the set $\{x \bot \mid x \in S\}$ can be enlarged to a partition $\mathcal{A}$ of $\mathcal{B}(L)$. If $C$ denotes the trivial partition $\{L\}$ of $\mathcal{B}(L)$, then $L \cong L_C$. Further the embedding $\varphi_C : L_C \rightarrow L_A$ is given by $\varphi_C(x) = ([x]_{\mathcal{A}} \bot \mid x \in S)$, for all $x \in L$. Note that $\text{Spt}(\varphi_C(x)) = x \bot$, for all $x \in L$, and so $\text{Spt}(\varphi_C(x)) \cap \text{Spt}(\varphi_C(y)) = \emptyset$, for $x \neq y$. Thus $\bigwedge\{\varphi_C(x) \mid x \in S\}$ exists in $L_C$.

More generally, if $S$ is a set of orthogonal elements of $L_C$ that have pairwise disjoint supports, then $\bigwedge^C S$ exists.

It can be shown that, given a family $S$ of orthogonal elements of $\mathcal{O}(L)$, there exists a partition $\mathcal{E}$ of $\mathcal{B}(L)$ such that: (i) every element of the disjoint family has a proxy at $\mathcal{E}$; (ii) these proxies have disjoint support in $L_\mathcal{E}$; and (iii) their meet in $L_\mathcal{E}$ is a proxy of the meet $\bigwedge^\mathcal{O}(L) S$. This establishes the lateral completeness of $\mathcal{O}(L)$.
Consider again the direct limit \((\bar{\varphi}_C : A_C \rightarrow O(L) \mid C \in \Pi)\) of the directed system \((\varphi_{CA} : L_C \rightarrow L_A \mid C \preceq A \in \Pi)\). Since all homomorphisms \(\varphi_{CA}\) are embeddings, the same is true for all \(\bar{\varphi}_C\). In particular, letting \(C = \{L\}\), we get an embedding \(\bar{\varphi} : L \rightarrow O(L)\).
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Claim: \(\bar{\varphi}\) is a dense embedding.

Indeed let \(p \in O(L)\) such that \(p < e\). We know that there is a proxy \(x = ([x_C]_{C \perp} \mid C \in C)\) for some partition \(C\). After proving a few lemmas, we can establish the existence of \(a \in L\) and \(B \in C\) such that \([xB]_{B \perp} < [aB]_{B \perp} < [eB]_{B \perp}\) and \([aC]_{C \perp} = [eC]_{C \perp}\) for \(C \neq B\). Thus \(p = \bar{\varphi}_C(x) \leq \bar{\varphi}_C([aC]_{C \perp} \mid C \in C) = \bar{\varphi}(a) < eO(L)\).
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Lastly, it can be shown that \(O(L)\) is projectable.
Recall that a **GMV-algebra** is an RL that satisfies the equations:

\[
x/ (y \backslash x \land e) \approx x \lor y \approx (x/y \land e)) \backslash x.
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\[ \frac{x}{(y \backslash x \land e)} \approx x \lor y \approx (x/y \land e)) \backslash x. \]

**Theorem**

Any member of a semilinear variety \( \mathcal{V} \) of GMV-algebras has a unique lateral completion in \( \mathcal{V} \).
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x/(y\setminus(x \land e)) \approx x \lor y \approx (x/y \land e))\setminus x.
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**Theorem**

Any member of a semilinear variety \( \mathcal{V} \) of GMV-algebras has a unique strongly projectable lateral completion in \( \mathcal{V} \).
Additional Results

Recall that a GMV-algebra is an RL that satisfies the equations:

\[ x/(y\backslash x \land e) \approx x \lor y \approx (x/y \land e))/y. \]

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**Theorem**

Any member of a semilinear variety $\mathcal{V}$ of $e$-cyclic RLs has a projectable completion in $\mathcal{V}$. 