Some problems on countable dense homogeneous spaces

Jan van Mill

University of Amsterdam

Las Cruces, January 2015
• A space $X$ is called *Countable Dense Homogeneous* (abbreviated: CDH) provided that for all countable dense subsets $D, E \subseteq X$ there is a homeomorphism $f : X \to X$ such that $f(D) = E$.

• Every *connected* CDH-space is homogeneous (essentially: Bennett 1972), so for connected spaces, CDH-ness can be thought of as a strong form of homogeneity.

• There are many CDH-spaces: Cantor set, irrational numbers, real line, Hilbert cube, manifolds without boundary, etc.

• A space $X$ is called *Strongly Locally Homogeneous* (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f : X \to X$ such that $f(x) = y$ and $f(z) = z$ for every $z \notin B$. 

(Bessaga and Pełczyński, 1970) Every Polish SLH-space is CDH.

(Hrušák and Zamora Avilés, 2005) Every Borel CDH-space is Polish.

(Farah. Hrušák and Martínez Ranero, 2005) There is a CDH-subset of $\mathbb{R}$ of size $\aleph_1$ (which is a $\lambda$-set).

Fitzpatrick and Zhou presented in 1990 in the *Open Problems in Topology Book* their open problems on countable dense homogeneity.

Its time for an update.
Problem 1

Is every connected, CDH, metric space SLH?

There is a connected, Polish, CDH-space $X$ that is not SLH. In fact, a homeomorphism on $X$ that is the identity on some nonempty open subset of $X$ must be the identity on all of $X$ (vM, 2005).

- Ungar proved in 1978 that that CDH metric continua are $n$-homogeneous for all $n$. Kennedy showed in 1984 that a 2-homogeneous metric continuum $X$ must be SLH, provided that $X$ admits a nontrivial homeomorphism that is the identity on some nonempty open set. Whether every 2-homogeneous metric continuum admits such a homeomorphism remains an open problem.

Problem 1’

Is every CDH continuum SLH?
It was claimed by Ungar in 1978 that every dense open subset of a locally compact separable metric CDH-space is again CDH. The proof is however incomplete. This prompted Fitzpatrick and Zhou to ask the following:

**Problem 2**

*If $X$ is CDH and metric and $U$ is open in $X$, must $U$ be CDH?*

This problem was solved for Polish spaces by a variation of the example on the previous slide (vM, 2005) which has the property that it has a dense open connected subset that is rigid. A space is *rigid* if the identity function is the only homeomorphism. For *locally compact* spaces the question is wide open since 1978.
Problem 2’

If $X$ is connected CDH, and metric, and $U$ is an open connected set in $X$, must $U$ be homogeneous? If $U$ is homogeneous, is it necessarily CDH?

- The first part of Problem 2’ was answered by the example on the previous slide.
Problem 3

If $X$ is a CDH, connected, Polish space, must $X$ be locally connected?

1. This is known to have an affirmative answer in case $X$ is also locally compact (Fitzpatrick, 1972). But for Polish spaces the question remains wide open.

2. There is some recent progress in another direction, though.

3. For a space $X$ and $x \in X$ we let $Q(x, X)$ denote the quasi-component of $x$ in $X$. That is, $Q(x, X)$ is the intersection of all clopen subsets of $X$ that contain $x$.

4. Observe that if $x \in X$, and $X$ is a subspace of $Y$, then $Q(x, X) \subseteq Q(x, Y)$. 
Some problems on countable dense homogeneous spaces

Theorem 4 (vM, 2014)

Let $X$ be a nonmeager connected CDH-space and assume that for some point $x$ in $X$ we have that for every open neighborhood $W$ of $x$, $Q(x, W) \setminus \{x\}$ is nonempty. Then $X$ is locally connected.

Theorem 11 implies that a counterexample to the Fitzpatrick-Zhou Problem promises to be very tricky. It is connected, yet its properties resemble those of complete Erdős space.

Corollary 5 (vM, 2014)

Every rimcompact connected CDH-space is locally connected.

No completeness assumptions!
Problem 4

For which 0-dimensional subsets $X$ of $\mathbb{R}$ is $X^\omega$ homogeneous? CDH?

1. This question was solved by Lawrence in 1998: all 0-dimensional subsets of $\mathbb{R}$ have this property.

2. Nontrivial generalization by Dow and Pearl in 1997. They proved that $X^\omega$ is homogeneous for every zero-dimensional first countable space $X$.

3. Nice related problem (van Douwen and vM, 1979): Is there a homogeneous subspace of the real line with the fixed-point property for homeomorphisms?
Some problems on countable dense homogeneous spaces

Problem 5

*Is the $\omega^\text{th}$ power of the Niemytzki plane homogeneous?*

- Open.

Problem 6

*Does there exist a CDH metric space that is not Polish?*

Problem 6’

*Is there an absolute example of a CDH metric space of cardinality $\omega_1$?*

- We already observed that Problems 6 and 6’ were solved in the affirmative by Farah, Hrušák and Martínez Ranero (2005).
Note that every countable CDH-space is discrete, hence $\aleph_1$ is the first cardinal where anything of CDH-interest can happen. Since $\mathbb{R}$ is CDH and has size $c$, it is an interesting open problem what can happen for cardinals greater than $\aleph_1$ but below $c$.

It was shown recently in Hernandez-Gutiérrez, Hrušák and van Mill (2013) that for every cardinal $\kappa$ such that $\omega_1 \leq \kappa \leq b$ there exists a CDH-subset of $\mathbb{R}$ of size $\kappa$.

In that same paper, there are two more examples that are of interest:

1. A Baire CDH-subspace of the real line that is not Polish.
2. A compact CDH-space of uncountable weight in ZFC. [Steprans and Zhou proved in 1988 that $2^\kappa$ is CDH for every $\kappa < p$.]

Some problems on countable dense homogeneous spaces
In 2003, it was shown that there is a compact space $X$ which is (topologically) homogeneous under $\text{MA}+\neg\text{CH}$ but not under $\text{CH}$ (vM).

This space $X$ has countable $\pi$-weight, character $\omega_1$ and weight $\mathfrak{c}$.

**Problem 7**

*Can there be a compact nowhere first countable homogeneous space of countable $\pi$-weight and weight less than $\mathfrak{c}$?*

This cannot be done by a straightforward modification of the earlier methods since Juhász proved in 1993 that under $\text{MA}$, every compact space of countable $\pi$-weight and weight less than $\mathfrak{c}$ is somewhere first countable.

Hence a homogeneous compactum of countable $\pi$-weight and weight less than $\mathfrak{c}$ is first countable under $\text{MA}$. 
Problem 8
What is the minimum weight of a nowhere first countable compactum of countable $\pi$-weight?

- Clearly, between $\omega_1$ and $c$.

Theorem 9
This number is equal to $\kappa_B$, the least cardinal $\kappa$ for which the real line $\mathbb{R}$ can be covered by $\kappa$ many nowhere dense sets.

Problem 10
1. Is there in ZFC a homogeneous nowhere first countable compact space of countable $\pi$-weight and weight $\kappa_b$?
2. If $\kappa_B^+ < c$, does there exist an example of a nowhere first countable compact space of countable $\pi$-weight and weight $\kappa_B^+$?
Some problems on countable dense homogeneous spaces

Theorem 11 (vM, 2014)

Let $X$ be a nonmeager connected CDH-space and assume that for some point $x$ in $X$ we have that for every open neighborhood $W$ of $x$, $Q(x, W) \setminus \{x\}$ is nonempty. Then $X$ is locally connected.

Corollary 12 (vM, 2014)

Every rimcompact connected CDH-space is locally connected.

- No completeness assumptions!
For a space $X$ we let $\mathcal{H}(X)$ denote its group of homeomorphisms. If $A \subseteq X$, then $\mathcal{H}(X; A)$ denotes $\{ f \in \mathcal{H}(X) : h \text{ restricts to the identity on } A \}$.

**Proposition 13 (vM, 2011)**

Let $X$ be CDH. If $F \subseteq X$ is finite and $D, E \subseteq X \setminus F$ are countable and dense in $X$, then there is an element $f \in \mathcal{H}(X; F)$ such that $f(D) \subseteq E$.

Let $X$ be any nonmeager CDH-space which is connected and contains a point $x$ such that for every open neighborhood $W$ of $X$, $Q(x, W) \setminus \{x\}$ is nonempty. By Bennett (1972), $X$ is homogeneous. Hence this property of the point $x$ is shared by all points.
Lemma 14

For every open neighborhood $V$ of a point $x$ in $X$ we have that the interior of $Q(x, V)$ is nonempty.

Assume that for some open $V$ in $X$ containing $x$ we have that $Q(x, V)$ has empty interior in $X$. Since $V$ is open in $X$, and $Q(x, V)$ is closed in $V$, this clearly implies that $Q(x, V)$ is nowhere dense in $X$.

For every $n$ pick an open neighborhood $U_n$ of $x$ such that $\text{diam } U_n < 2^{-n}$. The assumptions imply that for every $n$, there exists $y_n \in Q(x, U_n) \setminus \{x\}$.

Since $Q(x, V)$ is nowhere dense in $X$, we may pick a countable dense subset $E \subseteq X \setminus Q(x, V)$. Put $D = E \cup \{y_n : n \in \mathbb{N}\}$. By Proposition 13, there exists $f \in \mathcal{H}(X)$ such that $f(x) = x$ and $f(D) \subseteq E$. 
Pick $n$ so large that $f(U_n) \subseteq V$. Since $y_n \in Q(x, U_n) \setminus \{x\}$ we have that $f(y_n) \in Q(f(x), f(U_n)) \setminus \{f(x)\} = Q(x, f(U_n)) \setminus \{x\} \subseteq Q(x, V) \setminus \{x\}$. Since $f(y_n) \in E$ and $E \cap Q(x, V) = \emptyset$, this is a contradiction.

**Corollary 15**

*For every open subset $V$ of $X$ and $x \in V$, we have that the interior of $Q(x, V)$ is dense in $Q(x, V)$.*

Assume that the interior $W$ of $Q(x, V)$ is not dense in $Q(x, V)$. Then there are $y \in Q(x, V)$ and an open subset $U$ of $x$ such that $y \in U \subseteq V$ and $U \cap W = \emptyset$. By Lemma 14, the interior $P$ of $Q(y, U)$ is nonempty. However, $Q(y, U) \subseteq Q(y, V) = Q(x, V)$, hence $P \subseteq Q(x, V)$ and hence $P \subseteq W$. This is a contradiction since $\emptyset \neq P \subseteq U \cap W = \emptyset$. 
Lemma 16

There is a point $x \in X$ with the following property: for every open neighborhood $V$ of $x$, the quasi-component $Q(x, V)$ is a neighborhood of $x$.

Let $\mathcal{U}_1$ be a maximal pairwise disjoint collection of nonempty open subsets of $X$ each of diameter less than $2^{-1}$. Clearly, $\bigcup \mathcal{U}_1$ is dense. Fix $U \in \mathcal{U}_1$. Each quasi-component of $U$ has dense interior by Corollary 15. Hence the interiors of all the quasi-components of elements of $\mathcal{U}_1$ form a pairwise disjoint open (and hence countable) collection with dense union. Let $\mathcal{U}_2$ be a maximal pairwise disjoint collection of nonempty open subsets of $X$ each of diameter less than $2^{-2}$ and having the property that every element $V \in \mathcal{U}_2$ is contained in some quasi-component of some member from $\mathcal{U}_1$. It is clear that $\mathcal{U}_2$ has dense union.
Hence we can continue the same construction with all the quasi-components of members from $\mathcal{U}_2$, thus creating the family $\mathcal{U}_3$. Etc. At the end of the construction, we have a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of subfamilies of pairwise disjoint nonempty open subsets of $X$ such that for every $n$,

1. $\bigcup \mathcal{U}_n$ is dense in $X$,
2. if $V \in \mathcal{U}_{n+1}$, then there exist $U \in \mathcal{U}_n$ and $p \in U$ such that $V \subseteq Q(p, U)$,
3. $\text{mesh}\mathcal{U}_n < 2^{-n}$.

Since $X$ is nonmeager, the collection $\{X \setminus \bigcup \mathcal{U}_n : n \in \mathbb{N}\}$ does not cover $X$. Hence there is a point $x \in X$ for which there exists for every $n \in \mathbb{N}$ an element $U_n \in \mathcal{U}_n$ such that $x \in U_n$. We claim that $x$ is as required. To this end, let $V$ be any open neighborhood of $x$. By (3), there exists $n$ such that $x \in U_n \subseteq V$. Since by (2), $x \in U_{n+1} \subseteq Q(p, U_n)$ for some $p \in U_n$, we have $x \in U_{n+1} \subseteq Q(x, U_n)$. But $Q(x, U_n) \subseteq Q(x, V)$, and so $Q(x, V)$ is a neighborhood of $x$. 
Again by homogeneity, the property of the point $x$ in Lemma 16 is shared by all points.

**Corollary 17**

*Every quasi-component of an arbitrary open subset of $X$ is open.*

Now let $V$ be a nonempty open subset of $X$, and let $W$ be a quasi-component of $V$. Observe that $W$ is a clopen subset of $V$ since the quasi-components of $V$ form a pairwise disjoint family. If $W$ is not connected, then we can write $W$ as $A \cup B$, where $A$ and $B$ are disjoint nonempty open subsets of $W$. But then $A$ and $B$ are clearly clopen in $V$, which implies that $W$ is not a quasi-component. Hence quasi-components of open subsets of $X$ are both open and connected. So we arrive at the conclusion that $X$ is locally connected. This completes the proof of Theorem 11.
Proposition 18

Every meager CDH-space which has an open base $\mathcal{U}$ such that $\text{Fr } U$ is analytic for every $U \in \mathcal{U}$, is zero-dimensional.

- By a result of Fitzpatrick and Zhou (1992), it follows that $X$ is a $\lambda$-set. Observe that by the Baire Category Theorem, a countable dense subspace of a Cantor set $K$ is not a $G_\delta$-subset of $K$. This implies that $X$ does not contain a copy of the Cantor set. Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open basis for $X$ such that $\text{Fr } U_n$ is analytic for every $n$. Clearly, every $\text{Fr } U_n$ is countable since every uncountable analytic space contains a copy of the Cantor set. Let $D = \bigcup_n \text{Fr } U_n$. Then $D$ is countable and hence $G_\delta$ and so $X \setminus D$ can be written as $\bigcup_n F_n$, where every $F_n$ is closed in $X$. Since $F_n \cap \overline{U_m} = F_n \cap U_m$ for all $n$ and $m$, it follows that each $F_n$ is zero-dimensional.
So the cover

\[ \{\{d\} : d \in D\} \cup \{F_n : n \in \mathbb{N}\} \]

of \( X \) consists of countably many closed and zero-dimensional subsets. Hence \( X \) is zero-dimensional by the Countable Closed Sum Theorem.

Let \( X \) be any CDH-space which is connected and rimcompact. Hence \( X \) is nonmeager by the previous result.

Pick an arbitrary \( x \in X \).

**Lemma 19**

*For every open neighborhood \( V \) of \( x \) we have that \( Q(x, V) \setminus \{x\} \neq \emptyset \).*

Pick an open sets \( A \) such that \( x \in A \subseteq \overline{A} \subseteq V \) while moreover \( \text{Fr} \ A \) is compact. We claim that \( Q(x, V) \) meets \( \text{Fr} \ A \).
Indeed, pick an arbitrary (relatively) clopen $E \subseteq V$ that contains $x$. Then $E \cap \overline{A}$ is clopen in $\overline{A}$, hence closed in $X$, and contains $x$. Suppose that $(E \cap \overline{A}) \cap \text{Fr } A = \emptyset$. Then $E \cap \overline{A} = E \cap A$ is nonempty and clopen in $X$ which contradicts connectivity. Hence the collection

$$\{ E \cap \text{Fr } A : E \text{ is a (relatively) clopen subset of } V \text{ that contains } x \}$$

is a family of closed subsets of $\text{Fr } A$ with the finite intersection property. By compactness of $\text{Fr } A$, the set $Q(x, V)$ consequently meets $\text{Fr } A$.

So $X$ is as in Theorem 11, and we are done.