A COUNTERPART TO NAGATA IDEALIZATION

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ABSTRACT. Idealization of a module \( K \) over a commutative ring \( S \) produces a ring having \( K \) as an ideal, all of whose elements are nilpotent. We develop a method that under suitable field-theoretic conditions produces from an \( S \)-module \( K \) and derivation \( D : S \to K \) a subring \( R \) of \( S \) that behaves like the idealization of \( K \) but is such that when \( S \) is a domain, so is \( R \). The ring \( S \) is contained in the normalization of \( R \) but is finite over \( R \) only when \( R = S \). We determine conditions under which \( R \) is Noetherian, Cohen-Macaulay, Gorenstein, a complete intersection or a hypersurface. When \( R \) is local, then its \( m \)-adic completion is the idealization of the \( m \)-adic completions of \( S \) and \( K \).

1. Introduction

All rings in this article are commutative and have an identity. Let \( A \) be a ring, and let \( L \) be an \( A \)-module. The idealization, or trivialization, of the module \( L \) is the ring \( A \star L \), which is defined as an abelian group to be \( A \oplus L \), and whose ring multiplication is given by \( (a_1, \ell_1) \cdot (a_2, \ell_2) = (a_1a_2, a_1\ell_2 + a_2\ell_1) \) for all \( a_1, a_2 \in A \), \( \ell_1, \ell_2 \in L \). In particular, \((0, \ell_1) \cdot (0, \ell_2) = (0, 0)\), and hence the \( A \)-module \( L \) is encoded into \( A \star L \) as an ideal whose square is zero. Nagata introduced idealization in [15] to deduce primary decomposition of Noetherian modules from the primary decomposition of ideals in Noetherian rings. Among other positive uses of idealization are arguments for smoothness and the determination of when a Cohen-Macaulay ring admits a canonical module ([13, Section 25] and [22], respectively). But idealization also serves as a flexible source of examples in commutative ring theory, since it allows one to create a ring having an ideal which reflects the structure of a well-chosen module. However, as is clear from the construction, idealization introduces nilpotent elements, and hence the construction does not produce domains.

In this article we develop a construction of rings that behaves analytically like the idealization of a module. The structure of these rings \( R \) is determined entirely by an overring \( S \) of \( R \) and an \( S \)-module \( K \), and when \( S \) is a domain, so is \( R \). The extension \( R \subseteq S \) is integral, but is finite only if \( R = S \). Moreover, for certain elements \( r \) in the ring \( R \), \( R/rR \) is isomorphic to \( S/rS \star K/rK \), so that in sufficiently small neighborhoods, \( R \) is actually an idealization of \( S \) and \( K \). There are enough such elements \( r \) so that if \( R \) and \( S \) are quasilocal with finitely generated maximal ideals, then \( \hat{R} \) is isomorphic to \( \hat{S} \star \hat{K} \), where \( \hat{R} \), \( \hat{S} \) and \( \hat{K} \) are completions in relevant \( m \)-adic topologies. Thus when the ring \( R \) produced by the construction is a local Noetherian ring, it is analytically ramified, with ramification given in a clear way.
The construction of the ring $R$ from $S$ and $K$ is as a ring of “anti-derivatives” for a special sort of derivation from a localization of $S$ to a corresponding localization of $K$:

**Definition 1.1.** Let $S$ be a ring, let $K$ be an $S$-module, and let $C$ be a multiplicatively closed subset of nonzerodivisors of $S$ that are also nonzerodivisors on $K$. Then a subring $R$ of $S$ is twisted by $K$ along $C$ if there is a $C$-linear derivation $D : S_C \to K_C$ such that:

(a) $R = S \cap D^{-1}(K)$, (b) $D(S_C)$ generates $K_C$ as an $S_C$-module, and (c) $S \subseteq \ker D + cS$ for all $c \in C$. We say that $D$ twists $R$ by $K$ along $C$.

Note that for any derivation $D$ from a ring into a module $L$, the preimage $D^{-1}(K)$, with $K$ a submodule of $L$, is a ring. In particular, $\ker D$ is a ring. We use the term “twisted” to draw a loose analogy with the notion of the twist of a graded module or ring. In our case, however, rather than twist, or shift, a grading by an index, we twist the ring $S$ by a module $K$ to create $R$. This shifting is illustrated, to name just a few instances, by how $K$ shifts the Hilbert function of ideals of $R$ (see [21]), and also how $K$ ramifies the completion of $R$. Given $S$, $K$ and $C$, whether such a subring $R$ and derivation $D$ exist is in general not easy to determine, and we address this in Section 3. It is condition (c) that proves hard to satisfy. By contrast, condition (b) can be arranged by choosing $K$ so that $K_C$ is the $S_C$-submodule of the target of the derivation generated by $D(S_C)$, while (a) can be satisfied simply by assigning $R$ to be $S \cap D^{-1}(K)$.

We are specifically interested in when this construction produces Noetherian rings, and a stronger, absolute version of the notion is useful for this:

**Definition 1.2.** Let $S$ be a domain with quotient field $F$, and let $R$ be a subring of $S$. Let $K$ be a torsion-free $S$-module, and let $FK$ denote the divisible hull $F \otimes_S K$ of $K$. We say that $R$ is strongly twisted by $K$ if there is a derivation $D : F \to FK$ such that:

(a) $R = S \cap D^{-1}(K)$, (b) $D(F)$ generates $FK$ as an $F$-vector space, and (c) $S \subseteq \ker D + sS$ for all $0 \neq s \in S$. We say that $D$ strongly twists $R$ by $K$.

It is straightforward to check that strongly twisted implies twisted along the multiplicatively closed set $C = (S \cap \ker D) \setminus \{0\}$. In Theorem 3.5, we prove that when $F$ is a field of positive characteristic that is separably generated of infinite transcendence degree over a countable subfield, then for any domain $S$ having quotient field $F$ and torsion-free $S$-module $K$ of at most countable rank, there exists a subring $R$ of $S$ that is strongly twisted by $K$. Granted existence, we show in Theorem 5.2 that if $R$ is a subring of a domain $S$ that is strongly twisted by a torsion-free $S$-module $K$, then $R$ is Noetherian if and only if $S$ is Noetherian and certain homomorphic images of $K$ are finitely generated. In particular, if $S$ is a Noetherian ring and $K$ is a finitely generated torsion-free $S$-module, then $R$ is Noetherian. Along with the above existence result, this guarantees there are interesting examples which reflect in various ways the natures of $S$ and $K$.

The original idea of using pullbacks of derivations to construct Noetherian rings is due to Ferrand and Raynaud, who used it to produce three important examples: a one-dimensional local Noetherian domain $D$ whose completion when tensored with the quotient field of $D$ is not a Gorenstein ring (in other words, its generic formal fiber is not Gorenstein); a
two-dimensional local Noetherian domain whose completion has embedded primes; and a three-dimensional local Noetherian domain \( R \) such that the set of prime ideals \( P \) of \( R \) with \( R_P \) a Cohen-Macaulay ring is not an open subset of \( \text{Spec}(R) \) [5]. This last example was in fact constructed using the two-dimensional ring obtained in the second example, so the construction is known only to produce examples in Krull dimension 1 and 2. The method of Ferrand and Raynaud was further abstracted and improved on by Goodearl and Lenagan in the article [6], but again, only examples of dimension 1 and 2 were produced. Our variation on these ideas produces Noetherian rings without restriction on dimension. By developing the construction generally without much concern for the Noetherian case, we create more theoretical space for examples than the arguments of Ferrand and Raynaud permit. (For example, the construction of Ferrand and Raynaud requires \emph{a priori} that the pullback \( R \) of the derivation is Noetherian, a condition that can be hard to verify and one which seems to be the main obstacle to producing more examples with their method.) For more applications of some of these ideas to the case of dimension 1, see [19].

In later sections we use some elementary facts about derivations to prove many of our results. Let \( S \) be a ring, and let \( L \) be an \( S \)-module. A mapping \( D : S \to L \) is a \emph{derivation} if for all \( s, t \in S \), \( D(s + t) = D(s) + D(t) \) and \( D(st) = sD(t) + tD(s) \). If also \( A \) is a subset of \( S \) with \( D(A) = 0 \), then \( D \) is an \( A \)-linear derivation. The main properties of derivations we need are collected in (1.3).

(1.3) The module \( \Omega_{S/A} \) of Kähler differentials. Let \( S \) be a ring and let \( A \) be a subring of \( S \). There exists an \( S \)-module \( \Omega_{S/A} \) and an \( A \)-linear derivation \( d_{S/A} : S \to \Omega_{S/A} \), such that for every derivation \( D : S \to L \), there exists a unique \( S \)-module homomorphism \( \alpha : \Omega_{S/A} \to L \) such that \( D = \alpha \circ d_{S/A} \); see for example, [13, pp. 191-192]. The actual construction of \( \Omega_{S/A} \) is not needed here, but we do use the fact that the image of \( d_{S/A} \) in \( \Omega_{S/A} \) generates \( \Omega_{S/A} \) as an \( S \)-module [10, Remark 1.21]. The \( S \)-module \( \Omega_{S/A} \) is the \emph{module of Kähler differentials} of the ring extension \( A \subseteq S \), and the derivation \( d_{S/A} : S \to \Omega_{S/A} \) is the \emph{exterior differential} of \( A \subseteq S \).

We see in Theorem 4.1 that when \( R \) is a twisted subring of \( S \), then \( R \subseteq S \) is a special sort of integral extension, which in [20] is termed a “quadratic” extension:

(1.4) Quadratic extensions. An extension \( R \subseteq S \) is \emph{quadratic} if every \( R \)-submodule of \( S \) containing \( R \) is a ring; equivalently, \( st \in sR + tR + R \) for all \( s, t \in S \). In [20, Lemma 3.2], the following characterization is given for quadratic extensions in the sort of context we consider in this article. Let \( R \subseteq S \) be an extension of rings, and suppose there is a multiplicatively closed subset \( C \) of \( R \) consisting of nonzerodivisors in \( S \) such that every element of \( S/R \) is annihilated by some element of \( C \) and \( S = R + cS \) for all \( c \in C \). (In the next section we will express these two properties by saying that \( S/R \) is \( C \)-torsion and \( C \)-divisible.) Then \( R \subseteq S \) is a quadratic extension if and only there exists an \( S \)-module \( T \) and a derivation \( D : S \to T \) with \( R = \text{Ker} \, D \); if and only if \( S/R \) admits an \( S \)-module structure extending the \( R \)-module structure on \( S/R \).
2. Analytic extensions

In this section we introduce the notion of an analytic extension and show that twisted subrings are couched in such extensions, a fact that we rely heavily on in later sections. In framing the definition, and throughout this article, we use the following terminology. Let $S$ be a ring, let $L$ be an $S$-module and let $C$ be a multiplicatively closed subset of $S$ consisting of nonzerodivisors of $S$. The module $L$ is $C$-torsion provided that for each $\ell \in L$, there exists $c \in C$ with $c\ell = 0$; it is $C$-torsion-free if the only $C$-torsion element is 0. The module $L$ is $C$-divisible if for each $c \in C$ and $\ell \in L$, there exists $\ell' \in L$ such that $\ell = c\ell'$.

Following Weibel in [27], and as developed in [19], we use the following notion:

**Definition 2.1.** Let $\alpha : A \to S$ be a homomorphism of $A$-algebras, and let $C$ be a multiplicatively closed subset of $A$ such that the elements of $\alpha(C)$ are nonzerodivisors of $S$. Then $\alpha$ is an analytic isomorphism along $C$ if for each $c \in C$, the induced mapping $\alpha_c : A/cA \to S/cS : a \mapsto \alpha(a) + cS$ is an isomorphism. When $A$ is a subring of $S$ and the mapping $\alpha$ is the inclusion mapping, we say that $A \subseteq S$ is a $C$-analytic extension.

It follows that the mapping $\alpha$ is analytic along $C$ if and only if $S/\alpha(A)$ is $C$-torsion-free and $C$-divisible. For example, if $A$ is a ring and $X$ is an indeterminate for $A$, then $A[X] \subseteq A[[X]]$ is $C$-analytic with respect to $C = \{X^i : i > 0\}$. Similarly, if $A$ is a DVR with completion $\hat{A}$, then $A[X] \subseteq \hat{A}[X]$ is $C$-analytic for $C = \{t^i : i > 0\}$, where $t$ is a generator of the maximal ideal of $A$.

We also consider a stronger condition:

**Definition 2.2.** When $A$ and $S$ are domains, the map $\alpha$ is a strongly analytic isomorphism if $sS \cap \alpha(A) \neq 0$ for all $0 \neq s \in S$ and $\alpha$ is an analytic isomorphism along $C = A \setminus \{0\}$. When $A \subseteq S$ and the inclusion mapping is a strongly analytic isomorphism, then $A \subseteq S$ is a strongly analytic extension.

It follows that $A \subseteq S$ is a strongly analytic extension if and only if $S/A$ is a torsion-free divisible $A$-module and $P \cap A \neq 0$ for all nonzero prime ideals $P$ of $S$. The latter condition asserts that the generic fiber of $\text{Spec}(S) \to \text{Spec}(A)$ is trivial. Thus, following Heinzer, Rotthaus and Wiegand in [8], we say that the extension $A \subseteq S$ has trivial generic fiber (TGF). An immediate extension of DVRs is easily seen to be strongly analytic, but examples of strongly analytic extensions of Noetherian rings in higher dimensions are harder to find. One of the main goals of Section 3 is to give existence results for such extensions.

**Remark 2.3.** It is straightforward to verify that an extension of rings $A \subseteq S$ is $C$-analytic, where $C$ is a multiplicatively closed subset of $A$ consisting of nonzerodivisors of $S$, if and only if $S_C = A_C + S$ and $A = S \cap A_C$. Moreover, if $A \subseteq S$ is an extension of domains with quotient fields $Q$ and $F$, respectively, then $A \subseteq S$ is strongly analytic if and only if $F = Q + S$ and $A = S \cap Q$.

The following basic proposition shows that for a $C$-analytic extension $A \subseteq S$, the ideals of $A$ and $S$ meeting $C$ are related in a transparent way.
Proposition 2.4. Let $A \subseteq S$ be an extension of rings, and let $C$ be a multiplicatively closed subset of $A$ consisting of nonzerodivisors of $S$. Suppose that $A \subseteq S$ is a $C$-analytic extension. Then:

1. The mappings $I \mapsto IS$ and $J \mapsto J \cap A$ yield a one-to-one correspondence between ideals $I$ of $A$ meeting $C$ and ideals $J$ of $S$ meeting $C$. Prime ideals of $A$ meeting $C$ correspond to prime ideals of $S$ meeting $C$, and maximal ideals of $A$ meeting $C$ correspond to maximal ideals of $S$ meeting $C$.

2. If $J$ is a finitely generated ideal of $S$ meeting $C$ that can be generated by $n$ elements, then $J \cap A$ can be generated by $n + 1$ elements. If also $A$ is quasilocal, then $J \cap A$ can be generated by $n$ elements.

Proof. (1) Let $I$ be an ideal of $A$ meeting $C$, and let $c \in I \cap S$. Since $S = A + cS$ and $cS \cap A = cA$, it follows that $I = IS \cap A$. Similarly, if $J$ is an ideal of $S$ meeting $C$ and $c \in J \cap C$, then from $S = A + cS$, we deduce that $J = (J \cap A)S$. This proves that the mappings $I \mapsto IS$ and $J \mapsto J \cap A$ form a one-to-one correspondence. The second assertion regarding prime ideals is now clear, with one possible exception: If $P$ is a prime ideal of $A$ meeting $C$, then since $S/A$ is $C$-divisible, $S = A + PS$, so that $S/PS \cong A/(PS \cap A) = A/P$. Hence $S/PS$ is a domain, and $PS$ is prime ideal of $S$ meeting $C$. This same argument shows also that if $P$ is a maximal ideal of $A$, then $PS$ is a maximal ideal of $S$. And conversely, if $M$ is a maximal ideal of $S$ meeting $C$ and $P$ is a prime ideal of $A$ containing $M \cap A$, then the maximality of $M$ implies $M = (M \cap A)S = PS$, and hence by the correspondence, $M \cap A = P$, so that maximal ideals of $S$ contract to maximal ideals of $A$.

(2) Suppose that $J = (x_1, \ldots, x_n)S$ is a finitely generated ideal of $S$ such that $c \in J \cap C$. By (1), $J = (J \cap A)S$, so since $S = A + c^2 S$, it follows that $J = (J \cap A)S = (J \cap A) + c^2 S$, and hence for each $i$, there exist $a_i \in J \cap A$ and $s_i \in S$ such that $x_i = a_i + c^2 s_i$. Thus $J = (x_1, \ldots, x_n)S = (a_1, \ldots, a_n, c^2)S$, and by (1), $J \cap A = (a_1, \ldots, a_n, c^2)A$. To prove the last assertion, suppose that $A$ is quasilocal with maximal ideal $m$, and let $I = J \cap A$. Then since $c^2 \in mI$, Nakayama’s Lemma implies that $I = (a_1, \ldots, a_n)A$. Hence $I$ can be generated by $n$ elements. \qed

Some relevant technical properties of analytic extensions were studied in [19]. We quote these as needed throughout the article, beginning with the proof of the next theorem, which shows that twisted subrings occur within analytic extensions, and more interestingly, that a converse is also true.

Theorem 2.5. Let $R \subseteq S$ be an extension of rings, and let $C$ be a multiplicatively closed set of $R$ consisting of nonzerodivisors of $S$.

1. Suppose $R$ is twisted along $C$ by an $S$-module $K$ that is $C$-torsion-free. If $D$ is the derivation that twists $R$, then with $A = S \cap \text{Ker} \ D$, the extension $A \subseteq S$ is $C$-analytic, $R \subseteq S$ is quadratic and $S/R$ is $C$-torsion.
Since $R$ is the case, since if $S/R$ we show that $S$ is a domain and $A$ is $d$-
module. Consequently, $S/A$ is $d$-torsion free. Moreover, since $R$ is twisted by $D$, for each $c \in C$ we have $S \subseteq \ker D + cS$, which in turn implies that $S = A + cS$. Thus $S/A$ is $C$-divisible, and $A \subseteq S$ is a $C$-analytic extension. Finally, we show that $S/R$ is a $C$-torsion module and $R \subseteq S$ is a quadratic extension. The former is the case, since if $s \in S$, then since $K_C/K$ is $C$-torsion, there exists $c \in C$ such that $D(cs) = cD(s) \in K$, and hence $cs \in S \cap D^{-1}(K) = R$. To see that $R \subseteq S$ is quadratic, we use (1.4). Consider the derivation

$$D' : S \to K_C/K : s \mapsto D(s) + K,$$

where $s \in S$. Then $\ker D' = \{s \in S : D(s) \in K\} = S \cap D^{-1}(K) = R$, so that since $S/R$ is $C$-torsion (as we have verified) and $C$-divisible (it is the image of the $C$-divisible $A$-module $S/A$), we may apply (1.4) to conclude that $R \subseteq S$ is a quadratic extension.

(2) Write $d = d_{S_C/A_C}$ and $\Omega = \Omega_{S_C/A_C}$. Define $K$ to be the $S$-submodule of $\Omega$ generated by $d(R)$. We claim $R$ is twisted along $C$ by $K$, and the derivation that twists $R$ is $d$. Since $d$ is $C$-linear and $R_C = S_C$, we have:

$$K_C = \sum_{r \in R} S_C d(r) = \sum_{x \in R_C} S_C d(x) = \Omega.$$

Hence $K_C = \Omega$ and $K_C$ is generated as an $S_C$-module by $d(S)$. Moreover, $d : S_C \to K_C$ is a $C$-linear derivation. It is shown in [19, Proposition 3.3] that since $A \subseteq S$ is $C$-analytic, $R \subseteq S$ is quadratic and $S/R$ is $C$-torsion, then with $K$ defined as above, $R = S \cap d^{-1}(K)$ and $\Omega/K$ is $C$-torsion, and $K$ is the unique $S$-submodule of $\Omega$ satisfying these last two properties. Finally, since $A \subseteq S$ is $C$-analytic, $S = A + cS$ for all $c \in C$, and hence, since $A \subseteq \ker d$, we have $S \subseteq \ker d + cS$ for all $c \in C$. Thus $R$ is twisted along $C$ by the $S$-module $K$.

There is also a version of the theorem for strongly analytic extensions. Recall that if $S$ is a domain and $L$ is a torsion-free $S$-module, then a submodule $K$ of $L$ is full if $L/K$ is a torsion $S$-module.

**Corollary 2.6.** Let $R \subseteq S$ be an extension of domains, and let $F$ denote the quotient field of $S$.

(1) Suppose that $R$ is strongly twisted by a torsion-free $S$-module $K$. If $D$ is the derivation that strongly twists $R$, then with $A = S \cap \ker D$, the extension $A \subseteq S$ is strongly analytic, $R \subseteq S$ is quadratic and $R$ has quotient field $F$. 

Conversely, if there exists a subring $A$ of $R$ such that $A \subseteq S$ is strongly analytic, $R \subseteq S$ is quadratic and $R$ has quotient field $F$, then there is a unique full $S$-submodule $K$ of $\Omega_{F/Q}$ such that $R$ is strongly twisted by $K$ and $d_{F/Q}$ is the derivation that twists $R$.

Proof. (1) First observe that every nonzero ideal of $S$ contracts to a nonzero ideal of $A$. For if $s$ is a nonzero nonunit in $S$, then by assumption $S \subseteq \text{Ker } D + s^2S$, so that $s = a + s^2\sigma$ for some $a \in \text{Ker } D \cap S = A$ and $\sigma \in S$. Thus $s(1 - s\sigma) \in A$, and since $s$ is a nonzero nonunit in $S$, it follows that $sS \cap A \neq 0$. Therefore, the extension $A \subseteq S$ has TGF. Moreover, $R$ is twisted by $K$ along $C := A \setminus \{0\}$, so by Theorem 2.5, $A \subseteq S$ is $C$-analytic, $R \subseteq S$ is quadratic and $S/R$ is a torsion $R$-module. Since $A \subseteq S$ has TGF and is analytic along $C = A \setminus \{0\}$, it follows that $A \subseteq S$ is strongly analytic.

(2) Let $Q$ denote the quotient field of $A$, and let $C = A \setminus \{0\}$. Since $A \subseteq S$ has TGF, it follows that $QS = S_C = F$. Now by Theorem 2.5, there exists a unique full $S$-submodule $K$ of $\Omega_{F/Q}$ such that $R$ is twisted by $K$ along $C$ by the derivation $d_{F/Q}$. Clearly, $d_{F/Q}$ generates $K_C = FK = \Omega_{F/Q}$ as an $F$-vector space. Moreover, since $R$ is twisted by $D$ along $C$, we have $S \subseteq \text{Ker } D + aS$ for all $0 \neq a \in A$. Using again that $A \subseteq S$ has TGF, it follows that $S \subseteq \text{Ker } D + sS$ for all $0 \neq s \in S$, and hence $R$ is strongly twisted by the $S$-module $K$.

3. Existence of strongly twisted subrings

In this section we prove the existence of strongly twisted subrings of domains $S$ with sufficiently large quotient field $F$. When $S$ is, for example, a DVR, then this amounts to finding a subring $A$ of $S$ such that $A$ is a DVR and $A \subseteq S$ is an immediate extension, meaning that $A$ and $S$ have the same residue field and value group. For given such a DVR $A$ with quotient field $Q$, then as in Corollary 2.6, a full $S$-submodule of $\Omega_{F/Q}$ gives rise to a strongly twisted subring of $S$; see [19] for more on the special case of DVRs. What makes the case of DVRs simpler is that an immediate extension is easily shown to be strongly analytic. However, in higher dimensions it is more of a challenge to find subrings $A$ of a given domain $S$ that induce a strongly analytic extension $A \subseteq S$ because such extensions must not only be analytic along $C = A \setminus \{0\}$, but must also have trivial generic fiber. Satisfying these two conditions simultaneously is the obstacle.

The first proposition of the section characterizes, but does not guarantee, the existence of strongly twisted subrings, and we do not use it again in this section when we prove existence results. However, the proposition is a useful formulation for some classes of examples considered in Theorem 6.3. The proposition relies on a simple fact: Once a torsion-free module can be found that strongly twists a subring of $S$, then others also must exist, and it is really only the dimension of the $F$-vector space $FK$, i.e., the rank of $K$, that is essential in guaranteeing the existence of other twisting modules. This is the content of the following observation.
Lemma 3.1. Let $S$ be domain, and suppose $S$ has a subring that is strongly twisted by a torsion-free $S$-module $K$. Then for every torsion-free $S$-module $L$ with $\text{rank}(L) \leq \text{rank}(K)$, there exists a subring of $S$ that is strongly twisted by $L$.

Proof. Since $\text{rank}(L) \leq \text{rank}(K)$, there exists a projection of $F$-vector spaces $\alpha : FK \to FL$, so that $\alpha(FK) = FL$. Let $D$ be the derivation that strongly twists $K$, and let $D' = \alpha \circ D$. Then $D'$ is a derivation taking values in $FL$, and since $D(F)$ generates $FK$ as an $F$-vector space, it follows that $D'(F) = \alpha(D(F))$ generates $FL$ as an $F$-vector space. Moreover, $\text{Ker} D \subseteq \text{Ker} \alpha \circ D = \text{Ker} D'$, so that for all $0 \neq s \in S$, $S \subseteq \text{Ker} D + sS \subseteq \text{Ker} D' + sS$. Therefore, $T := S \cap D'^{-1}(L)$ is a subring of $S$ that is strongly twisted by $L$. □

Proposition 3.2. The following are equivalent for a domain $S$ with quotient field $F$.

1. $S$ has a strongly twisted subring.
2. There exists a nonzero derivation $D : F \to F$ such that $S \subseteq \text{Ker} D + sS$ for all $0 \neq s \in S$.
3. For each nonzero $S$-submodule $K$ of $F$, there exists a subring of $S$ that is strongly twisted by $K$.

Proof. (1) $\Rightarrow$ (2) Let $R$ be a subring of $S$ that is strongly twisted by a torsion-free $S$-module $K$, and let $D$ be the derivation that twists it. Let $\alpha : FK \to F$ be an $F$-linear transformation such that $\alpha \circ D : F \to F$ is nonzero. Then since $R$ is strongly twisted by $K$, we have for all $s \in S$ that $S \subseteq \text{Ker} D + sS \subseteq \text{Ker} \alpha \circ D + sS$, so $\alpha \circ D$ is a derivation that behaves as in (2).

(2) $\Rightarrow$ (3) With $D$ as in (2), since $D$ is a nonzero derivation and $F$ is a field, $D(F)$ generates $F$ as an $F$-vector space. Thus $S$ is a strongly twisted subring of itself (it is strongly twisted by $F$), so (3) follows from Proposition 3.1.

(3) $\Rightarrow$ (1) This is clear. □

It follows that $S$ is a strongly twisted subring of itself if and only if $S$ satisfies the equivalent conditions of the proposition.

The main focus of this section is a technique for producing strongly analytic extensions, and hence strongly twisted subrings. Not surprisingly, we need space to carve out such subrings, and hence we work with assumptions involving infinite differential bases.

Lemma 3.3. Let $F/k$ be an extension of fields, where $k$ has at most countably many elements.

1. When $F$ has characteristic 0, then $|F| = \dim_F \Omega_{F/k}$ if and only if $F$ has infinite transcendence degree over $k$.
2. If $F$ has positive characteristic, then in order that $|F| = \dim_F \Omega_{F/k}$, it suffices that $F/k$ is separably generated and of infinite transcendence degree.

Proof. In both (1) and (2), there exists a transcendence basis $\{s_i : i \in I\}$ of $F$ over $k$ such that $F$ is separably algebraic over the subfield $k(s_i : i \in I)$. Thus $\{d_{F/k}(s_i) : i \in I\}$ is a basis for the $F$-vector space $\Omega_{F/k}$ [4, Corollary A1.5(a), p. 567]. In particular, $|I| = \dim_F \Omega_{F/k}$. We claim that $|F| = |I|$. Since $F$ is algebraic over the infinite field $F_0 := k(s_i : i \in I)$,
then $|F| = |F_0|$ [16, Lemma 2.12.6]. Also, since $k$ is countable and $I$ is infinite, the field $F_0 = k(s_i : i \in I)$ has cardinality $|I|$ (see for example the proof of Theorem 2.12.5 in [16]). Therefore, $|F| = |I| = \dim_F \Omega_{F/k}$. □

Our interest in such field extensions is that in postive characteristic they give rise to analytic extensions:

**Lemma 3.4.** Let $F/k$ be a extension of fields with $|F| = \dim_F \Omega_{F/k}$, and suppose $k$ has at most countably many elements; $S$ is a $k$-subalgebra of $F$ having quotient field $F$; and $L$ is an $F$-vector space of at most countable dimension. Then for any $t \in S$ there exists a ring $A$ such that $k[t] \subseteq A \subseteq S$ and $S/A$ is a torsion-free divisible $A$-module. For this ring $A$, there exists an $A$-linear derivation $D : F \to L$ such that $D(S) = L$. If also $k$ has positive characteristic, then $A \subseteq S$ is a strongly analytic extension.

**Proof.** Since $S$ has quotient field $F$, the quotient rule for derivations implies that the $F$-vector space $\Omega_{F/k}$ is generated by the set $\{d_{F/k}(s) : s \in S\}$. Thus some subset of this collection is a basis for $\Omega_{F/k}$, and in fact if $d_{F/k}(t) \neq 0$, then we may choose this basis to contain the element $d_{F/k}(t)$. (We allow throughout the proof the possibility that $d_{F/k}(t) = 0$.) Therefore, there is a collection $\{s_i : i \in I\}$ of elements in $S$ such that the elements $d_{F/k}(s_i), i \in I$, form a basis of $\Omega_{F/k}$, and if $d_{F/k}(t) \neq 0$, we can assume that $t \in \{s_i : i \in I\}$.

Viewing $F$ as a $k$-vector space, we claim that $\dim_k F = |I| = |F|$. By assumption, $|F| = \dim_F \Omega_{F/k} = |I|$. Clearly, $\dim_k F \leq |F| = |I|$. On the other hand, since $d_{F/k}$ is a $k$-linear map and $\{d_{F/k}(s_i) : i \in I\}$ is linearly independent, then $\{s_i : i \in I\}$ is a $k$-linearly independent subset of $F$. Therefore, $\dim_k F = |I| = |F|$. Since $\dim_k F = |I|$, we may let $\{f_i : i \in I\}$ denote a basis for $F$ over $k$. We claim that $I$ can be partitioned into countably many sets $I_0, I_1, I_2, \ldots$ such that each $I_j$ has $|I|$ elements and if $d_{F/k}(t) \neq 0$, then $t = s_i$ for some $i \in I_0$. Indeed, first we may partition $I$ into a disjoint union $I = \bigcup_{\alpha \in A} X_{\alpha}$ of countably infinite sets $X_{\alpha}$ [9, Theorem 12, p. 40]. For each $\alpha \in A$, write $X_{\alpha} = \{x_{\alpha,j} : j \in \mathbb{N} \cup \{0\}\}$. Then for each $j \in \mathbb{N} \cup \{0\}$ we define $I_j = \{x_{\alpha,j} : \alpha \in A\}$. Now $|I| = |A| \cdot \aleph_0$, so since $|I|$ is infinite, $|I| = |\mathcal{A}|$ [9, Theorem 16, p. 40], and hence $|I| = |I_j|$ for all $j$. Thus for each $j \in \mathbb{N} \cup \{0\}$, there exists a bijection $\sigma_j : I_j \to I$. Moreover, if $d_{F/k}(t) \neq 0$, then after relabeling the $I_j$ we may assume that $t = s_i$ for some $i \in I_0$.

Let $L$ be an $F$-vector space of countably infinite dimension, and write $L = \bigoplus_{j=1}^{\infty} F e_j$, where $\{e_1, e_2, e_3, \ldots\}$ is a basis for $L$ over $F$. Define an $F$-linear mapping $\phi : \Omega_{F/k} \to L$ on our particular basis elements of $\Omega_{F/k}$ by

$$\phi(d_{F/k}(s_i)) = \begin{cases} 0 & \text{if } i \in I_0 \\ f_{\sigma_j(i)} e_j & \text{if } i \in I_j \text{ with } j \in \mathbb{N} \end{cases}$$

Recall that if $d_{F/k}(t) \neq 0$, then $t = s_i$ for some $i \in I_0$. Thus regardless of whether $d_{F/k}(t) \neq 0$, we have arranged it so that $\phi(d_{F/k}(t)) = 0$. 
Now \( D := \phi \circ d_{F/k} : F \to L \) is a \( k \)-linear derivation with \( D(t) = 0 \). Moreover, for each \( j \in \mathbb{N} \) (we are purposely excluding the case \( j = 0 \) here), we have:

\[
D(\sum_{i \in I_j} k \cdot s_i) = \sum_{i \in I_j} k \cdot \phi(d_{F/k}(s_i)) = \sum_{i \in I_j} k \cdot f_{\sigma_j(i)}e_j = \sum_{i \in I} k \cdot f_i e_j = Fe_j.
\]

Thus:

\[
L = \bigoplus_{j=1}^{\infty} Fe_j = \bigoplus_{j=1}^{\infty} D(\sum_{i \in I_j} k \cdot s_i) \subseteq D(\sum_{i \in I} k \cdot s_i) \subseteq D(S) \subseteq L,
\]

which proves that \( D(S) = L \).

Now set \( A = S \cap \text{Ker } D \). Then \( D \) is an \( A \)-module homomorphism, and since \( D(S) = L \), it follows that \( S/A \cong L \) as \( A \)-modules. Since \( F \) is a torsion-free divisible \( A \)-module and \( L \) is an \( F \)-vector space, then \( L \), and hence \( S/A \), are torsion-free divisible \( A \)-modules. Moreover, by the construction of \( D \), we have \( k[t] \subseteq \text{Ker } D \cap S = A \). Furthermore, in the case where \( k \) has characteristic \( p \neq 0 \), let \( I \) be a nonzero ideal of \( S \), and let \( 0 \neq s \in I \). Then \( D(s^p) = ps^{p-1}D(s) = 0 \), so that \( 0 \neq s^p \in A \cap I \). Thus \( A \subseteq S \) has TGF, and \( A \subseteq S \) is strongly analytic. Finally, if \( L' \) is a finite dimensional \( F \)-vector space, then there is a surjective \( F \)-linear transformation \( \psi : L \to L' \), so that \( \psi \circ D : F \to L' \) is an \( A \)-linear derivation with \( (\psi \circ D)(S) = \psi(L) = L' \).

The small detail in the lemma allowing us to assume that \( t \in A \) has an important consequence in that when \( t \) is chosen a nonunit in \( S \), then since \( t \in A \), there exist nonzero proper ideals in \( S \) that contract to nonzero proper ideals in \( A \). Therefore, when \( C = A \setminus \{0\} \), there exist proper ideals of \( S \) meeting \( C \), and in particular, \( A \) is not a field. We use this observation in Corollary 5.8 in a crucial way.

The following theorem is our main source of examples of strongly twisted subrings in high dimensions.

**Theorem 3.5.** Let \( F/k \) be a field extension such that \( k \) has positive characteristic and at most countably many elements. Suppose that \( F/k \) is a separably generated extension of infinite transcendence degree. If \( S \) is a \( k \)-subalgebra of \( F \) with quotient field \( F \) and \( K \) is a torsion-free \( S \)-module of at most countable rank, then there exists a subring \( R \) of \( S \) that is strongly twisted by \( K \).

**Proof.** Since \( F \) is a separably generated extension of infinite transcendence degree over \( k \), then by Lemma 3.3, \( |F| = \dim_F \Omega_{F/k} \). Let \( L = FK \), the divisible hull of \( K \). Then \( L \) is an \( F \)-vector space of at most countable dimension, so by Lemma 3.4 there exists a \( k \)-subalgebra \( A \) of \( S \) such that \( A \subseteq S \) is a strongly analytic extension and there exists an \( A \)-linear derivation \( D : F \to L \) such that \( D(S) = L \). Since \( A \subseteq S \) is strongly analytic, then \( S/A \) is a divisible \( A \)-module, and consequently, \( S = A + aS \) for all \( 0 \neq a \in A \). If \( 0 \neq s \in S \), then since strongly analytic extensions have TGF, there exists \( 0 \neq a \in A \cap sS \), so that \( S = A + sS \). Moreover, \( D \) is \( A \)-linear, so \( A \subseteq \text{Ker } D \), and hence \( S \subseteq \text{Ker } D + sS \) for all \( 0 \neq s \in S \). Thus, setting \( R = S \cap D^{-1}(K) \), we have that \( R \) is strongly twisted by \( K \). \( \square \)
Thus if \( k \) is a field of positive characteristic that is separably generated extension of infinite transcendence degree over a countable subfield, and we choose \( S \) between \( k[X_1, \ldots, X_n] \) and \( k(x_1, \ldots, x_n) \), then for every torsion-free \( S \)-module \( K \) of at most countable rank, there exists a subring \( R \) of \( S \) that is strongly twisted by \( K \). Note however that the theorem does not assert that \( R \) is a \( k \)-algebra.

4. Basic properties of twisted subrings

We give now a few basic properties of twisted subrings, the most fundamental of which is the assertion in Theorem 4.6 that if \( R \) is a subring of \( S \) twisted by \( K \), then \( R \) and \( S \star K \) are isomorphic analytically, in the sense of Section 2. Many of the results in this section depend on properties of analytic isomorphisms developed in [19], which we refer to in the course of the proofs. We assume throughout this section the following hypothesis.

\[ S \text{ is a ring; } C \text{ is a multiplicatively closed subset of nonzerodivisors of } S; \]
\[ R \text{ is a subring of } S \text{ that is twisted along } C \text{ by a } C\text{-torsion-free } S\text{-module } K; \]
\[ D \text{ is the derivation that twists } C. \]

We note first that \( R \subseteq S \) is a quadratic, hence integral, extension. In the case where \( S \) is a domain and \( R \) is strongly twisted, this extension is subintegral in the sense of Swan [25], meaning that \( R \subseteq S \) is integral, the contraction mapping \( \text{Spec}(S) \to \text{Spec}(R) \) is a bijection and the induced maps on residue field extensions are isomorphisms (so for every prime ideal \( P \) of \( S \), \( S_P = R_{P \cap R} + PS_P \)).

**Theorem 4.1.** The rings \( R \) and \( S \) share the same total ring of quotients, and the extension \( R \subseteq S \) is quadratic, and hence integral, but not finite unless \( R = S \). If also \( S \) is a domain, \( K \) is torsion-free and \( R \) is strongly twisted by \( K \), then \( R \subseteq S \) is a subintegral extension.

**Proof.** To see that \( R \) and \( S \) share the same total ring of quotients, it suffices to show that \( R_C = S_C \). Let \( s \in S \). Then since \( D(s) \in K_C \), there exists \( c \in C \) such that \( cD(s) \in K \). But \( D \) is \( C \)-linear, so \( D(cs) \in K \), and hence \( cs \in D^{-1}(K) \cap S = R \), proving that \( S_C = R_C \).

By Theorem 2.5(1), \( R \subseteq S \) is a quadratic extension. If this extension is also finite, say, \( S = R_{S_1} + \cdots + R_{S_n} \) for some \( S_i \in S \), then there exists \( c \in C \) such that \( cs_1, \ldots, cs_n \in R \), and hence \( cS \subseteq R \). But since \( S/R \) is \( C \)-divisible, \( S = R + cS \), so this forces \( S = R \).

Finally, suppose that \( S \) is a domain and \( R \) is strongly twisted by \( K \). To see that \( R \subseteq S \) is subintegral, we note first that by Proposition 2.4, the contraction mapping \( \text{Spec}(S) \to \text{Spec}(R) \) is a bijection. To complete the proof, let \( P \) be a prime ideal of \( S \). We claim that \( S_P = R_{P \cap R} + PS_P \). Since \( S/R \) is a divisible \( R \)-module (this follows from the definition of strongly twisted), \( S = R + P \), so it suffices to show that for each \( b \in S \setminus P \), \( b^{-1} \in R_{P \cap R} + PS_P \).

Let \( b \in S \setminus P \), and let \( A = S \cap \text{Ker} D \). We claim that \( bS \cap A \not\subseteq P \). For by Corollary 2.6, \( A \subseteq S \) is strongly analytic, so if \( bS \cap A \subseteq P \), then by Proposition 2.4, \( bS = (bS \cap A)/A \subseteq P \), contrary to assumption. Thus there exist \( a \in A \setminus P \) and \( s \in S \) such that \( a = bs \). Moreover, choosing \( 0 \neq c \in P \cap A \), we have since \( S = A + acS \) that there exists \( d \in A \) and \( \sigma \in S \) such that \( s = d + ac\sigma \), whence \( b^{-1} = sa^{-1} = da^{-1} + c\sigma \in R_P + PS_P \), as claimed. Therefore, \( R \subseteq S \) is a subintegral extension. \( \square \)
As a consequence of the fact that $R \subseteq S$ is integral, along with the fact that $S/R$ is $C$-torsion and $C$-divisible, we obtain information about $\text{Spec}(R)$:

**Theorem 4.2.** The mappings $P \mapsto PS$ and $Q \mapsto Q \cap R$ define a one-to-one correspondence between prime ideals $P$ of $R$ meeting $C$ and prime ideals $Q$ of $S$ meeting $C$. Under this correspondence, maximal ideals of $R$ meeting $C$ correspond to maximal ideals of $S$ meeting $C$. If also $S$ is a domain, then the contraction mapping $\text{Spec}(S) \to \text{Spec}(R)$ is a bijection.

*Proof.* In the proof, we use only that $R \subseteq S$ is integral (a fact given by Theorem 4.1) and that $S/R$ is $C$-divisible and $C$-torsion. Let $P$ be a prime ideal of $R$ meeting $C$. Since $S/R$ is $C$-divisible, $S = R + PS$, and hence $S/PS \cong R/(PS \cap R)$. If $Q$ is any prime ideal of $S$ lying over $P$, then $P \subseteq PS \cap R \subseteq Q \cap R = P$, so that $P = PS \cap R$. Hence $S/PS \cong R/P$, and it follows that $PS$ is a prime ideal of $S$. This shows also that if in addition $P$ is a maximal ideal of $R$, then $PS$ is a maximal ideal of $S$. On the other hand, if $L$ is a prime ideal of $S$ meeting $C$, then since for $c \in L \cap C$, $S = R + cS$, it follows that $L = (L \cap R)S$. Also, since maximal ideals of an integral extension contract to maximal ideals, it follows that maximal ideals of $R$ meeting $C$ correspond to maximal ideals of $S$ meeting $C$.

Now suppose that $S$ is a domain. We show that the mapping $\text{Spec}(S) \to \text{Spec}(R)$ is a bijection. To this end, we claim first that for each prime ideal $P$ of $R$ not meeting $C$, $R_P = S_{P_1}$, where $P_1$ is any prime ideal of $S$ lying over $P$, and hence there is only one such prime ideal $P_1$. (Note that since $S/R$ is $C$-torsion, it follows that $R$ and $S$ have the same quotient field so we may view all the localizations of $R$ and $S$ as subsets of this field.) Let $P$ be a prime ideal of $R$ not meeting $C$, and let $P_1$ be a prime ideal of $S$ lying over $P$. Since $P \cap C$ is empty, $R_C \subseteq R_P$, and since $R_C = S_C$, we have $S \subseteq R_P$. Thus $SR_P = R_P$, and it suffices to show that $SR_P = S_{P_1}$. In fact, since $R_P \subseteq S_{P_1}$, we need only show that $S_{P_1} \subseteq SR_P$, or, more precisely, that each $b \in S \setminus P_1$ is a unit in $SR_P$. Let $b \in S \setminus P_1$. Since $SR_P = R_P$, $SR_P$ is a quasilocal ring with maximal ideal $PR_P$. If $b \in PR_P$, then $b \in P_1S_{P_1} \cap S = P_1$, a contradiction. Therefore, $b$ is in the quasilocal ring $SR_P$, but not in its maximal ideal, so $b$ is a unit in $SR_P$, and we have proved that $R_P = S_{P_1}$.

Now we claim that $\text{Spec}(S) \to \text{Spec}(R)$ is a bijection. Since $R \subseteq S$ is integral, and hence each prime ideal of $R$ has a prime ideal of $S$ lying over it, the contraction mapping $\text{Spec}(S) \to \text{Spec}(R)$ is surjective. To see that it is injective, let $P$ be a prime ideal of $R$, and let $P_1$ be a prime ideal of $S$ lying over $P$. If $P \cap C$ is empty, then by what we have established above, $R_P = S_{P_1}$, so that $P_1$ is the only prime ideal of $S$ lying over $P$. Otherwise, if $P \cap C$ is nonempty, then by the first claim of the theorem, $P_1 = (P_1 \cap R)S = PS$, so that $P_1$ is the unique prime ideal of $S$ lying over $P$. This completes the proof. □

The next theorem, which relies in a crucial way on [19, Lemma 3.4], establishes a correspondence between submodules of $K_C/K$ and rings between $R$ and $S$.

**Theorem 4.3.** There is a one-to-one correspondence between intermediate rings $R \subseteq T \subseteq S$ and $S$-submodules $L$ of $K_C$ with $K \subseteq L \subseteq K_C$ given by:

$$T \mapsto \sum_{t \in T} SD(t) \quad \text{and} \quad L \mapsto S \cap D^{-1}(L).$$
Proof. Let $A = S \cap \mathrm{Ker} \, D$, and let $d = d_{SC/A_C}$. For each ring $T$ with $R \subseteq T \subseteq S$, define $\Omega(T) = \sum_{t \in T} Sd(t)$ and $L(T) = \sum_{t \in T} SD(t)$. By Theorem 2.5, $A \subseteq S$ is a $C$-analytic extension. Also, since $R$ is twisted by $K$ along $C$, then $R = S \cap D^{-1}(K)$ and $D(S_C)$ generates $K_C$ as an $S_C$-module. In [19, Lemma 3.4], it is shown that these two facts, along with the fact that $K$ is $C$-torsion-free, imply that there exists a surjective $S_C$-module homomorphism $\alpha : \Omega_{S_C/A_C} \to K_C$ such that $D = \alpha \circ d$ and for each $S$-module $L$ of $K_C$ with $L_C = K_C$, when $T = S \cap D^{-1}(L)$, then $\alpha(\Omega(T)) = L$ and $\Omega(T) = \alpha^{-1}(L)$.

To prove the theorem it suffices to show that for all rings $T$ with $R \subseteq T \subseteq S$, we have $T = S \cap D^{-1}(L(T))$, and for all $S$-modules $L$ with $K \subseteq L \subseteq K_C$, we have $L = L(S \cap D^{-1}(L))$. Let $T$ be a ring between $R$ and $S$. Clearly, $T \subseteq S \cap D^{-1}(L(T))$. Conversely, suppose that $s \in S \cap D^{-1}(L(T))$. Then there exist $s_1, \ldots, s_n \in S$ and $t_1, \ldots, t_n \in T$ such that $D(s) = \sum_i s_i D(t_i)$. Therefore, since $D = \alpha \circ d$, we have $\alpha(d(s)) = \alpha(\sum_i s_i d(t_i))$. Thus since $\mathrm{Ker} \, \alpha \subseteq \alpha^{-1}(K) = \Omega(R)$, we have $d(s) - \sum_i s_i d(t_i) \in \Omega(R) \subseteq \Omega(T)$. Therefore, $d(s) \in \Omega(T)$. As observed in the proof of Theorem 2.5(2), $T = S \cap D^{-1}(\Omega(T))$, so $s \in T$. This proves that $T = S \cap D^{-1}(L(T))$. Finally, let $L$ be an $S$-module between $K$ and $K_C$. We claim that $L = L(T)$, where $T = S \cap D^{-1}(L)$. But this is immediate, since as noted above, $L = \alpha(\Omega(T)) = \sum_{t \in T} S \alpha(d(t)) = \sum_{t \in T} SD(t) = L(T)$. □

Next we show, in what is our main structure theorem for twisted subrings, that a twisted subring behaves analytically like an idealization. The theorem is based on Proposition 3.5 in [19], and relies on the following lemmas.

**Lemma 4.4.** The derivation $D$ induces an isomorphism of $R$-modules given by

$$\alpha : S/R \to K_C/K : s + R \mapsto D(s) + K.$$ 

**Proof.** Since $R = S \cap D^{-1}(K)$, it is clear that $\alpha$ is well-defined and injective. To see that $\alpha$ is onto, let $y \in K_C$. Then since $K_C$ is generated as an $S_C$-module by $D(S_C)$ and $D$ is $C$-linear, we may write $y = \sum_i s_i D(x_i)$, where $s_i \in S$ and $x_i \in S_C$. Choose $c \in C$ such that $cD(x_i) \in K$ for each $i$. Since $S \subseteq \mathrm{Ker} \, D + cS$, we may for each $i$ write $s_i = a_i + c\sigma_i$, where $a_i \in \mathrm{Ker} \, D$ and $\sigma_i \in S$. Thus, since $a_1, \ldots, a_n \in \mathrm{Ker} \, D$, we have:

$$y + K = \sum_i D(a_i x_i) + \sum_i c \sigma_i D(x_i) + K = D(\sum_i a_i x_i) + K.$$ 

Therefore, $D$ maps onto $K_C/K$. □

**Lemma 4.5.** Let $f : R \to S \star K$ be the ring homomorphism defined by $f(r) = (r, D(r))$ for all $r \in R$. If $I$ is an ideal of $R$ meeting $C$ and $I = IS \cap R$, then $f(I)(S \star K) = (IS) \star K$.

**Proof.** Clearly, $f(I)(S \star K) \subseteq (IS) \star K$. To verify the reverse inclusion, let $x \in IS$, and let $k \in K$. We show that $(x, 0)$ and $(0, k)$ are both in $f(I)(S \star K)$, since this is enough to prove the claim. Let $A = S \cap \mathrm{Ker} \, D$. Choose $c \in I \cap C$. Then $S = A + cS$, so since $I = IS \cap R$, it follows that $x = i + cs$ for some $i \in I$ and $s \in S$. Since $S/A$-divisible, we may write $i = a + c\sigma$ for some $a \in A$ and $\sigma \in S$. Thus since we have assumed $I = IS \cap R$, we have ...
\[ a = i - c\sigma \in IS \cap A = (IS \cap R) \cap A = I \cap A. \] Consequently, setting \( t = \sigma + s \), we have \( x = i + cs = a + c(\sigma + s) = a + ct. \) Since \( D \) is \( A \)-linear, then \( D(a) = 0 \), and:

\[ (x,0) = (a,0) + (ct,0) = f(a) + f(c)(t,0) \in f(I)(S \ast K). \]

Next, we show that \( (0,k) \in f(I)(S \ast K) \). By Lemma 4.4, \( K_C = D(S) + K \), so since \( D \) is \( C \)-linear, there exist \( s_2 \in S \) and \( k_2 \in K \) such that \( k = D(c s_2) + c k_2. \) Since \( R = S \cap D^{-1}(K) \), it follows that \( y := c s_2 \in IS \cap R = I. \) Thus \( f(y) = (y, D(y)) \in f(I)(S \ast K). \) Consequently:

\[ (0,k) = (y, D(y)) + (-y, c k_2) = (y, D(y)) + f(c)(-s_2, k_2) \in f(I)(S \ast K). \]

This proves \( f(I)(S \ast K) = (IS) \ast K. \)

**Theorem 4.6.** The mapping \( f : R \to S \ast K : r \mapsto (r, D(r)) \) is an analytic isomorphism along \( C \). If also \( S \) is a domain, \( K \) is torsion-free and \( R \) is strongly twisted by \( K \), then this map is faithfully flat.

**Proof.** Let \( A = S \cap \text{Ker} \, D \). Then by Theorem 2.5, \( A \subseteq S \) is \( C \)-analytic. It is shown in [19, Proposition 3.5] that this fact along with the following assumptions imply that \( f \) is an analytic isomorphism along \( C \): (a) \( K \) is a \( C \)-torsion-free \( S \)-module; (b) \( D : S_C \to K_C \) is an \( A_C \)-linear derivation; (c) \( D(S_C) \) generates \( K_C \) as an \( S_C \)-module, and (d) \( R = S \cap D^{-1}(K) \). All of these conditions are satisfied since \( R \) is twisted by \( K \) along \( C \), and \( D \) is the derivation that twists it. If also \( S \) is a domain, \( K \) is torsion-free and \( R \) is strongly twisted by \( K \), then \( f \) is an analytic isomorphism along \( C = A \setminus \{0\} \), so that \( \text{Coker} \, f \) is a torsion-free divisible \( R \)-module, a fact which implies that \( f \) is flat. If \( M \) is a maximal ideal of \( R \), then since by Theorem 4.1, \( R \subseteq S \) is integral, it follows that \( M = MS \cap R. \) Moreover, since \( A \subseteq S \) is strongly analytic (Corollary 2.6), and hence has TGF, the extension \( A \subseteq R \) has TGF since \( S/R \) is a torsion \( R \)-module. If \( R \) is a field, then clearly \( f \) is faithful. Otherwise, if \( R \) is not a field, then the maximal ideals of \( S \) meet \( C = A \setminus \{0\} \), and hence by Lemma 4.5, \( f(M)(S \ast K) = MS \ast K \neq S \ast K \), so that \( f \) is faithful. \( \square \)

**Corollary 4.7.** If \( R \) and \( S \) are quasilocal, each with finitely generated maximal ideal meeting \( C \), then \( f : R \to S \ast K \) lifts to an isomorphism of rings, \( \hat{R} \to \hat{S} \ast \hat{K} \), where \( \hat{R} \) is the completion of \( R \) in its \( m \)-adic topology, while \( \hat{S} \) and \( \hat{K} \) are the completions of \( K \) in the \( m \)-adic topology of \( S \).

**Proof.** Let \( M \) and \( N \) denote the maximal ideals of \( R \) and \( S \), respectively. By Theorem 4.2, \( N = MS. \) Let \( A = S \cap \text{Ker} \, D \), and let \( m = M \cap A \). By Theorem 2.5, \( A \subseteq S \) is \( C \)-analytic. Thus by Proposition 2.4, \( m \) is a maximal ideal of \( A \), so necessarily, \( m = MS \cap A. \) We claim that \( m R \) is an \( M \)-primary ideal of \( R \). Indeed, if \( P \) is a prime ideal of \( R \) such that \( m R \subseteq P \), then \( P \) meets \( C \) since \( M \) meets \( C \) and \( m = M \cap A. \) Thus \( m S \subseteq PS \), and by Proposition 2.4, \( m S \) is the maximal ideal of \( S \), so \( m S = PS \). This shows that for every prime ideal \( P \) of \( R \) containing \( m R \), we have \( PS = m S \), and hence by Theorem 4.2, \( P = PS \cap R = m S \cap R = M. \) Therefore, \( M \) is the unique prime ideal of \( R \) containing \( m R \), and hence \( m R \) is \( M \)-primary. Now since \( M \) is finitely generated, some power of \( M \) is contained in \( m R \), and hence \( \hat{R} \cong \text{lim}_\rightarrow R/m^i R. \) Similarly, since \( m S \) is the maximal ideal of \( S \),
we have $\hat{S} \cong \lim_\rightarrow S/m^iS$ and $\hat{K} \cong \lim_\rightarrow K/m^iK$. Now let $c \in m \cap C$. Then by Theorem 4.6 we have that for each $i$, the induced mapping $R/c^iR \to S/c^iS \star K/c^iK$ is an isomorphism. It follows that the induced mapping $R/m^i \to S/m^iS \star K/m^iK$ is an isomorphism. This in turn implies that $f$ lifts to an isomorphism $\hat{R} \to \hat{S} \star \hat{K}$.

**Remark 4.8.** Since the maximal ideal $N$ of $S$ is extended from that of $R$, the $N$-adic and $M$-adic topologies on $S$-modules agree. Hence $\hat{S}$ and $\hat{K}$ can also be viewed as the completions of $S$ and $K$ in the $M$-adic topology.

5. Noetherian rings

In this section we characterize when strongly twisted subrings are Noetherian, and consider also a special situation when being twisted along a multiplicatively closed subset is enough to guarantee the subring is Noetherian.

**Lemma 5.1.** Let $S$ be a ring, let $C$ be a multiplicatively closed subset of nonzerodivisors of $S$, and let $K$ be a $C$-torsion-free $S$-module. Suppose that $R$ is a subring of $S$ that is twisted by $K$ along $C$, and let $I$ be an ideal of $R$ meeting $C$. If $IS \cap R$ is a finitely generated ideal of $R$, then $K/IK$ is a finitely generated $S$-module. Conversely, if $I = IS \cap R$, $IS$ is a finitely generated ideal of $S$ and $K/cK$ is a finitely generated $S$-module for some $c \in I \cap C$, then $I$ is a finitely generated ideal of $R$.

**Proof.** Let $D$ be the derivation that twists $R$, and let $A = S \cap \text{Ker } D$. Suppose that $IS \cap R$ is a finitely generated ideal of $R$, and write $IS \cap R = (x_1, \ldots, x_n)R$. Observe that since $K$ is an $S$-module, $IK \subseteq (IS \cap R)K \subseteq IK$, so that $(IS \cap R)K = IK$. We claim that $K = SD(x_1) + \cdots + SD(x_n) + IK$, and we prove this indirectly. First we show that $K$ is generated as an $S$-module by $D(R)$. For if $x \in K$, then since $K_C$ is generated by $D(S_C)$ as an $S_C$-module and $D$ is $C$-linear, there exist $s_i, \sigma_i \in S$ and $c \in C$ such that $x = \sum_i s_i D(\sigma_i)$. Since $S/A$ is $C$-divisible, there exist $a_i \in A$ and $\tau_i \in S$ such that $\sigma_i = a_i + c\tau_i$. Since $D(a_i) = 0$, it follows that $x = \sum_i s_i D(\tau_i)$. Now choose $c \in C$ such that for all $i$, $cD(\tau_i) \in K$. For each $i$, write $s_i = b_i + e\tau_i$ for some $b_i \in A$ and $t_i \in S$. Then $x = \sum_i s_i D(\tau_i) = D(\sum_i b_i t_i) + \sum_i t_i D(e\tau_i)$. Now $e\tau_i \in D^{-1}(K) \cap S = R$, and $x \in K$, so that $\sum_i b_i t_i \in D^{-1}(K) \cap S = R$, which proves that $K$ is generated as an $S$-module by $D(R)$.

Now let $y \in IS \cap R$, and write $y = x_1 r_1 + \cdots + x_n r_n$ for $r_1, \ldots, r_n \in R$. Then

\[
D(y) = r_1 D(x_1) + \cdots + r_n D(x_n) + x_1 D(r_1) + \cdots + x_n D(r_n) \\
\in RD(x_1) + \cdots + RD(x_n) + IK.
\]

Therefore, $D(IS \cap R) \subseteq SD(x_1) + \cdots + SD(x_n) + IK$. Now since $S/A$ is $C$-divisible, then $S = A + IS$, and since $A \subseteq R$, we have then $R = A + (IS \cap R)$. Thus

\[
D(R) = D(A) + D(IS \cap R) = D(IS \cap R).
\]

Since, as we have shown, $K$ is generated as an $S$-module by $D(R) = D(IS \cap R)$, we conclude that $K = SD(x_1) + \cdots + SD(x_n) + IK$. Therefore, $K/IK$ is a finitely generated $S$-module.
Conversely, suppose that \( I = IS \cap R, IS \) is a finitely generated ideal of \( S \) and \( K/aK\) is a finitely generated \( S\)-module for some \( c \in I \cap C \). Then by Theorem 4.6, the mapping \( f \) induces an isomorphism \( f_c : R/cR \cong S/cS \star K/cK \), and by Lemma 4.5,
\[
I/cR \cong f_c(I/cR)(S/cS \star K/cK) = IS/cS \star K/cK.
\]
Now \( IS/cS \) and \( K/cK \) are finitely generated \( S\)-modules. But \( S = A + cS \), so it follows that \( IS/cS \) and \( K/cK \) are finitely generated \( A\)-modules. Therefore, \( I/cR \) is a finitely generated \( A\)-module, and hence \( I \) is a finitely generated ideal of \( R \). □

**Theorem 5.2.** Suppose that \( S \) is a domain and \( R \) is a subring of \( S \) strongly twisted by a torsion-free \( S\)-module \( K \). Let \( D \) be the derivation that strongly twists \( R \). The ring \( R \) is a Noetherian domain if and only if \( S \) is a Noetherian domain and for each \( 0 \neq a \in S \cap \text{Ker} D \), \( K/aK \) is a finitely generated \( S\)-module.

**Proof.** Let \( A = S \cap \text{Ker} D \). If \( R \) is a Noetherian domain, then since every prime ideal of \( S \) is extended from \( R \) (Theorem 4.2), every prime ideal of \( S \) is finitely generated, and hence \( S \) is Noetherian. Moreover, if \( R \) is Noetherian, then for every \( 0 \neq a \in A \), \( aS \cap R \) is a finitely generated ideal of \( R \), and so by Lemma 5.1, \( K/aK \) is a finitely generated \( S\)-module. Conversely, if \( S \) is Noetherian and \( K/aK \) is a finitely generated \( S\)-module for every \( 0 \neq a \in A \), then by Lemma 5.1 every ideal \( I \) of \( R \) of the form \( I = IS \cap R \) is finitely generated. Since \( R \subseteq S \) is an integral extension, every prime ideal of \( R \) has this form, and hence every prime ideal of \( R \) is finitely generated, and \( R \) is Noetherian. □

In the setting of Theorem 5.2, the Noetherian rings between \( R \) and \( S \) correspond by Theorem 4.3 to the \( S\)-submodules \( L \) of \( FK \) that contain \( K \) for which \( L/aL \) is a finitely generated \( S\)-module for all \( 0 \neq a \in S \cap \text{Ker} D \). Of course, when \( R \) has dimension 1, then every ring between \( R \) and \( S \) must be Noetherian, but in higher dimensions, the preceding observations make it easy to find non-Noetherian rings between \( R \) and \( S \) when \( K \) is finitely generated. By contrast, if \( K \) is not finitely generated, we see below that it can happen that there are no non-Noetherian rings between \( R \) and \( S \) when \( S \) has dimension \( > 1 \). But in the case where \( K \) is finitely generated, non-Noetherian rings must occur:

**Proposition 5.3.** Let \( S \) be a Noetherian domain of Krull dimension \( > 1 \), and suppose \( R \) is a subring of \( S \) strongly twisted by a finitely generated torsion-free \( S\)-module \( K \). Then there exists a non-Noetherian ring between \( R \) and \( S \).

**Proof.** Define \( T = S \cap D^{-1}(K_P) \). Then \( T \) is a ring with \( R \subseteq T \subseteq S \) and \( T \) is strongly twisted by \( K_P \). Suppose by way of contradiction that \( T \) is a Noetherian ring. Let \( 0 \neq c \in P \). Then by Theorem 5.2, \( K_P/cK_P \) is a finitely generated \( S\)-module. Moreover, since \( K \) is a finitely generated \( S\)-module and \( c \in P \), Nakayama’s Lemma implies that \( K_P/cK_P \) is a nonzero \( S\)-module. Let \( E \) be a nonzero cyclic \( S_{P}\)-submodule of \( K_P/cK_P \). Then since \( S \) is a Noetherian ring and \( K_P/cK_P \) is a finitely generated \( S\)-module, \( E \) is also a finitely generated \( S\)-module. Now \( E \cong S_P/(0 :_{S_P} E) \), and since \( E \neq 0 \), then \( S_P/PS_P \) is a homomorphic image of \( E \), and hence \( S_P/PS_P \) must also be a finitely generated \( S\)-module. But \( S_P/PS_P \) is isomorphic to the quotient field of the domain \( S/P \), so the finite generation of \( S_P/PS_P \) forces \( S_P = S \),
which in turn implies that $P$ is a maximal ideal of $S$, a contradiction. Therefore, $T$ is a non-Noetherian ring between $R$ and $S$.

The assumption that $K/aK$ is a finitely generated $S$-module for all $0 \neq a \in S \cap \ker D$ is weaker than simply requiring $K$ itself to be finitely generated. This subtlety leaves room for an interesting class of examples where although $K$ is not finitely generated, it produces Noetherian subrings $R$ of $S$. We illustrate this in Example 5.5, which uses the following observation.

**Lemma 5.4.** Let $V$ be a DVR with maximal ideal $\mathfrak{m}$, let $K$ be a torsion-free finite rank $V$-module and let $r = \dim_{V/\mathfrak{m}} K/\mathfrak{m}K$. Then $r \leq \text{rank}(K)$ and for all proper nonzero ideals $J$ of $V$, $K/JK$ is a free $V/J$-module of rank $r$.

**Proof.** Let $\mathcal{F}$ be the set of all $V$-submodules $H$ of $K$ such that $K/H$ is a nonzero torsion-free divisible $V$-module. Suppose first that $\mathcal{F}$ is empty. Since $K$ is torsion-free, we may view $K$ as contained in an $F$-vector space $L$ of the same rank, say $n$, as $K$. Write $L = Fe_1 \oplus \cdots \oplus Fe_n$, where $e_1, \ldots, e_n$ is a basis for $L$. Then since $\mathcal{F}$ is empty, the image of the projection map $\pi_i : K \to F$ of $K$ onto the $i$-th coordinate is not all of $F$. Thus since $V$ is a DVR, there exists $t \in V$ such that $\pi_i(K) \subseteq t^{-1}V$ for all $i = 1, \ldots, n$. Therefore, $K \subseteq t^{-1}Ve_1 \oplus \cdots \oplus t^{-1}Ve_n$, and hence $K$ is a submodule of a finitely generated free $V$-module. Since $V$ is a DVR, $K$ is a free $V$-module of rank $r$, and hence in the case where $\mathcal{F}$ is empty, the lemma is proved.

Next suppose that $\mathcal{F}$ is nonempty, and let $m$ be the maximum of the ranks of the torsion-free divisible $R$-modules $K/H$, where $H$ ranges over the members of $\mathcal{F}$. Choose $H \in \mathcal{F}$ such that $K/H$ has rank $m$. Let $J$ be a proper nonzero ideal of $V$, and write $J = vV$ for some $v \in V$. We claim that $K/vK \cong H/vH$, and that $H$ is a free $V$-module (necessarily of rank no more than the rank of $K$). Now, since $K/H$ is torsion-free, we have $vK \cap H = vH$, and hence there is an embedding $H/vH \to K/vK$ defined by $h + vH \mapsto h + vK$ for all $h \in K$. Moreover, since $K/H$ is a divisible $V$-module, $K = H + vK$, and hence the mapping is an isomorphism. If there does not exist a $V$-submodule $G$ of $H$ such that $H/G$ is a nonzero torsion-free divisible $V$-module, then, as above, $H$ is a free $V$-module of rank $\leq n$. In this case, since for any $0 \neq v \in \mathfrak{m}$, we have shown that $H/vH \cong K/vK$, it follows that $H/\mathfrak{m}H \cong K/\mathfrak{m}K$, so that since $H$ is a free $V$-module, $\text{rank}(H) = r$.

Thus the only case that remains to rule out is where there exists a $V$-submodule $G$ of $H$ such that $H/G$ is a nonzero torsion-free divisible $V$-module. Assuming the existence of such a $V$-submodule $G$ of $H$, there is an exact sequence of $V$-modules:

$$0 \to H/G \to K/G \to K/H \to 0.$$ 

Since $H/G$ is a divisible torsion-free $V$-module, this sequence splits, and hence $K/G \cong H/G \oplus K/H$. Since $H/G$ is a nonzero divisible torsion-free module and $K/H$ is a divisible torsion-free $V$-module of rank $m$, this means that $K/G$ is a divisible torsion-free $V$-module of rank $> m$, contradicting the choice of $H$, and the lemma is proved.

**Example 5.5.** Let $k$ be a field of positive characteristic that is separably generated and of infinite transcendence degree over a countable subfield, let $X_1, \ldots, X_d$ be indeterminates
for \( k \), and let \( S = k[X_1, \ldots, X_d]/(X_1, \ldots, X_d) \). Matsumura has shown that the generic formal fiber of a local domain essentially of finite type over a field has dimension one less than the domain [14, Theorem 1], and Heinzer, Rotthaus and Sally have shown that this condition on the generic formal fiber guarantees the existence of a birationally dominating DVR having residue field finite over the residue field of the base ring [7, Corollary 2.4]. Thus \( S \) is birationally dominated by a DVR \( V \) having residue field finite over the residue field \( k \) of \( S \). Let \( K \) be a nonzero torsion-free finite rank \( V \)-module that is not divisible. By Theorem 3.5, there exists a subring \( R \) of \( S \) strongly twisted by \( K \). Also, with \( N \) the maximal ideal of \( S \), the fact that \( V \) is a DVR implies that \( K/NK \) is a finitely generated \( V \)-module (Lemma 5.4). If \( a \in S \cap \text{Ker} \, D \), where \( D \) is the derivation that strongly twists \( R \), then since \( V \) is a DVR dominating \( S \) and \( K \) is a \( V \)-module, \( a^iK = N^jK \) for some \( i, j \geq 0 \). Since \( V \) is a DVR and the residue field of \( V \) is a finitely generated \( S \)-module, it follows that \( V/xV \) is a finitely generated \( S \)-module for all \( 0 \neq x \in V \). Thus since \( K/NK \) is a finitely generated \( V/NV \)-module and \( V/NV \) is a finitely generated \( S \)-module, then \( K/NK \) is a finitely generated \( S \)-module. Now since \( N \) is a finitely generated ideal of \( S \), it follows that \( K/NK = K/aK \) is a finitely generated \( S \)-module. Therefore, \( K/aK \) is a finitely generated \( S \)-module, and by Theorem 5.2, \( R \) is a Noetherian ring. Moreover, if the dimension of \( S \) is more than 1 and \( K = V \), then \( K \) is not a finitely generated \( S \)-module. Twisted subrings arising in this manner are considered later in this section and the next, as well as in [21].

The next propositions contrasts the sort of twisted subrings occurring in the example with those in Proposition 5.3.

**Proposition 5.6.** Let \( S \) be a local Noetherian domain with maximal ideal \( N \). Suppose that \( S \) is birationally dominated by a DVR \( V \) such that \( V = S + NV \), and that there exists a subring \( R \) of \( S \) that is strongly twisted by a torsion-free finite rank \( V \)-module \( K \). Then every ring between \( R \) and \( S \) is a local Noetherian ring that is strongly twisted by some \( V \)-module \( L \) with \( K \subseteq L \subseteq FK \).

**Proof.** Let \( T \) be a ring such that \( R \subseteq T \subseteq S \), and let \( F \) denote the quotient field of \( S \). Then by Theorem 4.3, there exists an \( S \)-module \( L \) such that \( K \subseteq L \subseteq FK \) and \( T = S \cap D^{-1}(L) \). We claim that \( L \) is in fact a \( V \)-module (not just an \( S \)-module). Let \( E \) be a free \( V \)-submodule of \( L \) having the same rank as \( L \), and let \( e_1, \ldots, e_n \) be a basis for \( E \), so that \( E = Ve_1 \oplus \cdots \oplus Ve_n \). Then since \( E \) and \( L \) have the same rank, we may view \( L \) as an \( S \)-submodule of \( Fe_1 \oplus \cdots \oplus Fe_n \). To show that \( L \) is a \( V \)-module, it suffices to show that for each \( v \in V \) and \( y \in L \), \( vy \in E + Sy \). Let \( v \in V \) and \( y \in L \), and write \( y = x_1e_1 + \cdots + x_ne_n \), where \( x_1, \ldots, x_n \in F \). Let \( I = (V :_F x_1) \cap \cdots \cap (V :_F x_n) \). Since \( V \) is an overring of \( S \), there exists \( 0 \neq t \in I \cap N \). Moreover, since \( V = S + NV \), it follows that \( V = S + N^jV \) for all \( j > 0 \). Hence, since \( V \) is a DVR and \( 0 \neq t \in N \), it must be that \( V = S + tV \). Thus there exists \( s \in S \) such that \( v - s \in tV \subseteq I \). Consequently, \( vx_i - sx_i \in V \) for all \( i = 1, 2, \ldots, n \). Returning now to the claim that \( vy \in E + Sy \), observe that \( vy - sy = (vx_1 - sx_1)e_1 + \cdots + (vx_n - sx_n)e_n \in E \), so the claim is proved. Therefore, \( L \) is a \( V \)-module. By Lemma 5.4, \( L/sL \) is a finitely generated \( V \)-module for all \( 0 \neq s \in S \). Since \( V = S + NV \), then \( V/NV \) is a cyclic \( S \)-module. This, along with the fact that \( V \) is
a DVR, implies that $V/sV$ is a finitely generated $S$-module for all $0 \neq s \in S$. Therefore, $L/sL$ is a finitely generated $S$-module for all $0 \neq s \in S$, and hence by Theorem 5.2, $T$ is a local Noetherian domain.

The characterization of when strongly twisted subrings are Noetherian in Theorem 5.2 depends on the subring being strongly twisted rather than twisted along $C$. However, there is a specific circumstance when being twisted along a multiplicatively closed subset suffices to give Noetherianness. The idea behind the following theorem originates with Ferrand and Raynaud [5, Proposition 3.3] and the version we give here is a generalization of a result of Goodearl and Lenagan [6, Proposition 7]. Our formulation and approach to the theorem are different, but ultimately, as we point out in the proof, a key step depends on an argument from [5]. Also, unlike the strongly twisted Noetherian rings produced using Theorems 3.5 and 5.2, the next theorem produces examples in arbitrary characteristic, as Corollary 5.8 illustrates.

**Theorem 5.7.** Let $(S, N)$ be a two-dimensional local Noetherian UFD that is birationally dominated by a DVR $V$ having the same residue field as $S$, and such that there is $t \in N$ with $tV$ the maximal ideal of $V$. If $R$ is a subring of $S$ that is twisted along $C = \{t^i : i > 0\}$ by an $S$-submodule $K$ of a finitely generated free $\hat{V}$-module $K'\! /\! K$ a $C$-torsion-free $S$-module, then $R$ is a two-dimensional local Noetherian ring such that for every height 1 prime ideal $P$ of $R$, $R_P = S_Q$ for some height 1 prime ideal $Q$ of $S$.

**Proof.** Let $K$ be the $S$-submodule of $\hat{V}$ by which $R$ is twisted, and let $D$ be the derivation that twists $R$. Let $A = S \cap \text{Ker } D$, and let $Q$ be the quotient field of $A$. Since by Theorem 4.1, $R \subseteq S$ is an integral extension and $S$ is local, $R$ has a unique maximal ideal $M := N \cap R$. To prove that $R$ is Noetherian, we show that every prime ideal of $R$ is finitely generated. First we claim that every ideal of $R$ containing a power of $t$ is finitely generated. Indeed, for each $i > 0$, we can argue as in Example 5.5 that $V/t^iV$ is a finitely generated $S$-module. Now since $K'/K$ is $C$-torsion-free, we have for each $i > 0$, $t^iK' \cap K = t^iK$, and hence $K/t^iK \cong (K + t^iK')/t^iK' \subseteq K'/t^iK'$. But $K'/t^iK'$ is a finitely generated module over $V/t^iV$, and since $V = S + tV$, it follows that $K'/t^iK'$ is a finitely generated $S$-module. Hence $K/t^iK$, since it is isomorphic to an $S$-submodule of $K'/t^iK'$, is also a finitely generated $S$-module. Now by Theorem 4.6, $R/t^iR$ is isomorphic as a ring to $S/t^iS \ast K/t^iK$, so that since $S/t^iS$ is a Noetherian ring and $K/t^iK$ is a finitely generated $S$-module, $R/t^iR$ is a Noetherian ring. Therefore, it follows that every ideal of $R$ containing a power of $t$ is finitely generated. In particular, $M$ is a finitely generated ideal of $R$.

Since $R \subseteq S$ is an integral extension, $R$ has Krull dimension 2, and hence to prove that $R$ is Noetherian, all that is left to show is that each height 1 prime ideal of $R$ is finitely generated. In fact, if $P$ is a height 1 prime ideal of $R$, then since $R \subseteq S$ is integral, there is a height 1 prime ideal of $S$ lying over $P$. Thus since $S$ is an UFD, there exists $f \in S$ such that $P = fS \cap R$, and so to complete the proof of the theorem it is enough to show that $fS \cap R$ is a finitely generated ideal of $R$. Our proof of this fact is adapted from Ferrand and Raynaud [5, Proposition 3.3].
Define $I = \{ s \in S : fs \in R \}$. Then $I$ is a fractional ideal of $R$ such that $fS \cap R = fI$. Thus to prove that $fS \cap R$ is a finitely generated ideal of $R$, it suffices to show that $I$ is a finitely generated fractional ideal of $R$. Now $D(f) \in K_C$, so there exists $c \in C$ such that $D(cf) = cD(f) \in K$, and hence $cf \in R$. Consequently, $c \in I$, and so if there exists $b \in C$ such that $bI \subseteq R$, then $bc \in bI \subseteq R$, so that by what we have established above, since $bI$ contains an element of $C$, the ideal $bI$, and hence the fractional ideal $I$, is finitely generated. Thus it remains to show that there exists $b \in C$ such that $bI \subseteq R$.

Let $e_1, \ldots, e_n$ be a basis for the free $\widehat{V}$-module $K'$, so that $K \subseteq \widehat{V}e_1 \oplus \cdots \oplus \widehat{V}e_n$. For each $i = 1, \ldots, n$, let $\pi_i$ be the projection of $K$ onto the $i$-th coordinate of this direct sum; i.e., $\pi_i(v_1e_1 + \cdots + v_ne_n) = v_i$ for all $v_1, \ldots, v_n \in \widehat{V}$. Let $s \in I$. Then $sf \in R$, so that $sD(f) + fD(s) = D(sf) \in K$. Since $D(f) \in K_C$, there exists $c \in C$ such that $cD(f) \in K$. Thus since $csD(f) + cfD(s) \in K$ and $cD(f) \in K$, we have that $cD(s) \in K$. Since the choice of $s \in I$ was arbitrary, we have for each $i = 1, \ldots, n$ and $s \in I$ that $\pi_i(D(s)) \in (cf)^{-1}\widehat{V}$. Since $t\widehat{V}$ is the maximal ideal of $\widehat{V}$, there exists $b \in C$ such that $b(cf)^{-1} \in \widehat{V}$. Consequently, for each $i = 1, \ldots, n$ and $s \in I$, $\pi_i(D(bs)) = b\pi_i(D(s)) \in b(cf)^{-1}\widehat{V} \subseteq \widehat{V}$, and hence for all $s \in I$, $D(bs) \in K' \cap K_C = K$, where this last assertion follows from the fact that $K'/K$ is $C$-torsion-free. This then implies that $bI \subseteq S \cap D^{-1}(K) = R$, which proves the claim, and hence verifies that $fS \cap R$ is a finitely generated ideal of $R$. We conclude that $R$ is a Noetherian domain.

To prove the final assertion, let $P$ be a height 1 prime ideal of $R$. Since $K'/K$ is $C$-torsion-free, $\bar{K} = K' \cap K_C$, so that $K_P = \bar{K}_P \cap (K_C)_P$. Now $R_P \not\subseteq V$, for since $R_P$ has dimension 1, the maximal ideal of any overring of $R_P$ other than $F$ must contract in $R$ to $P$, yet the maximal ideal of $V$ contracts to $M$. Thus $R_P \not\subseteq V$, and since $V$ is a DVR, it must be that $V_P$ is the quotient field of $V$, and hence $\bar{K}_P$ is a vector space over the quotient field of $\widehat{V}$. Therefore, $K_P = (K_C)_P$. By Lemma 4.4, $K_C/K$ is isomorphic as an $R$-module to $S/R$, so it follows that $S \subseteq R_P$, and hence $R_P$ is a localization of $S$ at a height 1 prime ideal of $S$.

With this theorem and the existence result, Lemma 3.4, we give an example in characteristic 0 of a Noetherian twisted subring of dimension > 1:

**Corollary 5.8.** Let $k$ be a field of characteristic 0 that has infinite transcendence degree over its prime subfield, let $X$ and $Y$ be indeterminates for $k$, and let $S = k[X,Y]_{(X,Y)}$. Then for each $n \geq 1$, there exists an analytically ramified local Noetherian ring $R$ having normalization $S$, embedding dimension $2 + n$, multiplicity 1, and an isolated singularity.

**Proof.** Let $F$ denote the quotient field of $S$. Since $k((X))$ has infinite transcendence degree over $k$, it follows that $S$ embeds into $k[[X]]$ in such a way that the image of $(X,Y)$ is contained in $Xk[[X]]$; see [28, p. 220]. Viewing $S$ as a subring of $k[[X]]$, let $V = k[[X]] \cap k(X,Y)$. Then $V$ is a DVR with residue field $k$ and maximal ideal generated by $X$. Let $K$ be a rank $n$ free $V$-module. Then by Lemma 3.4, there exists a ring $A$ such that $A \subseteq S$ is a $C$-analytic extension with $C = \{ X^i : i \geq 1 \}$, and there exists an $A$-linear derivation $D : F \to FK$ such that $D(S) = FK$. Let $R = D^{-1}(K) \cap S$. Then $R$ is twisted by $K$. 


along C, and since the cokernel of the canonical mapping $K \rightarrow \hat{V} \otimes_V K$ is $C$-torsion-free, then by Theorem 5.7, $R$ is a Noetherian ring. By Theorem 4.1, $R \subseteq S$ is integral, so that also $R$ is a local ring. By Corollary 4.7, the $M$-adic completion of $R$ is $\hat{R} = \hat{S} \star \hat{K}$. Since $N/N^2$ has dimension 2 as an $S/N$-vector space, while $\hat{K}/N\hat{K}$ has dimension $n$, it follows that $\hat{S} \star \hat{K}$ has embedding dimension $2 + n$. Similar calculations show that the multiplicity of $\hat{S} \star \hat{K}$ is 1; we omit these calculations because in [21] we describe the Hilbert polynomials and multiplicity of twisted subrings in detail, and from this description the multiplicity in the present context can easily be deduced. Therefore, since embedding dimension and multiplicity of twisted subrings in detail, and from this description the multiplicity of $\hat{S} \star \hat{K}$ is 1. By Theorem 5.7, each localization of $R$ at a height 1 prime ideal is a DVR, so $R$ has an isolated singularity.

As another application of the theorem, we reframe an example due to Goodearl and Lenagan.

**Example 5.9.** (Goodearl and Lenagan [6, p. 494]) Let $k$ be a field, and let $U = k[[x]]$, with $x$ an indeterminate for $k$. Choose $y, z \in xU$ such that $y$ and $z$ are algebraically independent over $k(x)$ (see [28, p. 220] for a constructive argument that there are infinitely many such choices for $y$ and $z$). Let $A = k[x, y]_{(x, y)}$, $W = k(x, z) \cap U$, $S = W[y]_{(x, y)}$, and $F = k(x, y, z)$. The definition of $S$ makes sense, since $W$ is a DVR with quotient field $k(x, z)$ whose maximal ideal is $k(x, z) \cap xU$, and since $xU$ is the maximal ideal of $U$, it follows that $xW$ is the maximal ideal of $W$. Thus $(x, y)W[y]$ is a maximal ideal of $W[y]$. Moreover, this shows also that $S$ is a regular local ring of Krull dimension 2 with quotient field $F = k(x, y, z)$. With $C = \{x^i : i > 0\}$, the extension $A \subseteq S$ is $C$-analytic. Now let $D = \frac{\partial}{\partial x}$, and note that $D$ is $A$-linear. Let $L$ be the $S_C$-submodule of $k((x))$ generated by $D(S)$. Define $K = L \cap U$ and $R = S \cap D^{-1}(K)$. Then since $U$ is a DVR with maximal ideal $xU$, it follows that $K_C = L \cap UC = L$. Therefore, $R$ is twisted by $K$ along $C$. Moreover, since $K_C = L$, we have $U \cap K_C = U \cap L = K$, and hence $U/K$ is a $C$-torsion-free $S$-module. Thus by Theorem 5.7, $R$ is a local Noetherian ring having the properties of the theorem.

In the example, $k[x, y, z] \subseteq R \subseteq S \subseteq k(x, y, z)$, and since $x, y$ and $z$ are algebraically independent, we can work backwards from any field of the form $k(X, Y, Z)$ and view $k[X, Y, Z]$ as embedded in $k[[X]]$, with $Y, Z \in Xk[[X]]$. Then we obtain a ring $R$ as in the example:

**Corollary 5.10.** Let $k$ be a field, and let $X, Y, Z$ be indeterminates for $k$. Then there exists a two-dimensional analytically ramified local Noetherian domain between $k[X, Y, Z]$ and $k(X, Y, Z)$ having an isolated singularity and normalization a regular local ring. □

6. **COHEN-MACAULAY RINGS**

We consider now when strongly twisted local subrings are Cohen-Macaulay, Gorenstein, a complete intersection or a hypersurface. Since these properties are invariant under completion, the following theorem is a consequence of well-known facts applied to the idealization $\hat{R} \cong \hat{S} \star \hat{K}$ given by Corollary 4.7.
**Theorem 6.1.** Let $S$ be a quasilocal domain, and let $R$ be a subring of $S$ strongly twisted by a finitely generated torsion-free $S$-module $K$. Then the following statements hold for $R$.

1. $R$ is a Cohen-Macaulay ring if and only if $S$ is a Cohen-Macaulay ring and $K$ is a finitely generated maximal Cohen-Macaulay module.

2. $R$ is a Gorenstein ring if and only if $S$ is a Cohen-Macaulay ring that admits a canonical module $\omega_S$ and $K \cong \omega_S$.

3. If $S$ is a Gorenstein ring, then for $K = S$, the ring $R$ is a Gorenstein ring.

4. $R$ is a complete intersection if and only if $S$ is a complete intersection and $K \cong S$.

5. $R$ is a hypersurface if and only if $S$ is a regular local ring and $K \cong S$.

**Proof.** First observe that since $K$ is finitely generated, then by Theorem 5.2, $R$ is Noetherian if and only if $S$ is Noetherian.

(1) Since a local ring is Cohen-Macaulay if and only if its completion is Cohen-Macaulay, it is enough to determine when $\hat{R}$ is Cohen-Macaulay [3, Corollary 2.1.8, p. 60]. But by Corollary 4.7, $\hat{R} \cong \hat{S} \ast \hat{K}$, so this is easy to do. Indeed, properties of idealizations show that $\hat{R}$ is Cohen-Macaulay if and only if $\hat{S}$ is Cohen-Macaulay and $\hat{K}$ is a maximal Cohen-Macaulay $S$-module (meaning that the depth of $\hat{K}$, its dimension and the dimension of $\hat{S}$ are all the same); see [2, Corollary 4.14] or [26, p. 52]. Since $K$ is a finitely generated torsion-free $S$-module, $K$ is a maximal Cohen-Macaulay $S$-module if and only if $\hat{K}$ is a maximal Cohen-Macaulay module [3, Corollary 2.1.8, p. 60].

(2) A Cohen-Macaulay ring admits a canonical module if and only if it is the homomorphic image of a Gorenstein ring [3, Theorem 3.3.6]. We collect several other facts: (a) a local ring is Gorenstein if and only if its completion is Gorenstein [3, Proposition 3.1.19, p. 95]; (b) $\hat{\omega}_S = \omega_S$ [3, Theorem 3.3.5, p. 110]; (c) since $K$ and $\omega_S$ are finitely generated torsion-free modules, $\hat{K} \cong \omega_S$ if and only if $K \cong \omega_S$ (this can be deduced from Example 7.5(i) and 8.11 of [13]); and (d) when $A$ is a local Noetherian ring and $M$ is an $A$-module, then $A \ast M$ is a Gorenstein ring if and only if $A$ admits a canonical module $\omega_A$ and $M \cong \omega_A$ (apply [22, Theorem 7] and [26, p. 52]). Thus, combining these observations with the fact that $\hat{R} \cong \hat{S} \ast \hat{K}$, we have that $R$ is Gorenstein if and only if $\hat{S} \ast \hat{K}$ is Gorenstein; if and only if $S$ admits a canonical module and $K \cong \omega_S$.

(3) The ring $S$ is Gorenstein if and only if $S$ is Cohen-Macaulay and $\omega_S \cong S$ [3, Theorem 3.3.7, p. 112]. Now apply (2).

(4) Given a local ring $A$ and $A$-module $M$, the local ring $A \ast M$ is a complete intersection if and only if $A$ is a complete intersection and $M \cong A$ [26, p. 52]. Thus since a local ring is clearly a complete intersection if and only if its completion is a complete intersection, we may use the fact that $\hat{R} \cong \hat{S} \ast \hat{K}$ to obtain (4).

(5) Whether $R$ is a hypersurface (meaning that $\hat{R}$ is isomorphic to regular local ring modulo a principal ideal) is deduced from [26, p. 52]: With $A$ a local ring and $M$ an $A$-module, the ring $A \ast M$ is a hypersurface if and only if $A$ is a regular local ring and $M \cong A$. □
Let one proper subring of $R$ and hence there is a complete intersection between $R$ and $S$ if and only if rank $K = 1$.

**Proof.** If $T$ is a ring between $R$ and $S$, then by Theorem 4.3, $T$ is strongly twisted by a unique $S$-module $L$ with $K \subseteq L \subseteq FK$. By Theorem 6.1, $T$ is a Cohen-Macaulay ring if and only if $L$ is a maximal Cohen-Macaulay $S$-module. Moreover, by the theorem, $T$ is a complete intersection if and only if $L$ is a rank one free $S$-module. If rank $K = 1$, then since $K$ is a finitely generated $S$-module, there is a rank one free $S$-module between $K$ and $FK$, and hence there is a complete intersection between $R$ and $S$. \(\square\)

Next we consider the case where $S$ is a local Noetherian domain and there exists at least one proper subring of $S$ that is strongly twisted by an $S$-module. From Proposition 3.2 this then leads to an abundance of Noetherian subrings strongly twisted by $S$-modules, namely one for each fractional ideal of $S$.

**Theorem 6.3.** Let $S$ be a local Noetherian domain having a strongly twisted subring, and let $D : F \to F$ be the derivation given by Proposition 3.2(2). For each fractional ideal $I$ of $S$, let $R_I = S \cap D^{-1}(I)$, so that $R_I$ is the subring of $S$ strongly twisted by $I$. Then:

1. The set of all $R_I$ is convex: If $I$ and $J$ are fractional ideals of $S$, and $T$ is a ring with $R_I \subseteq T \subseteq R_J$, then there exists a fractional ideal $K$ of $T$ such that $T = RK$.
2. If $S$ has Krull dimension $> 1$, then for each fractional ideal $I$ of $S$, there is a non-Noetherian ring between $R_I$ and $S$.
3. The set of $R_I$ forms a lattice (without top or bottom element): For each pair of fractional ideals $I$ and $J$ of $S$, $R_{I+J} = R_I + R_J$; $R_{I \cap J} = R_I \cap R_J$; and $I \subseteq J$ if and only if $R_I \subseteq R_J$.
4. The ring $R_I$ is a Cohen-Macaulay ring if and only if $S$ is Cohen-Macaulay and the fractional ideal $I$ is a maximal Cohen-Macaulay $S$-module. Thus when $S$ is Cohen-Macaulay, then for each $N$-primary ideal $I$ of $S$, $R_I$ is a Cohen-Macaulay ring.
5. Suppose that $S$ is a Cohen-Macaulay ring that admits a canonical module $\omega_S$ (which is necessarily isomorphic to an ideal of $S$). Then for each fractional ideal $I$ of $S$, the ring $R_I$ is a Gorenstein ring if and only if $I \cong \omega_S$.
6. The ring $R_I$ is a complete intersection (resp., hypersurface) if and only if $S$ is a complete intersection (resp., regular local ring) and $I$ is a principal fractional ideal of $S$.

**Proof.** (1) Let $K$ be the $S$-submodule of $F$ generated by $D(T)$. Then by Theorem 4.3, $T = D^{-1}(K) \cap S$. Also, $D(R_I) \subseteq D(T) \subseteq D(R_J)$, and again by Theorem 4.3, $I$ is generated
as an $S$-module by $D(R_I)$, while $J$ is generated as an $S$-module by $D(R_J)$. Thus $I \subseteq K \subseteq J$, so that $K$ is a fractional ideal of $S$ with $T = R_K$.

(2) Apply Proposition 5.3.

(3) Since $R_I = D^{-1}(I) \cap S$ and, as noted in the proof of (1), $I$ is the $S$-submodule of $F$ generated by $D(R_I)$, it follows that $I \subseteq J$ if and only if $R_I \subseteq R_J$. Thus $R_I + R_J \subseteq R_{I+J}$. Also, since $R_I + R_J$ is a ring (Theorem 4.1), we have by (2) that $R_I + R_J = R_K$ for some fractional ideal $K$ with $I + J \subseteq K$. On the other hand, since $R_K \subseteq R_{I+J}$, it must be that $K \subseteq I + J$, and hence $K = I + J$.

(4), (5) and (6): Apply Theorem 6.1. □

We return now to the case considered at the end of Section 5 in which $R$ is strongly twisted by a $V$-module, where $V$ is a DVR overring of $S$. We see below that this case never produces Cohen-Macaulay rings, except in dimension 1. First we note that the fact that $K$ is a $V$-module has an interesting consequence for $\text{Spec}(R)$. Everywhere off the closed point $\{M\}$ of $\text{Spec}(R)$, the local rings of the points of $\text{Spec}(R)$ and $\text{Spec}(S)$ are the same.

**Proposition 6.4.** Let $S$ be a local Noetherian domain, and suppose that there exists a DVR $V$ birationally dominating $S$ and having residue field finite over $S$. If $K$ is a nonzero torsion-free finite rank $V$-module and $R$ is a subring of $S$ strongly twisted by $K$, then $R$ is a local Noetherian domain and for each nonmaximal prime ideal $P$ of $R$, $R_P = S_P'$, where $P'$ is the unique prime ideal of $S$ lying over $R$.

**Proof.** An argument such as that in Example 5.5 shows that $R$ is a local Noetherian domain. Let $P$ be a nonmaximal prime ideal of $R$. Then since the maximal ideal of $V$ contracts to $M$ and $V$ is a DVR, it must be that that $V_P = F$, and hence since $K$ is a $V$-module, $K_P = FK$. Consequently, by Lemma 4.4, $R_P = S_P$. Let $P'$ be a prime ideal of $S$ lying over $P$. Since $P' \cap S = P$, we have that $P'S_P \neq S_P$, and hence $P'S_P = PS_P$. To see that this implies $S_{P'} \subseteq S_P$, let $x \in S_{P'}$. Then $S \cap x^{-1}S P'$. If $x \notin S_P$, then $R \cap x^{-1}S \subseteq P$. But then $S \cap x^{-1}S \subseteq R_P \cap x^{-1}S_P \subseteq PR_P = P'S_P$, so that $S \cap x^{-1}S \subseteq P'S_P \cap S = P'$, a contradiction that implies $S_{P'} = S_P = R_P$. Thus the proposition is proved. □

**Corollary 6.5.** With $R$ and $S$ as in the proposition, if $S$ is a regular local ring, then $R$ has an isolated singularity. □

It follows from the proposition that if $S$ has Krull dimension $> 1$ and $K \neq FK$ (so that $R \subseteq S$), then $R$ is not Cohen-Macaulay (compare to Theorem 6.1). Certainly if it was, then $S$ could not be integrally closed, since unmixedness would force Serre’s condition $S_2$ on $R$, which, along with the regularity condition $R_1$ on $S$, and hence $R$, would imply $R$ is integrally closed, contradicting the fact that $R \subseteq S$ is integral. But regardless of whether $S$ is integrally closed, unmixedness fails in a strong way for $R$ when $S$ has Krull dimension $> 1$:

**Proposition 6.6.** With $R$, $K$ and $S$ as in Proposition 6.4 and $K \neq FK$, the maximal ideal $M$ of $R$ is the associated prime of a nonzero principal ideal.
Proof. It suffices to exhibit an element \( s \in (R :_FR M) \) that is not in \( R \), for then \( M = R \cap s^{-1}R \) and the proposition follows. Let \( t \in M \) such that \( tV = MV \). Observe that \( tK \neq K \), for otherwise since \( V \) is a DVR and \( tV \neq V \), it follows that \( K \) is a divisible \( V \)-module and hence \( K = FK \), contrary to assumption. Therefore, \( K \subseteq t^{-1}K \subseteq FK \), and since \( R = D^{-1}(K) \cap S \) and by Theorem 4.3, \( R \neq D^{-1}(t^{-1}K) \cap S \), there exists \( s \in S \) such that \( D(s) \in t^{-1}K \setminus K \). (Here, \( D \) is the derivation that twists \( R \).) Now, since \( D \) is a derivation, \( K \) is a \( V \)-module and \( MV = tV \), we have

\[
D(sM) = sD(M) + MD(s) \subseteq K + Mt^{-1}K = K.
\]

Thus \( sM \subseteq D^{-1}(K) \cap S = R \), and we have \( s \in (R :_FR M) \setminus R \), as claimed. \qed

If \( A \) is a local Cohen-Macaulay ring of Krull dimension \( d \), then an inequality due to Abhyankar in [1] places a lower bound on the multiplicity \( e(A) \) of \( A \):

\[
e(A) \geq \text{emb.dim } A - d + 1.
\]

To contrast this with the non-Cohen-Macaulay case, Abhyankar constructs in [1] for each pair of integers \( n > d > 1 \) a local ring of embedding dimension \( n \), Krull dimension \( d \) and multiplicity 2. Example 5.5 can be used to accomplish something similar:

Example 6.7. Let \( n > d > 1 \), and let \( S = k[X_1, \ldots, X_d](X_1, \ldots, X_d) \) and \( V \) be as in Example 5.5. Let \( K \) be a free \( V \)-module of rank \( n \), and let \( R \) be the subring of \( S \) that is strongly twisted by \( K \). Then, as in the example, \( R \) is a local Noetherian domain. Since \( R \subseteq S \) is an integral extension, \( R \) has Krull dimension \( d \). Moreover, as in Corollary 5.8, the fact that \( \hat{R} \) is isomorphic as a ring to \( \hat{S} \hat{\otimes} \hat{K} \) implies that \( R \) has multiplicity 1 and embedding dimension \( d + n \). Also, by Corollary 6.5, \( R \) has an isolated singularity, and by Proposition 6.6, the maximal ideal of \( R \) is associated to a principal ideal of \( R \). \qed

7. Non-Noetherian rings

Although our focus is mainly on the Noetherian case, we make in this section a few remarks on twisted subrings of not-necessarily-Noetherian domains. Specifically, we characterize the twisted subrings of \( S \), where \( S \) is allowed to be either a Prüfer domain, a Dedekind domain or a Krull domain, and we see that various degrees of “stability” are necessitated by such assumptions on \( S \). Following Lipman [11] and Sally and Vasconcelos [24], an ideal \( I \) of a domain \( R \) is stable if \( I \) is projective over its ring of endomorphisms. In case \( R \) is quasilocal, \( I \) is stable if and only if \( I^2 = iI \) for some \( i \in I \) [17, Lemma 3.1]. A domain is finitely stable if every nonzero finitely generated ideal is stable; it is stable if every ideal is stable. We use the following two facts; the first is due to Rush [23, Theorem 2.3], and the second, which can be found in [18, Corollary 2.5], is based on similar ideas.

(a) If \( R \) is a finitely stable domain, then \( R \subseteq \overline{R} \) is a quadratic extension and \( \overline{R} \) is a Prüfer domain. Conversely, if \( R \subseteq S \) is a quadratic extension and \( \overline{R} \) is a Prüfer domain such that at most two maximal ideals of \( \overline{R} \) lie over each maximal ideal of \( R \), then \( R \) is a finitely stable domain.
(b) A domain $R$ is one-dimensional and stable if and only if $R \subseteq \overline{R}$ is a quadratic extension; $\overline{R}$ is a Dedekind domain; and there are at most two maximal ideals of $\overline{R}$ lying over each maximal ideal of $R$.

**Theorem 7.1.** Let $S$ be an integrally closed domain, and let $C$ be a multiplicatively closed subset of $S$. Suppose that $R$ is a subring of $S$ that is twisted along $C$ by some $C$-torsion-free $S$-module. Then:

1. $S$ is a Prüfer domain if and only if $R$ is a finitely stable domain.
2. $S$ is a Dedekind domain if and only if $R$ is a stable domain of Krull dimension 1.
3. $S$ is a Krull domain if and only if $S$ is the intersection of its localizations at height 1 prime ideals; the set of height one prime ideals of $R$ has finite character; and for each such prime ideal $P$, $R_P$ is a stable domain.

**Proof.** First note that by Theorem 4.1, $R \subseteq S$ is a quadratic extension and $R$ and $S$ share the same quotient field. By Theorem 4.2 every prime ideal of $R$ has a unique prime ideal of $S$ lying over it. Moreover, since $R \subseteq S$ is integral and $S$ is integrally closed, the ring $S$ is the integral closure of $R$ in its quotient field. Thus to prove (1) we may apply (a) above to obtain that $R$ is a finitely stable domain if and only if $S$ is a Prüfer domain. Moreover, by (b), $S$ is a Dedekind domain if and only if $R$ is a stable domain of Krull dimension 1, and this proves (2).

To prove (3), observe first that since each height 1 prime ideal of $R$ has a unique height 1 prime ideal of $S$ lying over it, it follows that the set of height 1 prime ideals of $S$ has finite character if and only if the set of height 1 prime ideals of $R$ has finite character. Suppose that $S$ is a Krull domain, and let $P$ be a height 1 prime ideal of $S$. We claim that $S_{P \cap R} = S_P$. Indeed, since $S$ is a Krull domain, $S = \bigcap_Q S_Q$, where $Q$ ranges over the height 1 prime ideals of $S$. Since this intersection has finite character, it follows that $S_{P \cap R} = \bigcap_Q (S_Q)_{P \cap R}$. Since $S_Q$ is a DVR and there is a unique prime ideal of $S$ lying over $P \cap R$, then $(S_Q)_{P \cap R}$ is the quotient field of $S$ for all $Q \neq P$. Thus $S_{P \cap R} = S_P$, and from the fact that $R \subseteq S$ is a quadratic extension, we obtain that for each height 1 prime ideal $P$ of $S$, $R_{P \cap R} \subseteq S_P$ is a quadratic extension. Therefore, by (b) above, $R_{P \cap R}$ is a stable domain.

Conversely, suppose that $S$ is the intersection of its localizations at height 1 prime ideals; the set of height 1 prime ideals of $R$ has finite character; and for each such prime $P$, $R_P$ is a stable domain. Then, as we have already noted, the set of height 1 prime ideals of $S$ has finite character, so it remains to show that $S_P$ is a DVR for each prime ideal $P$ of $S$. Now by assumption $R_{P \cap R}$ is a stable domain of Krull dimension 1, and hence by (b), the integral closure of $R_{P \cap R}$ in its quotient field is a Dedekind domain. But the quasilocal domain $S_P$, as an integrally closed overring of $R_{P \cap R}$, must contain this Dedekind domain and hence $S_P$ must be a DVR. Thus $S$ is a Krull domain. \qed

As an example of how to apply Theorem 7.1, as well as Theorem 3.5 (the theorem on the existence of strongly twisted subrings), we build in Corollary 7.4 a one-dimensional stable domain that has infinitely many maximal ideals $M_n$, each of which has a generating set.
of prescribed size. The existence of such rings is a consequence of a general fact, which we establish in Proposition 7.3, regarding Dedekind domains that have a strongly twisted subring. The proposition relies on the following technical observation.

**Lemma 7.2.** Let $S$ be a domain with quotient field $F$, and let $K$ be a nonzero torsion-free $S$-module. If $R$ is a subring of $S$ that is strongly twisted by $K$, then for each nonzero prime ideal $P$ of $S$, the subring $R_{P \cap R}$ of $S_P$ is strongly twisted by $K_P$.

**Proof.** Let $D$ be the derivation that twists $R$, let $A = S \cap \text{Ker} D$, and note that by Corollary 2.6, $A \subseteq S$ is a strongly analytic extension. First we show that $S_P = S_{P \cap A}$, where the second localization is with respect to $A \setminus (P \cap A)$. We need only verify that $S_P \subseteq S_{P \cap A}$, since the reverse inclusion is clear. In fact, it suffices to verify that $s^{-1} \in S_{P \cap A}$ for each $s \in S \setminus P$. To this end, let $s \in S \setminus P$. Then $s^{-1} \in S_{P \cap A}$ if and only if $A \cap sS \not\subseteq P$. If $A \cap sS \subseteq P$, then applying Proposition 2.4(a) we have $sS = (A \cap sS)S \subseteq P$, contrary to the choice of $s$. Hence $A \cap sS \not\subseteq P$, and the claim that $S_P = A_{P \cap A}$ follows.

Next we claim that $R_{P \cap R} = D^{-1}(K_P) \cap S_P$. Let $r \in R$ and $b \in R \setminus P$. Then since $R = D^{-1}(K) \cap S$, we have $D(r/b) = (bD(r) - rD(b))/b^2 \in K_P$, so that $D(R_{P \cap R}) \subseteq K_P$. Thus $R_{P \cap R} \subseteq D^{-1}(K_P) \cap S_P$. To see that the reverse inclusion holds, suppose that $x \in S_P$ such that $D(x) \in K_P$. By our above argument, $S_P = S_{P \cap A}$, so there exist $s \in S$ and $c \in A \setminus (P \cap A)$ such that $x = \frac{c}{s}$. By assumption $D(\frac{c}{s}) \in K_P$, and since $D$ is $A$-linear, we have $\frac{1}{c}D(s) = D(\frac{c}{s}) \in K_P$. Thus, since $c \not\in P$, we conclude $D(s) \in K$. Since $R = S \cap D^{-1}(K)$, this implies that $s \in R$, and hence $x = \frac{s}{c} \in R_{P \cap R}$. This proves the claim that $R_{P \cap R} = D^{-1}(K_P) \cap S_P$.

Finally we claim that $R_{P \cap R}$ is strongly twisted by $K_P$. Indeed, we have verified that $R_{P \cap R} = D^{-1}(K_P) \cap S_P$. Also, since $R$ is strongly twisted by $K$, $D(F)$ generates $FK$ as an $F$-vector space. Moreover, $S \subseteq \text{Ker} D + sS$ for all $0 \neq s \in S$, and as noted above $S_P = S_{P \cap A}$, so since $A_{P \cap A} \subseteq \text{Ker} D$ we have that $S_P \subseteq \text{Ker} D + sS_P$ for all $0 \neq s \in S$. Thus $R_{P \cap R}$ is strongly twisted by $K_P$. \hfill $\square$

**Proposition 7.3.** Suppose that $S$ is a Dedekind domain with quotient field $F$ having countably many maximal ideals, and that $S$ has a subring that is strongly twisted by a torsion-free $S$-module of infinite rank. If $\{e_n\}_{n=1}^{\infty}$ is a sequence for which each $e_n \in \mathbb{N} \cup \{\infty\}$, then there exists a subring $R$ of $S$ having countably many maximal ideals $M_1, M_2, \ldots$ such that:

1. $R$ is a stable domain having normalization $S$ and quotient field $F$.
2. For each $n > 0$, $M_n$ is minimally generated by $e_n + 1$ elements.
3. If each $e_n$ is finite, then $R$ is a Noetherian domain.

**Proof.** List the maximal ideals of $S$ as $N_1, N_2, \ldots$, and for each $t \geq 1$, define $K_t = \bigoplus_{i=1}^{\infty} S_{N_i}$. Then define $K = \bigoplus_{t=1}^{\infty} K_t$. By Lemma 3.1, $K$ is a strongly twisting module for $S$. Let $D : F \to FK$ be the corresponding derivation that twists $R := S \cap D^{-1}(K)$. Let $A = S \cap \text{Ker} D$. Then for each $0 \neq a \in A$, since $a$ is contained in at most finitely many of the $N_i$'s, there exist positive integers $t_1, t_2, \ldots, t_m$ such that $K/aK \cong K_{t_1}/aK_{t_1} \oplus \cdots \oplus K_{t_m}/aK_{t_m}$ as $S$-modules. For each $t_i$, since $S_{N_{t_i}}$ is a DVR, it follows that $S_{N_{t_i}}/aS_{N_{t_i}}$ is a cyclic $S$-module. (This is because for each maximal ideal $N$ of $S$ and $k > 0$, $S_N = S + N^k S_N$, so that since $S_N$ is a
DVR, $S_N = S + aS_N$ for each $0 \neq a \in S$.) Thus if $e_n$ is finite for all $n > 0$, then $K/aK$ is a finite $S$-module. Therefore, in this case by Theorem 5.2, $R$ is a Noetherian domain, proving (3). Also, regardless of whether all the $e_n$’s are finite, Theorem 7.1 implies that $R$ is a stable domain with quotient field $F$ and normalization $S$, and this proves (1).

To prove (2), for each $n$, let $M_n = N_n \cap R$. Then each $M_n$ is a maximal ideal of $R$, and since $R \subseteq S$ is integral, every maximal ideal of $R$ is accounted for in this way. Fix $n$, and to simplify notation, let $M = M_n$ and $N = N_n$. By Lemma 7.2, $R_M$ is a subring of $S_N$ that is strongly twisted by $K_N$. If the maximal ideal of $R_M$ is finitely generated, so that $R_M$ is a Noetherian domain, then since by Theorem 4.7, $(R_M) \cong (S_N) \ast (K_N)$ and $S_N$ is a DVR, the embedding dimension of $R_M$ is given by the following calculation (recall our notation $M = M_n$ and $N = N_n$):

$$\text{emb.dim } R_M = 1 + \dim_{S_N/NK_N} K_N/NK_N = 1 + \dim_{S_N/NK_N} K_n/NK_n = 1 + e_n.$$ 

Thus for each $n$, either $R_{M_n}$ is a non-Noetherian ring or $R_{M_n}$ is Noetherian and its maximal ideal can be generated by $e_n + 1$ but no fewer elements. Since every nonzero ideal of $R$ is contained in at most finitely many maximal ideals of $R$, an ideal of $R$ can be generated by $k$ elements, with $k \geq 2$, if it can be locally generated by $k$ elements [12, Theorem 26, p. 35]. Therefore, if $M_n$ is finitely generated, it can be minimally generated by $e_n + 1$ elements. This proves (2).

Corollary 7.4. Assume that:

(a) $k$ is a countable field of prime characteristic that is a separably generated extension of infinite transcendence degree over a subfield, and

(b) $\{e_n\}_{n=1}^{\infty}$ is a sequence for which each $e_n \in \mathbb{N} \cup \{\infty\}$.

Then there exists a subring $R$ of $k[X]$ having quotient field $k(X)$ such that $R$ is a stable domain with normalization $k[X]$ and the set of maximal ideals of $R$ can be written $\{M_1, M_2, \ldots\}$, where for each $n$, $M_n$ is minimally generated by $e_n + 1$ elements.

Proof. Since $k$ is countable, $k[X]$ is a PID having countably many maximal ideals. Therefore, we may apply Theorem 3.5 and Proposition 7.3 to obtain a stable subring $R$ of $S$ whose maximal ideals behave accordingly. □

The proposition and its corollary concern one-dimensional non-local twisted subrings. The one-dimensional local case is treated extensively in [19], while more on local stable domains can be found in [20].

References

A COUNTERPART TO NAGATA IDEALIZATION


