Noetherian rings without finite normalization

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Abstract. A number of examples and constructions of local Noetherian domains without finite normalization have been exhibited over the last seventy-five years. We discuss some of these examples, as well as the theory behind them.

Keywords. Noetherian ring, normalization, completion.


1 Introduction

The publication of Emmy Noether’s seminal works of 1921 and 1927 on ideal decompositions led to a number of fundamental properties of Noetherian rings being swiftly established. Within five years of the second of these articles, Ore remarked in a survey on ideal theory, perhaps a bit optimistically: “The theory itself to a large extent is still in an evolutionary stage and has not reached the harmonious form it will probably assume later on. Only for domains in which the finite chain condition [i.e., ascending chain condition] holds does it seem to have arrived at some degree of perfection” [47, p. 728]. Five years later, in a review of Krull’s *Idealtheorie*, he would refer to “the maze of material accumulated in recent years in the field of abstract ideal theory” [48, p. 460]. In these two articles, Noether introduced the ascending chain condition to axiomatize the ideal theory of finitely generated algebras over a field and orders in algebraic number fields. The importance of local rings, as well as their completions, was quickly realized and emphasized by Krull [21] and Chevalley [7]. Shortly thereafter, in 1946, Cohen provided a structure theory for complete local rings: Every such ring is the homomorphic image of a power series ring over a field or a rank one discrete valuation ring [9]. Thus, locally and at the limit, Noetherian rings behave like the rings arising from the geometric and arithmetic contexts which Noether sought to capture with the axiom of the ascending chain condition.

However, non-complete local Noetherian rings prove to be more elusive and capable of betraying the geometric and arithmetic intuition in which the theory is rooted, and over the ninety years since the introduction of the ascending chain condition into commutative ideal theory, there has developed along with the axiomatic theory a large bestiary of examples that behave in one fundamental way or another differently than the “standard” rings that motivate and drive the subject. Thus the need for additional hypotheses such as “excellence” to capture, for example, geometric features like desingularization [27]. Similarly, there are a number of penetrating variations on excellence,
such as $G$-rings, Japanese rings, Nagata rings and geometric rings. Miles Reid comments on the delicacy of this part of the theory: “Grothendieck (in [15, IV, 7.8]) has developed the theory of ‘excellent rings’ (following Akizuki, Zariski and Nagata), that assembles everything you might ever need as a list of extra axioms, but it seems that this will always remain an obscure appendix in the final chapter of commutative algebra textbooks: ‘Le lecteur notera que le résultats les plus delicats du §7 ne serviront qu’assez peu dans la suite’” [54, p. 136].

Nevertheless, regardless of whether one views these issues as foundational or obscure, there do exist Noetherian rings at a far remove from excellence. Unlike complete local rings, these rings behave in essential ways differently than rings in arithmetic and geometric applications. Such rings earn the appellation “bad” in Nagata’s famous appendix to his 1962 monograph, Local rings. As early as 1935, bad Noetherian rings were known: in that year Akizuki and Schmidt each published examples of bad one-dimensional local Noetherian domains. What made their examples bad was the failure to have finite normalization; that is, the integral closure of the ring in its total quotient ring is not a finitely generated module over the base ring. We discuss the examples of Akizuki and Schmidt in Section 3.

In what follows we survey examples and constructions of Noetherian rings that are bad in the sense that they fail to have finite normalization. Finite normalization is, as we recall in Section 2, closely related to the absence of nontrivial nilpotents in the completion, and thus there is a parallel emphasis on the completion of the rings in the constructions we discuss.

The designation “bad,” as well as the technical ingenuity behind some of the examples, reinforce a notion that these examples are esoteric and hard to produce\(^1\). In Sections 3, 4 and 5, we review, beginning with the early discoveries of Akizuki and Schmidt, examples of one-dimensional local Noetherian domains without finite normalization, and we also look to more recent constructions, including in Section 5 a geometric example. While the theory behind the constructions is often involved, the constructions themselves are often conceptually straightforward, and show that there is a certain inevitability to the examples: one encounters them with standard operations such as a finite extension of a DVR, as the kernel of a differential, or as an intersection of an image of the completion of a local domain with its quotient field. From this point of view, the examples are not so exotic, and one of the motivations behind this survey is to describe how Noetherian domains without finite normalization occur between (Section 3), above (Sections 4 and 5) and below (Section 6) naturally occurring Noetherian rings.

In Section 3 we consider examples of one-dimensional local Noetherian rings with-

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\(^1\) Miles Reid: “The catch-phrase ‘counterexamples due to Akizuki, Nagata, Zariski, etc., are too difficult to treat here’ when discussing questions such as Krull dimension and chain conditions for prime ideals, and finiteness of normalization is a time-honoured tradition in commutative algebra textbooks...This does little to stimulate enthusiasm for the subject, and only discourages the reader in an already obscure literature” [54, p. 136].
out finite normalization that occur in an immediate extension of rank one discrete valuation rings (DVRs), and in Corollary 3.11 we give a simple field-theoretic criterion for when a such an example can be sandwiched into an immediate extension of DVRs. Well-known examples due to Akizuki, Schmidt, Nagata, and Ferrand and Raynaud, all fit into this framework.

Section 4 also focuses exclusively on one-dimensional examples. However, the emphasis here is on when such examples can be found birationally dominating a given local Noetherian domain of dimension possibly larger than 1. The examples in this section occur in a particularly transparent way, either as a kernel of a certain exterior differential or cut out of the quotient field of a local Noetherian domain by an image of its completion. The latter construction leads to the striking theorem of Heinzer, Rotthaus and Sally that every local Noetherian domain of Krull dimension greater than one which is essentially of finite type over a field is birationally dominated by a one-dimensional local Noetherian domain without finite normalization (see Theorem 4.3).

Section 5 is devoted to a geometric example, due to Reguera, of a one-dimensional Noetherian domain without finite normalization. This example stands out from the rest in that it has a direct geometric interpretation: it is the local ring of a point, not on a curve, but on the space of arcs associated to the curve.

Section 6 discusses mainly a technique from [43] for locating Noetherian domains without finite normalization as subintegral extensions of “known” Noetherian domains. This technique produces such Noetherian rings in any Krull dimension, and affords a good deal of control over the traits of the examples it yields. The tradeoff however is that strong demands are made on the quotient field of the domain in order to guarantee existence of these well-behaved subrings.

The literature on bad Noetherian rings is diverse and scattered, since these rings often arise as counterexamples on frontiers of various topics. What follows is an attempt to organize and point out a few central ideas in some of these examples. The focus remains narrowly on local Noetherian domains without finite normalization, and does not touch in any detail on other classes of bad Noetherian rings. For more on other sorts of nonstandard Noetherian rings, see [29] and its references.

Terminology and notation. All rings are commutative and have an identity. A quasilocal ring is a not-necessarily-Noetherian ring with a unique maximal ideal; in general our focus is on the Noetherian case, and for this we follow tradition and refer to a quasilocal Noetherian ring as a local Noetherian ring. The completion of a local Noetherian ring \( R \) with maximal ideal \( m \) in its \( m \)-adic topology is denoted \( \hat{R} \). A rank one discrete valuation ring (DVR) is a local principal ideal domain. The integral closure of a ring \( R \) in its total ring of quotients is denoted \( \overline{R} \). The ring \( \overline{R} \) is also referred to more succinctly as the normalization of \( R \). The ring \( R \) has finite normalization if \( \overline{R} \) is a finitely generated \( R \)-module; \( R \) is normal if \( R = \overline{R} \). An overring of a domain \( R \) is a ring between \( R \) and its quotient field.

Acknowledgment. I thank the referee for helpful comments and corrections.
2 Normalization and completion

In this section we sketch some of the history regarding the relationship between normalization and completion of a local Noetherian ring. Krull proved in 1938 that when a local Noetherian domain $R$ has Krull dimension 1, then $R$ has finite normalization if and only if the completion of $R$ in its $m$-adic topology has no nonzero nilpotent elements; that is, $R$ is analytically unramified [21]. Ten years later, Zariski noted that when a local domain $R$ has finite normalization, then $R$ is analytically unramified if and only if its normalization is analytically unramified [64, p. 360]. Thus, granted finite normalization, whether $R$ is analytically unramified depends entirely on the analytic ramification of the normalization $\hat{R}$. Motivated by this, as well as Krull’s theorem above, Zariski proved that if the local ring of a point on an irreducible variety is normal, then the completion of this local ring is a domain; i.e., the local ring is analytically irreducible [64, Théorème 2]. Two years later, in 1950, he reached a stronger conclusion: Such a normal local domain is analytically normal, meaning that its completion is a normal local domain [65, Théorème 2]. That the local rings of points on a normal variety are analytically normal can be viewed as a version of Zariski’s Main Theorem; see [33, pp. 207–214], [40] and [66].

In the article of 1948, Zariski asked then whether every normal Noetherian local domain is analytically irreducible [64, p. 360]. (Nagata later answered this question in the negative, as we mention in Section 3.) Zariski identified two conditions on a normal local Noetherian domain $R$ that are sufficient but not necessary for $R$ to be analytically irreducible [64, p. 360]:

1. for each prime ideal $P$ of $R$, $R/P$ is analytically unramified, and
2. if $P_1$ and $P_2$ are distinct associated prime ideals of $\hat{R}$, then $(P_1 + P_2) \cap R \neq 0$.

Statement (2) implies the fiber of the prime ideal $\{0\}$ under the mapping $\text{Spec}(\hat{R}) \to \text{Spec}(R)$ is connected. For if the fiber of $\{0\}$ is not connected, then there exist ideals $I$ and $J$ of $\hat{R}$ such that $(I + J) \cap R \neq 0$ and $I$ and $J$ each are intersections of minimal prime ideals of $\hat{R}$. But then if $P_1 \supseteq I$ and $P_2 \supseteq J$ are minimal, hence associated, prime ideals of $\hat{R}$, $(P_1 + P_2) \cap R \neq 0$.

A geometric version of condition (1) winds through the literature under different guises: In his 1962 monograph, Local rings, Nagata defines a Noetherian ring $R$ to be pseudo-geometric if for each prime ideal $P$ of $R$ and each finite extension field $L$ of the quotient field $\kappa(P)$ of $R/P$, the integral closure of $R/P$ in $L$ is finite over $R/P$ (it is the persistence of (1) across finite extensions that earns the adjective “geometric” here) [36, p. 131]. Grothendieck later replaced the terminology of pseudo-geometric rings with that of anneaux universellement japonais; Bourbaki and Matsumura, whom we follow, in turn refer to pseudo-geometric rings as Nagata rings.

In any case, Nagata proves that a local Noetherian Nagata domain is analytically unramified [36, (36.4), p. 131], and that a ring essentially of finite type over a Nagata ring is also a Nagata ring [36, (36.5), p. 132]. Consequently, since fields and the ring of integers are Nagata rings, we arrive at the fundamental fact:
A finitely generated algebra over a field or the ring of integers has finite normalization.

This answered in a strong way an earlier question of Zariski: Is the local ring of a point of an irreducible algebraic variety analytically unramified? (Chevallay had provided the first positive answer to Zariski’s specific question about local rings of points on varieties [8, Lemma 9 on p. 9 and Theorem 1 on p. 11], and Zariski himself also later gave another proof of Chevallay’s theorem in [64, p. 356].)

In summary, a local Noetherian Nagata domain is analytically unramified. Nagata also proved a partial converse in an article of 1958: If $R$ is a reduced local Noetherian ring and $R$ is analytically unramified, then $R$ has finite normalization [35, Proposition 1, p. 414]. Of course, Nagata rings demand something quite a bit stronger than simply finite normalization. Indeed, Nagata gave an example in 1955 of a normal local Noetherian domain that is analytically ramified [34, p. 111]; thus a local domain having finite normalization need not be a Nagata ring or analytically unramified.

Finally in 1961, Rees made precise the connection between analytic ramification and normalization:

**Theorem 2.1.** (Rees [51, Theorems 1.2 and 1.5]) Let $R$ be a reduced local Noetherian ring, and let $F$ be its total ring of quotients. Then $R$ is analytically unramified if and only if for all $x_1, \ldots, x_n \in F$, the ring $R[x_1, \ldots, x_n]$ has finite normalization.

In light of the connection between analytic ramification and normalization, one source of local Noetherian domains without finite normalization are those local domains (or possibly overrings of such domains) whose completions have nilpotent elements. There is a rich and extensive literature on realizing complete local rings as the completion of local Noetherian rings having specified properties. In 1981, Larfeldt and Lech gave one such method: Let $K$ be a field, let $X_1, \ldots, X_n$ be indeterminates for $K$ and let $I$ be an ideal of $K[X_1, \ldots, X_n]$ that is primary for $(X_1, \ldots, X_n)$. Set $A = K[X_1, \ldots, X_n]/I$. Then using a notion of flat couples, it is shown that there exists a one-dimensional local Noetherian domain $R$ such that $\hat{R} = A[[T]]$, where $T$ is an indeterminate for $A$ [22, p. 201]. In a later article, Lech proved a much more general existence theorem:

**Theorem 2.2.** (Lech [23, Theorem 1]) A complete local ring $R$ with maximal ideal $m$ is the completion of a local Noetherian domain if and only if (i) $m = (0)$ or $m \notin \text{Ass}(R)$, and (ii) no nonzero integer of $R$ is a zero-divisor.

Lech’s article initiates a deep sequence of papers characterizing the completions of local UFDs (Heitmann [19, Theorems 1 and 8]), reduced local rings (Lee, Leer, Pilch and Yasufuku [24, Theorem 1]) and excellent local domains containing the ring of integers (Loepp [28, Theorem 9]). These results all demand remarkably little of a complete local ring in order to realize it as the completion of a specific sort of local Noetherian domain. That the completions of more abstract local Noetherian domains
could behave so differently from those arising in algebraic geometry and algebraic number theory occasions Eisenbud’s remark, “This is one of the ways in which the Noetherian property is ‘too general’” [12, p. 193].

In any case, the construction of Lech guarantees the existence of many examples of one-dimensional local Noetherian domains without finite normalization: If \( R \) is a one-dimensional complete local Cohen-Macaulay ring that is not reduced, its prime subring is a domain and \( R \) is torsion-free over its prime subring, then there exists a one-dimensional local Noetherian domain whose completion is \( R \), and hence this ring does not have finite normalization.

### 3 Examples between DVRs

As discussed in Section 2, a one-dimensional local Noetherian domain has finite normalization if and only if it is analytically ramified. In this section we give an overview of some examples of one-dimensional analytically ramified local Noetherian domains, as well as some of the general theory regarding the existence of such examples. This section concerns one-dimensional local rings sandwiched into an immediate extension of discrete rank one valuation rings (DVRs), while the next section deals with the existence of one-dimensional analytically ramified local rings birationally dominating a local Noetherian domain.

An extension \( U \subseteq V \) of DVRs is **immediate** if \( U \) and \( V \) have the same value group and residue field; equivalently, with \( \mathfrak{m} \) the maximal ideal of \( U \), \( \mathfrak{m}V \neq V \) and \( V = U + \mathfrak{m}V \). It is easy to see that an extension \( U \subseteq V \) of DVRs is immediate if and only if the inclusion \( U \subseteq V \) lifts to an isomorphism of \( U \)-algebras \( \hat{U} \rightarrow \hat{V} \). Thus for a DVR \( U \), the extension \( U \subseteq \hat{U} \) is immediate. We use the following lemma to explain why some classical examples due to Akizuki, Schmidt and Nagata, all of which occur within an immediate extension of DVRs, fail to have finite normalization.

**Lemma 3.1.** If \( U \subseteq R \subseteq V \) is an extension of rings such that \( U \subseteq V \) is an immediate extension of DVRs, then \( V/R \) is a divisible \( R \)-module, and hence \( V \) is not a finite \( R \)-module.

**Proof.** Let \( 0 \neq r \in R \). To see that \( V = R + r \mathfrak{m} \), use the fact that \( V \) is a DVR to find nonnegative integers \( i \) and \( j \) such that \( r^iV = \mathfrak{m}^jV \), where \( \mathfrak{m} \) is the maximal ideal of \( U \). Since \( U \subseteq V \) is an immediate extension, \( V = U + \mathfrak{m}^jV = U + r^iV \subseteq R + r^iV \), proving that \( V = R + r \mathfrak{m} \) and that \( D := V/R \) is a divisible \( R \)-module. Since \( D \neq 0 \), there exists a maximal ideal \( M \) of \( R \) such that \( D_M \neq 0 \). But since \( D \) is divisible, \( D_M = MD_M \), so that by Nakayama’s Lemma, \( D \) is not a finite \( R \)-module, and hence \( V \) is not a finite \( R \)-module. \( \square \)

We now outline without proof the 1935 example of Akizuki. With the exception of our use of Lemma 3.1, we follow Reid [53, Section 9.5] and [54], where commentary and a complete justification for the example can be found.
Example 3.2. (Akizuki [2]) Let $U$ be a DVR with maximal ideal $\mathfrak{m} = tU$ and quotient field $K$ such that there exists $z = \sum_{i=0}^{\infty} u_i t^{e_i} \in \hat{U}$ (with each $u_i$ a unit in $U$) that is transcendental over $K$ and $e_i \geq 2e_{i+1} - e_i$ for all $i \geq 1$, where $e_0 = 0$. For $i \geq 0$, let $z_i = u_i + u_{i+1} t^{e_i+1} - e_i + u_{i+2} t^{e_i+2} - e_i + \cdots$. Let $R$ be the localization of $U[t(z_0 - u_0)], \{(z_i - u_i)^2\}_{i=0}^{\infty}$ at the maximal ideal generated by $t$ and $t(z_0 - u_0)$. Calculations show that $R$ is a one-dimensional local Noetherian domain with quotient field $K(z)$ and normalization $V = K(z) \cap \hat{U}$ properly containing $R$ (for details, see [53, 54]). Thus since $U \subseteq V$ is an immediate extension of DVRs, Lemma 3.1 shows that $R$ does not have finite normalization, a fact which can also be proved by a direct argument [53, 54].

More can be said about this example: Since the maximal ideal $M$ of $R$ is generated by 2 elements and $M^2 = tM$ [53, Exercise 9.5, p. 148], it follows that $R$ has multiplicity 2, and hence every ideal of $R$ can be generated by 2 elements [57, Theorem 1.1, p. 49]. We discuss rings for which every ideal can be generated by 2 elements later in this section.

Another example, this one due to Schmidt, appeared in the same year as Akizuki’s example. Unlike Akizuki’s example, Schmidt’s example requires positive characteristic, but this in turn makes the justification for the example easier. We give this example next, but presented and justified differently than the original. For another approach, one that makes explicit the valuation theory in the example, see Zariski [63, pp. 23–25].

Example 3.3. (Schmidt [58]) Let $k$ be a field of characteristic $p > 0$, let $X$ be an indeterminate for $k$, and let $z \in Xk[[X]]$ such that $z$ is transcendental over $k(x)$. (Such an element $z$ must exist; see [67, p. 220].) Then $k[X, z] \subseteq k[[X]]$. Let $U = k(X, z^p) \cap k[[X]]$, and consider the ring $R = U[z] \subseteq k(X, z)$. Then $U$ is a DVR with maximal ideal generated by $X$, and $R$ has quotient field $k(X, z)$. Since $z$ is integral over $U$, $R$ is a finite extension of $U$, so $R$ is a one-dimensional Noetherian domain. Let $V = k(X, z) \cap k[[X]]$. Then $V$ is a DVR with maximal ideal generated by $X$. Moreover, $V$ is integral over $R$, since $V^p \subseteq U$. Thus $R$ is a local ring with normalization $V$. Since $R$ is a finite $U$-module and $U \nsubseteq V$, Lemma 3.1 implies that $R \neq V$, so again by Lemma 3.1, $R$ does not have finite normalization.

The next example is from Nagata’s appendix, “Bad Noetherian domains” [36]. Nagata considers the ring $A = k^p[[X_1, \ldots, X_n]][k]$, where $k$ is a field of characteristic $p > 0$ such that $[k : k^p] = \infty$ and $X_1, \ldots, X_n$ are indeterminates for $k$. He proves that $A$ is a regular local ring and a proper subring of its completion $k[[X_1, \ldots, X_n]]$ [36, (E3.1), p. 206]. By varying the choices of $n$ and $z \in k[[X_1, \ldots, X_n]]$ appropriately, he fashions the ring $A[z]$ such that it is (depending on the choice of $n$ and $z$): (a) a one-dimensional analytically ramified local Noetherian domain; (b) a two-dimensional local Noetherian domain without finite normalization having a non-Noetherian ring between itself and its normalization; (c) a three-dimensional local Noetherian domain.
without finite normalization and whose normalization is not Noetherian; (d) a normal local Noetherian domain which is analytically ramified; and (e) a subring of a one-dimensional Noetherian domain without finite normalization but such that each localization at a prime ideal has finite normalization [36, Examples 3–6 and 8, p. 205–212]. While all these examples are relevant here, we mention only how the choice is made in case (a), since it fits into the present discussion of one-dimensional rings. We differ from Nagata’s original justification of the example in that we rely on Lemma 3.1; see also Reid [53, pp.136–137] for a general approach to the example.

Example 3.4. (Nagata [36, Example 3, p. 205]) With notation and assumptions as above, let \( U = k^p[[X]][k] \), so that \( U \) is a DVR. (Thus in the notation above, \( U = A \) with \( n = 1 \).) Since \( [k : k^p] = \infty \), there exist countably many distinct \( p \)-independent elements \( \alpha_1, \alpha_2, \ldots \) of \( k \). Set \( z = \sum_{i=0}^{\infty} \alpha_i X^i \in k[[X]] \). Then since \( z \) is integral over \( U \), the ring \( R = U[z] \) is a one-dimensional Noetherian domain. Also, since the DVR \( V = U(z) \cap k[[X]] \) is integral over \( R \) (indeed, \( V^p \subseteq R \)), it follows that \( V \) is the normalization of \( R \), and hence \( R \) is a local ring. The choice of coefficients of \( z \) forces \( z \notin U \), so \( U \subset R \subset V \). Also, \( R \neq V \), since otherwise \( V/U \) is a finite \( U \)-module, contrary to Lemma 3.1. Thus \( U \subset R \subset V \), and another application of Lemma 3.1 shows that \( R \) does not have finite normalization.

In all of the above examples, the one-dimensional analytically ramified local Noetherian domain \( R \) has the property that the inclusion \( R \to \hat{V} \) lifts to a surjection \( \hat{R} \to \hat{V} \). (This follows from the fact that since \( V/R \) is a divisible \( R \)-module, \( V = R + M^i V \), where \( M \) is the maximal ideal of \( R \), for all \( i > 0 \).) Since \( \hat{V} \) is a domain and \( R \) is not a DVR, the kernel \( P \) of this mapping is a nonzero prime ideal. Moreover, since the integral closure of \( R \) is a local ring, the ring \( \hat{R} \) has a unique height 1 prime ideal \( P \) [30, Theorem 7.9, p. 77], which therefore, since \( \hat{R} \) has dimension 1, must be nilpotent. In summary: If \( U \subset R \subset V \) is an extension of rings such that \( U \subset V \) is an immediate extension of DVRs, \( V \) is the normalization of \( R \) and \( R \) is a Noetherian ring (all these requirements are satisfied by the examples of Akizuki, Schmidt and Nagata), then there is a nonzero nilpotent prime ideal \( P \) of \( \hat{R} \) such that \( \hat{R}/P \cong \hat{V} \). In [4], Bennett proves a converse for positive characteristic:

Theorem 3.5. (Bennett [4, Theorem 1, p. 133]) Let \( R \) be a one-dimensional local Noetherian domain of characteristic \( p > 0 \). If there is a nilpotent prime ideal \( P \) of \( \hat{R} \) such that \( \hat{R}/P \) is a DVR, then there is a DVR \( U \) such that \( U \subset R \subset \hat{U} \) and \( R^q \subset U \) for some \( q = p^e \).

A one-dimensional analytically ramified local Noetherian domain can always be modified by finitely many quadratic transformations to produce a ring \( R \) whose completion has a nilpotent prime ideal with residue ring a DVR [4, Section 1]. Bennett refers to the extension \( U \subset R \subset \hat{U} \) in the theorem, with \( R \) purely inseparable over \( U \), as a presentation of \( R \). The positive characteristic examples of Nagata and Schmidt
occur within such a presentation. As discussed above, a presentation \( U \subseteq R \subseteq \hat{U} \) forces the existence of a prime ideal \( P \) of \( \hat{R} \) such that \( \hat{R}/P \) is a DVR, and if also \( R \) has a local normalization, then \( P \) is a nilpotent prime ideal of \( \hat{R} \). Turning this around, Bennett considers when a complete local ring having a nonzero nilpotent prime ideal with residue ring a DVR arises from a presentation of an analytically ramified local ring:

**Theorem 3.6.** (Bennett [4, Theorem 6.0.4]) Let \( V \) be a complete DVR of positive characteristic, and let \( C \) be a one-dimensional local Noetherian ring that is a flat finite \( V \)-algebra with nilpotent ideal \( P \) such that \( C/P \cong V \). Then there exists a local Noetherian domain \( R \) and a DVR \( U \) such that \( \hat{U} = V, U \subseteq R \subseteq V \) and \( \hat{R} \cong C \).

Since \( V \) is a DVR, the assertion here that \( C \) is flat means only that the nonzero elements of \( V \) are nonzerodivisors on \( C \). The idea behind the construction, which Bennett terms “quasi-algebraization,” is to begin with a suitable choice for the DVR \( U \), select carefully a finite \( U \)-subalgebra \( S \) of \( V = \hat{U} \), and then perform an infinite sequence of operations on \( S \) to produce \( R \), all the while staying in the quotient field of \( S \). The quotient field of \( R \) is then finite over that of \( U \), but \( R \) is not finite over \( U \).

Bennett’s article was partly inspired by examples of Ferrand and Raynaud, who in 1970 introduced a method based on derivations to construct analytically ramified local Noetherian domains [13]. We will have more to say about their method in Section 6, but we mention here one of the examples produced with their construction, an example which occurs in an immediate extension of DVRs. In that sense it fits within the sequence of examples considered so far; however, it does not fit directly within Bennett’s framework because it has characteristic 0.

The example of Ferrand and Raynaud uses the notion of idealization of a module. When \( A \) is a ring and \( L \) is an \( A \)-module, we denote by \( A \star L \) the Nagata idealization (or, trivialization) of \( L \). This ring is defined as a set by

\[
A \star L = \{(a, \ell) : a \in A, \ell \in L\},
\]

where for all \( a_1, a_2 \in A \) and \( \ell_1, \ell_2 \in L \), addition and multiplication in the ring are defined by:

\[
(a_1, \ell_1) + (a_2, \ell_2) = (a_1 + a_2, \ell_1 + \ell_2)
\]

\[
(a_1, \ell_1)(a_2, \ell_2) = (a_1a_2, a_1\ell_2 + a_2\ell_1).
\]

The completion of the local ring in the following proposition is the Nagata idealization of a module; the completions considered later in this section also have this form.

**Proposition 3.7.** (Ferrand and Raynaud [13, Proposition 3.1]) Let \( \mathbb{C}\{X\} \) be the ring of convergent power series with complex coefficients, and let \( F \) be the quotient field of \( \mathbb{C}\{X\} \). Then for each \( e \in \mathbb{N} \) there exists a subring \( R \) of \( \mathbb{C}\{X\} \) such that the following statements hold for \( R \).
(1) \( R \) is a Noetherian domain with quotient field \( F \), embedding dimension \( e + 1 \) and normalization \( \mathbb{C}\{X\} \).

(2) \( \widehat{R} \cong \mathbb{C}[[X]] \star J \), for some \( \mathbb{C}[[X]] \)-module \( J \), and when \( e \) is finite (so that \( R \) is Noetherian), \( J \) is free of rank \( e \).

(3) If \( e > 1 \), then \( R \) does not have a canonical ideal and the generic formal fiber of \( R \) is not Gorenstein.

The ring \( R \) is sandwiched into the immediate extension \( \mathbb{C}[X](X) \subseteq \mathbb{C}\{X\} \). It arises as \( D^{-1}(K) \), where \( D : \mathbb{C}\{X\} \to L \) is a well-chosen derivation, \( L \) is a \( \mathbb{C}((X)) \)-vector space and \( K \) is a finite rank free \( \mathbb{C}[[X]] \)-submodule of \( L \). In the terminology of Section 6, \( R \) is a “strongly twisted” subring of \( \mathbb{C}\{X\} \). It also can be deduced from (2) that the multiplicity and embedding dimension of \( R \) agree.

Goodearl and Lenagan generalize this idea to incorporate higher-order differentials, and in so doing permit multiplicity and embedding dimension to differ. The rings they construct with this method are differentially simple, meaning there is a derivation \( D \) from the ring to itself such that no proper nonzero ideal is invariant under \( D \).

**Theorem 3.8.** (Goodearl–Lenagan [14, Proposition 6 and Example D]) For any pair of positive integers \( m \) and \( t \), there exists a one-dimensional analytically ramified local Noetherian domain \( R \) containing the field of rational numbers such that \( R \) has embedding dimension \( m + 1 \), multiplicity \( m + t \) and is differentially simple.

The issue of differential simplicity is relevant to our themes here because Posner showed in 1960 that a differentiably simple ring of characteristic zero finitely generated over its field of constants (“constants” here being the constants for the relevant derivation; the ring of all such constants forms a field in a differentially simple ring) is normal [49, Theorem 1], and he asked implicitly in a subsequent paper whether a differentiably simple ring of characteristic zero must be normal [50, p. 1421]. This question is made more compelling by the 1966 observation of Seidenberg that if a Noetherian domain \( R \) contains the rational numbers, then the conductor \( (R :_{R} \widehat{R}) \) of the normalization of \( R \) into itself is invariant under the derivation [59, p. 169]. Thus if \( R \) is a differentiably simple local Noetherian domain containing the rational numbers, then either \( R \) is normal or \( R \) does not have finite normalization, and hence arises the question of whether the latter case can occur.

Moreover, Vasconcellos showed that if \( R \) is a one-dimensional analytically unramified local Noetherian ring containing the rational numbers and having a derivation \( D \) such that \( D(x) = 1 \) for some nonunit \( x \) of \( R \), then \( R \) is a DVR, and he asked whether this remains true if \( R \) is assumed to be reduced but analytically ramified [62, p. 230]. Lequain answered Vasconcellos’ question in the negative by modifying Akizuki’s construction, Example 3.2, to produce a one-dimensional analytically ramified

\(^2\) There is a mistake in the proof of the first lemma of Posner’s article which is corrected in [59].
local Noetherian domain $R$ having a nonunit $x$ such that $D(x) = 1$ for a deriva-
tion $D : R \to R$ [25, Theorem 2.1 and Example 2.2]. Since $R$ is local and one-
dimensional, the condition $D(x) = 1$ then forces $R$ to be differentially simple with
respect to $D$. Another construction in mixed characteristic given later by de Souza
Doering and Lequain produced similar examples, but of arbitrary large embedding di-
mension and multiplicity [10, Proposition 1]. However, as noted in [14, p. 479], the
calculations to justify these examples are “long and technical.” The construction of
Goodearl and Lenagan, in addition to showing every possible pair $2 \leq m \leq e$ can
occur as embedding dimension and multiplicity, respectively, of such an example, has
the advantage of being computationally simpler than the previous examples, as well as
having an obvious choice for the derivation to decide the differential simplicity of the
ring, namely the derivation that defines the ring as a pullback.

Proposition 3.7, which is the point of departure for the construction of Goodearl
and Lenagan, can be viewed as a particular instance of a more general method of
extracting analytically ramified local rings as subrings of rings possessing a special sort
of derivation. As discussed in Section 6, this method can be applied to produce rings
of any Krull dimension. But in dimension 1, some consequences of the method can
be described without mention of the notion of what is termed in Section 6 a “strongly
twisted” subring. So in keeping with the focus on one-dimensional rings, we discuss
some of these consequences now and postpone till Section 6 an explanation of the
method behind the results.

In dimension 1, the method of strongly twisted subrings produces stable domains.
An ideal $I$ of a domain $R$ is stable if it is projective over $\text{End}_R(I)$, its ring of en-
domorphisms. The terminology here is due to Lipman [26]; see the survey [41] for
background and an explanation of the terminology. In this section we are interested
exclusively in the case in which $R$ is a quasilocal ring, and in this situation stable ide-
als are simply ideals having a principal reduction of reduction number $\leq 1$; that is, an
ideal $I$ of a quasilocal domain $R$ is stable if and only if $I^2 = iI$ for some $i \in I$ (see
[26] and [42, Lemma 3.1]). The domain $R$ is stable provided every nonzero ideal is
stable.

Bass proved that the class of stable domains includes the 2-generator domains, those
domains for which every ideal can be generated by 2 elements [3]. A local Noethe-
rian domain with finite normalization is a 2-generator ring if and only if it is a stable
domain; see Drozd and Kiričenko [11] or Sally and Vasconcelos [55, Theorem 2.4].
The latter authors noted that a stable Noetherian domain must have Krull dimension 1
[56], and they proved that a stable local Noetherian domain of embedding dimension
2 must be a 2-generator ring [56, Lemma 3.2]. Using the method of Ferrand and Ray-
naud they also gave an example in characteristic 2 of a local Noetherian stable domain
that does not have the 2-generator property, and hence does not have finite normaliza-
tion [56, Example 5.4]. Heinzer, Lantz and Shah modified this example to show that
every embedding dimension $> 1$ was possible for a local Noetherian stable domain
without finite normalization [17, (3.12)]. (It is easy to see that for a local Noetherian
stable domain, since its maximal ideal has reduction number 1, the embedding dimension and multiplicity of the ring agree.) A consequence of Theorem 3.13 below is that examples of large embedding dimension exist in any characteristic.

Thus analytically unramified local Noetherian stable domains are simply 2-generator rings, but the class of analytically ramified local Noetherian stable domains properly includes the class of 2-generator rings, and whether an analytically ramified local Noetherian stable domain $R$ is a 2-generator ring is conditioned on whether its embedding dimension is $\leq 2$. The embedding dimension of $R$ reflects how far away $R$ is from its normalization:

**Theorem 3.9.** [44, Theorem 4.2] Let $R$ be a quasilocal domain with normalization $\overline{R}$ and quotient field $F$, and let $n > 1$. Then $R$ is an analytically ramified local Noetherian stable domain of embedding dimension $n$ if and only if $\overline{R}$ is a DVR and $\overline{R}/R \cong \bigoplus_{i=1}^{n-1} F/\overline{R}$ as $R$-modules.

More characterizations in this same spirit can be found in [44]. The case $n = 1$ implies that the analytically ramified local 2-generator domains are precisely those quasilocal domains $R$ for which $\overline{R}$ is a DVR and $\overline{R}/R \cong F/\overline{R}$ as $R$-modules [44, Corollary 4.5]. It follows that when $R$ is an analytically ramified local 2-generator domain, then $R \subseteq \overline{R}$ is what is termed in [16] a $J$-extension, meaning that every proper $R$-subalgebra of $\overline{R}$ is a finitely generated $R$-module. It is in fact the unique $J$-extension of $R$ in its quotient field [16, Proposition 3.1]. More generally, when $R$ is an analytically ramified local Noetherian domain, then the $J$-extensions $R \subseteq S$, where $S$ is an overring of $R$, are in one-to-one correspondence with the minimal prime ideals of the total quotient ring of $\overline{R}$ [16, Theorem 2.1]. An analytically ramified local Noetherian domain, even a Gorenstein one, may have more than one $J$-extension in its quotient field [16, Example 3.8].

Analytically ramified local Noetherian stable domains arise from immediate extensions of DVRs, as we see in the next theorem, which uses the notion of the exterior differential of a ring extension. Given an $A$-linear derivation $D : S \to L$, with $L$ an $S$-module, there exists a universal module through which $D$ must factor. More precisely, for an extension $R \subseteq S$ of rings, there exists an $S$-module $\Omega_{S/R}$, and an $R$-linear derivation $d_{S/R} : S \to \Omega_{S/R}$, such that for every derivation $D : S \to L$, there exists a unique $S$-module homomorphism $\alpha : \Omega_{S/R} \to L$ with $D = \alpha \circ d_{S/R}$; see for example [12, 20]. The $S$-module $\Omega_{S/R}$ is the module of Kähler differentials of the ring extension $R \subseteq S$, and the derivation $d_{S/R} : S \to \Omega_{S/R}$ is the exterior differential of the extension $R \subseteq S$. If $L$ is a torsion-free $S$-module, then a submodule $K$ of $L$ is full if $L/K$ is a torsion $S$-module.

**Theorem 3.10.** [45, Theorems 4.1 and 4.4] Let $U \subseteq V$ be an immediate extension of DVRs with quotient fields $Q$ and $F$, respectively. Suppose $K$ is a full $V$-submodule of $\Omega_{F/Q}$ such that $n := \dim_{V/\mathfrak{M}} K/\mathfrak{M}K$ is positive, where $\mathfrak{M}$ is the maximal ideal of $V$. Then $R = V \cap d_{F/Q}^{-1}(K)$ is an analytically ramified quasilocal stable domain
with normalization $V$. Every quasilocal stable domain containing $U$ and having normalization $V$ must arise this way for a unique choice of $K$, and satisfy the following properties.

1. $R$ is a Noetherian domain (with embedding dimension $n + 1$) if and only if $n$ is finite.

2. If $R$ is a Noetherian domain, then $\hat{R} \cong \hat{V} \times J$, where $J$ is a free $\hat{V}$-module of rank $n$.

With some basic facts about the module of Kähler differentials, the theorem yields a criterion for when an analytically ramified local Noetherian domain can be sandwiched into an immediate extension of DVRs.

**Corollary 3.11.** [45, Corollary 4.2] Let $U \subseteq V$ be an immediate extension of DVRs having quotient fields $Q$ and $F$, respectively. Then there exists an analytically ramified local Noetherian domain containing $U$ and having normalization $V$ if and only if either (a) $F$ has characteristic 0 and is not algebraic over $Q$, or (b) $F$ has characteristic $p > 0$ and $F \neq Q[F^p]$.

**Proof.** The proof depends on the fact that statements (a) or (b) hold precisely when $\Omega_{F/Q} \neq 0$ [20, Proposition 5.7]. Suppose that $R$ is an analytically ramified local Noetherian domain containing $U$ and having normalization $V$. Then there exists an analytically ramified local ring between $R$ and $V$ that has normalization $V$ and is a stable ring (see Theorem 4.2 below). Thus by Theorem 3.10, $\Omega_{F/Q} \neq 0$, as claimed. Conversely, if $\Omega_{F/Q} \neq 0$, then since $\Omega_{F/Q}$ is an $F$-vector space, we may choose any proper full $S$-submodule $K$ of $\Omega_{F/Q}$ such that $\dim V/MK$ is finite and apply Theorem 3.10 to obtain the ring in the corollary. 

**Remark 3.12.** In positive characteristic, every analytically ramified local Noetherian stable domain must arise as in Theorem 3.10. For if $R$ is an analytically ramified local Noetherian stable domain, then there exists a prime ideal $P$ of $\hat{R}$ such that $P^2 = 0$ and $\hat{R}/P$ is a DVR [44, Corollary 3.5]. Thus when $R$ has positive characteristic, there exists by Theorem 3.5 a DVR $U$ such that $U \subseteq R \subseteq \hat{U}$. With $F$ the quotient field of $R$, we have that $U \subseteq V := \hat{U} \cap F$ is an immediate extension, and so we are in the setting of the theorem. See also [45, Theorem 6.4].

Corollary 3.11 helps explain why the examples of Akizuki, Schmidt and Nagata given earlier in this section are couched as they are.

- **Akizuki’s example.** The ring in this example is sandwiched into an immediate extension of DVRs $U \subseteq V$ with quotient fields $K$ and $K(z)$, respectively, where $z$ is transcendental over $K$. Thus $\Omega_{K(z)/K}$ has dimension 1 as a $K(z)$-vector space, and as in the proof of Corollary 3.11, there exists an analytically ramified local Noetherian stable domain containing $U$ and having normalization $V$. As discussed after Akizuki’s example, the ring produced in the example is a 2-generator ring, so in fact
Theorem 3.10 captures this same ring from another point of view, as a pullback of a derivation.

- **Schmidt’s example.** This ring is sandwiched into a characteristic $p$ immediate extension $U \subseteq V$, where $U$ and $V$ have quotient fields $Q := k(X, z^p)$ and $F := k(X, z)$, respectively, and $X$ and $z$ are transcendental over $k$. Since $F \neq Q[F^p]$, Corollary 3.11 shows there is an analytically ramified local Noetherian domain containing $U$ and having normalization $V$.

- **Nagata’s example.** This example involves an immediate extension $U \subseteq V$ of DVRs in characteristic $p > 0$, where (in the notation of the example) the quotient field of $U$ is $Q = k^p((X))[k]$ and the quotient field of $V$ is $F = k^p((X))[k](z)$. Now $z \in F$ but $z \notin Q[F^p] = Q$, so by Corollary 3.11, there must be an analytically ramified local Noetherian stable domain containing $U$ and having normalization $V$.

Theorem 3.10, along with technicalities involving separability and valuation theory, leads to an existence theorem in the setting of function fields which shows that analytically ramified local Noetherian stable domains of arbitrarily large embedding dimension exist in every characteristic (compare to the discussion preceding Theorem 3.9). By a DVR in $F/k$ we mean a DVR that is a $k$-algebra having quotient field $F$. A divisorial valuation ring in $F/K$ is a DVR $V$ in $F/k$ such that $\text{trdeg}_k V/\mathfrak{M} = \text{trdeg}_k F - 1$, where $\mathfrak{M}$ is the maximal ideal of $V$.

**Theorem 3.13.** [45, Theorem 7.3] Let $F/k$ be a finitely generated field extension, and let $V$ be a DVR in $F/k$ with maximal ideal $\mathfrak{M}$ such that $V/\mathfrak{M}$ is a finitely generated extension of $k$. Then the following statements are equivalent.

1. $V$ is the normalization of an analytically ramified local Noetherian domain containing $k$.
2. $V$ is the normalization of an analytically ramified Noetherian stable ring containing $k$ and having embedding dimension $d = \text{trdeg}_k F - \text{trdeg}_k V/\mathfrak{M}$.
3. $V$ is not a divisorial valuation ring in $F/k$.

If $F/k$ is a finitely generated field extension of transcendence degree $d > 1$, then there exists a DVR $V$ in $F/k$ with maximal ideal $\mathfrak{M}$ such that $V/\mathfrak{M}$ is a finite algebraic extension of $k$ (see Theorem 4.3). Thus by Theorem 3.13, $V$ is the normalization of an analytically ramified Noetherian stable ring containing $k$ and having embedding dimension $d$.

At the other extreme from function fields are those local Noetherian domains whose normalizations are complete DVRs. The following theorem is obtained by sandwiching local Noetherian domains into an immediate extension of DVRs.

**Theorem 3.14.** [45, Theorem 7.6] Let $V$ be a complete DVR with residue field $k$. 


(1) If $V$ has characteristic $p \neq 0$ and $k$ is perfect, then there does not exist an analytically ramified local Noetherian domain containing $k$ whose normalization is $V$.

(2) If either (a) $V = \widehat{\mathbb{Z}_p}$, (b) $V$ and $k$ have characteristic 0, or (c) $V$ has characteristic $p \neq 0$ and $[k : k^p]$ is uncountable, then for every $d > 1$ there exists an analytically ramified local Noetherian stable domain of embedding dimension $d$ whose normalization is $V$. There also exists a non-Noetherian stable domain whose normalization is $V$.

If $R$ is an analytically ramified local Noetherian domain of dimension 1 whose normalization is a DVR, then by Theorem 4.2 below, there exists an analytically ramified 2-generator overring of $R$ having the same normalization as $R$. Using Matlis’ theory of $Q$-rings, such rings which have normalization a complete DVR are classified in [45]. (An integral domain $R$ with quotient field $Q$ is a $Q$-ring if $\text{Ext}_R^1(Q, R) \cong Q$; see [31].)

\textbf{Theorem 3.15.} [45, Theorem 8.4] A one-dimensional quasilocal domain $R$ with quotient field $Q$ is an analytically ramified local 2-generator ring whose normalization $\overline{R}$ is a complete DVR if and only if $\overline{R}$ has rank 2 as a torsion-free $R$-module, $\overline{R}/R$ is a nonzero divisible $R$-module and there are no other proper nonzero divisible $R$-submodules of $Q/R$.

\section{Examples birationally dominating a local ring}

The analytically ramified local Noetherian domains considered in the last section are sandwiched into an immediate extension of DVRs. We depart now from this approach, and while the analytically ramified rings we next consider have normalization a DVR, we do not need that they have a DVR subring which anchors an immediate extension. (Although by Theorem 3.5, in positive characteristic such a DVR is always present modulo an adjustment by finitely many quadratic transformations.) To frame this next sequence of results, we require some terminology: If $B \subseteq R$ is an extension of quasilocal domains, then $R$ dominates $B$ if the maximal ideal of $B$ is a subset of the maximal ideal of $R$. The ring $R$ birationally dominates $B$ if $R$ dominates $B$ and has the same quotient field as $B$. When $B$ is a quasilocal domain with maximal ideal $m$, we say the quasilocal ring $R$ finitely dominates $B$ if $R$ birationally dominates $B$ and $R/mR$ is a finite $B$-module. When also $R/mR$ is a nonzero cyclic $B$-module, we say $R$ tightly dominates $B$; i.e., the quasilocal ring $R$ tightly dominates $B$ if $R$ birationally dominates $B$ and $R = B + mR$. Thus $R$ tightly dominates the subring $B$ if and only if $mR$ is the maximal ideal of $R$ and $B$ and $R$ share the same residue field and quotient field.

Tightly dominating DVRs arise from analytic arcs. For example, if $k$ is a field and $X_1, \ldots, X_n$ are indeterminates for $k$, then the ring $A = k[X_1, \ldots, X_n]$ embeds into $k[[X_1]]$ as a $k[X_1]$-algebra in such a way that the images of $X_2, \ldots, X_n$ are in
Identifying $A$ with its image in $k[[X_1]]$, the DVR that is the intersection of $k[[X_1]]$ with the quotient field of $A$ tightly dominates $A$. More generally, if $A$ is a local Noetherian domain, then there exists a DVR $V$ that tightly dominates $A$ if and only if there is a prime ideal $P$ of $\hat{A}$ such that $P \cap A = 0$ and $\hat{A}/P$ is a DVR [44, Corollary 5.5].

If $A$ is a local Noetherian domain and $P$ is a prime ideal of $\hat{A}$ such that $P \cap A = 0$ and $\dim(\hat{A}/P) = 1$ (i.e., the dimension of the generic formal fiber of $A$ is $\dim(A) - 1$), then there is a finitely generated birational extension of $A$ that is essentially of finite type over $k$ and tightly dominated by this DVR. Thus blowing up at an ideal of $A$ produces a ring that is tightly dominated by a DVR. As the next theorem shows, in such a situation it is conceptually easy to locate analytically ramified local Noetherian domains. Recall from Section 3 that $d_{V/A}$ is the exterior differential of the ring extension $V/A$.

**Theorem 4.1.** [45, Theorems 5.1 and 5.3 and Corollary 5.4] Let $U \subseteq A \varsubsetneq V$ be an extension of local Noetherian domains, where $U \subseteq V$ is an immediate extension of DVRs with quotient fields $Q$ and $F$, respectively, and $V$ birationally dominates $A$. The ring $R = \ker d_{V/A}$ is an analytically ramified local Noetherian stable ring that tightly dominates $A$; $R$ has normalization $V$; $R$ has maximal ideal $mR$ extended from the maximal ideal $m$ of $A$; and $R$ is contained in every stable ring between $A$ and $V$. If also $A$ is essentially of finite type over $U$ with Krull dimension $d > 1$ and $P$ is the kernel of $\hat{A} \to \hat{V}$, then

$$\text{emb.dim } R = 1 + \text{dim}_F \Omega_{F/Q} = 1 + \text{emb.dim } \hat{A}_P.$$

Moreover, if $F$ is separable over $Q$, then $\text{emb.dim } R = d$ and the ring $\hat{A}_P$ is a regular local ring.

Thus in the context of the theorem, when $A$ is essentially of finite type over $U$, then the embedding dimension of $R$ is a measure of the regularity of a corresponding prime ideal in the generic formal fiber. If $U$ is an excellent DVR, then $A$ is excellent, and hence the generic formal fiber of $A$ is regular. Therefore, in this case, $\text{emb.dim } R = d$. This occurs, for example, in the following circumstance. Let $k$ be a field, and let $A$ be a local domain of Krull dimension $d > 1$ that is essentially of finite type over $k$. Then $A$ is finitely dominated by a DVR (see Theorem 4.3). If in fact $A$ is tightly dominated by a DVR $V$, then, choosing $t \in A$ such that $tV$ is the maximal ideal of $V$, we have that $U := k[t]/(t) \subseteq V$ is an immediate extension of DVRs, and $A$ is essentially of finite type over $U$. Therefore, the theorem is applicable to $U \subseteq A \varsubsetneq V$, and since $A$ is excellent, we conclude also that $\text{emb.dim } R = d$.

There is also a version of the theorem for dimension 1, but in this case the base ring $A$ must necessarily be analytically ramified, since otherwise every overring of
A has finite normalization. More precisely, let \( A \) be an analytically ramified local Noetherian domain whose normalization \( \overline{A} \) is a DVR that tightly dominates \( A \). Then \( R = \ker d_{\overline{A}/A} \) is an analytically ramified local Noetherian stable domain such that every stable overring of \( A \) contains \( R \); this is established in the proof of [44, Theorem 5.11]. The assumption here that \( \overline{A} \) is a DVR that tightly dominates \( A \) is equivalent to the assertion that \( \overline{A}/A \) is a divisible \( A \)-module. Divisible submodules of \( \overline{A}/A \) play an important role in Matlis’ approach to one-dimensional analytically ramified Cohen-Macaulay rings in the monograph [30], where the subtleties of the analytically ramified case are dealt with in some detail. These ideas also lead in [44, Theorem 5.11] to another proof of the following theorem of Matlis.

**Theorem 4.2.** (Matlis [30, Theorem 14.16]) Every one-dimensional analytically ramified local Noetherian domain is finitely dominated by an analytically ramified local 2-generator ring \( R \).

If \( R \) is a one-dimensional analytically unramified local ring, then every overring of \( R \) is analytically unramified. But in higher dimensions, there is much more room between a Noetherian domain \( A \) and its quotient field for pathological behavior, and even though \( A \) may be a natural enough Noetherian ring, say, a polynomial ring over a field, there can exist analytically ramified local Noetherian overrings of \( A \). This is a consequence of the following theorem, which shows that under mild hypotheses, such analytically ramified overrings must exist. The equivalence of statements (2), (4) and (5) can be deduced from Heinzer, Rotthaus and Sally [18, Corollaries 1.27 and 2.4]. That the other statements are equivalent to these three, as well as the assertion about the case in which \( A \) is excellent, is proved in [44, Corollary 5.13]. The final assertion of the theorem is due to Matsumura, who proved that (5) holds for a local domain of dimension \( d > 1 \) that is essentially of finite type over a field [32, Theorem 2].

**Theorem 4.3.** Let \( A \) be a local Noetherian domain with Krull dimension \( d > 1 \). Then the following statements are equivalent.

1. \( A \) is finitely dominated by an analytically ramified local Noetherian stable ring.
2. \( A \) is finitely dominated by an analytically ramified one-dimensional local Noetherian ring.
3. \( A \) is tightly dominated by an analytically ramified one-dimensional local Noetherian ring.
4. \( A \) is finitely dominated by a DVR.
5. The dimension of the generic formal fiber of \( A \) is \( d - 1 \).

If also \( A \) is excellent, then the stable ring in (1) can be chosen to have embedding dimension \( d \) but no bigger. Moreover, these five equivalent conditions are satisfied when \( A \) is essentially of finite type over a field and has dimension \( d > 1 \).
The ring in (2) is obtained using a theorem, stated below, of Heinzer, Rotthaus and Sally, which involves intersecting a homomorphic image of the completion of a local domain with the quotient field of the domain. This is a third source of one-dimensional analytically ramified local rings (the other two being immediate extensions of DVRs and kernels of exterior differentials). To formalize this idea, let $A$ be a local Noetherian domain, and suppose that $I$ is an ideal of $\hat{A}$ such that every associated prime $P$ of $I$ satisfies $A \cap P \neq 0$. Then the canonical mapping $A \to \hat{A}/I$ is an embedding, and we can identify $A$ with its image in $\hat{A}/I$. Under this identification, since the associated primes of $I$ contract to 0 in $A$, it follows that the nonzero elements of $A$ are nonzerodivisors in $\hat{A}/I$. Therefore, the quotient field $F$ of $A$ can be viewed as a subring of the total quotient ring of $\hat{A}/I$, and hence we may consider the ring $R = F \cap (\hat{A}/I)$.

**Theorem 4.4.** (Heinzer-Rotthaus-Sally [18, Corollary 1.27]) Let $A$ be a local Noetherian domain with quotient field $F$, and let $I$ be an ideal of $\hat{A}$ such that $\dim(\hat{A}/I) = 1$ and every associated prime $P$ of $I$ satisfies $A \cap P \neq 0$. Then $R = \hat{A}/I \cap F$ is a one-dimensional local Noetherian domain with $\hat{R} \cong \hat{A}/I$, and if $I$ is properly contained in its radical, then $R$ is analytically ramified.

Thus if $A$ has dimension $d > 1$ and the generic formal fiber of $A$ has dimension $d - 1$ (as is the case in the context of Theorem 4.3), then we may choose a prime ideal $P$ of $\hat{A}$ such that $\hat{A}/P$ has dimension 1. For any $P$-primary ideal $I$ of $\hat{A}$ properly contained in $P$, the theorem shows that $R = \hat{A}/I \cap F$ is a one-dimensional analytically ramified local Noetherian ring. If also $P^2 \subseteq I$, then $R$ is a stable domain [44, Theorem 5.3]. Moreover, if the generic formal fiber of $A$ has $n$ distinct prime ideals $P_1, \ldots, P_n$ of dimension 1, then choosing $I = Q_1 \cap \cdots \cap Q_n$, where each $Q_i$ is $P_i$-primary and at least one $Q_i$ is not prime, yields a one-dimensional analytically ramified local domain whose normalization has $n$ maximal ideals. Such an example can be realized for example whenever $A$ is a countable local Noetherian domain of Krull dimension $>1$. For in this case, there exists for each $n > 0$, $n$ distinct prime ideals of dimension 1 in the generic formal fiber of $A$ [18, Proposition 4.10]. Lech’s construction in Theorem 2.2 is another way to produce one-dimensional analytically ramified local Noetherian domains with normalization not a local ring.

**5 A geometric example**

As discussed in Section 2, the local rings of points of varieties are analytically unramified, and hence one does not encounter local Noetherian domains without finite normalization in a direct way in geometric contexts. However, Reguera has recently shown that local rings of certain points in the space of arcs of an irreducible curve are analytically ramified. The space of arcs of a variety, introduced by Nash in [37] to study the geometry of the singular locus of a variety, encodes information about the
exceptional divisors of desingularizations of the variety. This space, which is in fact a scheme, is somewhat mysterious, since it need not be of finite type over a field, or even Noetherian. We discuss in a very limited way the space of arcs of an affine scheme in order to give some context for the example of Reguera.

Let $X$ be a separated scheme of finite type over a perfect field $k$, and let $K/k$ be a field extension. A $K$-arc on $X$ over $k$ is a $k$-morphism $\text{Spec } K[[T]] \to X$. Associated to $X$ is a scheme $X_\infty$, the space of $K$-arcs of $X$ over $k$, whose $K$-rational points are the $K$-arcs on $X$. The scheme $X_\infty$ is constructed as a direct limit of spaces of “truncated arcs,” but we omit this description here, and give instead an interpretation of $X_\infty$ in the case where $X$ is affine; see [52] for the more general version.

If $X$ is the affine space $\text{Spec } k[X_1, \ldots, X_m]/I$, with $I$ an ideal of $k[X_1, \ldots, X_m]$, then $X_\infty$ can described as follows. For each $i \geq 0$, let $X_i^\infty = (X_{1,i}, \ldots, X_{m,i})$ be an $m$-tuple of indeterminates for $k$. For each $f \in I$, we have

$$f(\sum_{i=0}^{\infty} X_{1,i}T^i, \ldots, \sum_{i=0}^{\infty} X_{m,i}T^i) = \sum_{i=0}^{\infty} F_iT^i,$$

for some $F_i \in k[X_0, \ldots, X_i, \ldots]$. Then $X_\infty$ is the affine scheme $X_\infty = \text{Spec } A$, where

$$A = k[X_0, \ldots, X_i, \ldots]/(\{F_i\}_{i \geq 0, f \in I}).$$

The space $X_\infty$ parameterizes the arcs of $X$ in the following way. For $P$ a prime ideal of $A$, let $\kappa(P)$ denote the residue field of $P$, and let $\phi_P : A \to \kappa(P)$ be the canonical homomorphism. Then corresponding to $P$ is the $\kappa(P)$-arc $\alpha_P : \text{Spec } \kappa(P)[[T]] \to X$ induced by the ring homomorphism

$$\alpha_P^\# : k[X_1, \ldots, X_m]/I \to \kappa(P)[[T]] : X_t + I \mapsto \sum_{i=0}^{\infty} \phi_P(X_{t,i})T^i.$$

Thus there is a morphism of schemes $j : X_\infty \to X$ which sends $P \in X_\infty$ to the image under $\alpha_P$ of the closed point of $\text{Spec } \kappa(P)[[T]]$; i.e., $j$ sends $P$ to the center of the arc $\alpha_P$ on $X$.

Reguera considers the structure of the ring $A$, as well as $A/N(A)$, with $N(A)$ the nilradical of $A$, and shows that for the cusp $f(X, Y) = X^3 - Y^2$, there is a localization $A_P$ of $A$ at a height one prime ideal $P$ such that $B = A_P/N(A_P)$ is a one-dimensional analytically ramified local Noetherian domain of embedding dimension 2. This example is abstracted into a more general result, which we state below in Theorem 5.2. We first sketch the example, and refer to [52, Corollary 5.6] for the extensive theory and calculations which justify the example.

**Example 5.1.** (Reguera [52]) Let $X$ and $Y$ be indeterminates for $\mathbb{C}$, let $f(X, Y) = X^3 - Y^2$ and let $C$ be the affine curve $\text{Spec } \mathbb{C}[X, Y]/(f)$. Then $C_\infty = \text{Spec } A$, where

$$A = \mathbb{C}[X_0, Y_0, X_1, Y_1, \ldots]/(F_0, F_1, \ldots)$$
and as above,

\[
\left( \sum_{i=0}^{\infty} X_i T_i \right)^3 - \left( \sum_{i=0}^{\infty} Y_i T_i \right)^2 = \sum_{i=0}^{\infty} F_i T_i.
\]

In particular, \( F_0 = X_0^3 - Y_0^2 \) and \( F_1 = 3X_0^2X_1 - 2Y_0Y_1 \). Let

\[
P = \sqrt{(X_0, X_1, Y_0, Y_1, Y_2)}.\]

Then \( P \) is a prime ideal of \( A \) [52, Example 3.16]. The ring \( A_P \) is not a Noetherian ring, but \( A_P/N(A_P) \) is a one-dimensional analytically ramified Noetherian domain of embedding dimension 2 [52, Corollary 5.6].

The prime ideal \( P \) in the example also has geometric significance: it is the generic point of the closed subset of \( C_\infty \) consisting of the arcs on \( C \) centered at the origin [52, Corollary 5.6]. The ideas behind the example yield a more general result:

**Theorem 5.2.** (Reguera [52, Corollary 5.7]) Let \( C \) be an irreducible formal curve of multiplicity \( e \geq 2 \) over a field \( k \) of characteristic 0, and let \( P \) be the generic point of the irreducible closed subset consisting of the arcs centered at a singular point of \( C \). Then for \( B = O_{C_\infty, P} \), the ring \( B/N(B) \) is a one-dimensional analytically ramified local Noetherian domain of embedding dimension \( e \).

### 6 Strongly twisted subrings of local Noetherian domains

Let \( S \) be a ring, and let \( L \) be an \( S \)-module. When \( D : S \to L \) is a derivation, then for each \( S \)-submodule \( K \) of \( L \), the pullback \( D^{-1}(K) \) is a subring of \( S \). This simple observation is used in a sophisticated way by Ferrand and Raynaud in their 1970 article to construct analytically ramified local Noetherian domains of dimension 1 and 2. (A third example in dimension 3 was also constructed, but it was produced from the two-dimensional example rather than directly from a derivation.) The ring in Theorem 3.7 is the one-dimensional example. In that case \( S = \mathbb{C}\{X\} \), \( L \) is a finite dimensional vector space over the field \( \mathbb{C}((X)) \), and \( K \) is a free module over \( \mathbb{C}[[X]] \) of the same rank as the dimension of \( L \). The ring \( R \) in the theorem is then \( R = D^{-1}(K) \), and the fact that \( R \) works as advertised in the theorem is the real content of the construction. That the ring \( R \) is Noetherian is subtle and sensitive to the setting here, and in general, because the derivation is not an \( R \)-module map, the connection between module-theoretic properties of \( K \) and ideal-theoretic properties of \( R \) is opaque. Ferrand and Raynaud’s approach to this difficulty is to single out in a technical lemma what specifically makes their example work [13, Lemme 2.1]. Because of its length we do not reproduce their lemma here, but ultimately it asserts that \( \hat{R} \cong \hat{S} \ast \hat{K} \) (where \( \ast \) represents Nagata idealization, as discussed in Section 3). To do so it demands much of \( S, K \) and \( L \), and even the resulting pullback \( R \); e.g., it requires \textit{a priori} that \( R \) is a
Noetherian ring, which presents significant challenges to applying the lemma. In any case, the example in Theorem 3.7 just meets the requirements of their lemma.

Remarkably, however, Ferrand and Raynaud find a two-dimensional UFD $S$ that also satisfies the criteria of the lemma, and as a consequence they produce a two-dimensional local Noetherian domain $R$ whose completion fails to be analytically unramified in a dramatic way, in that it has embedded primes. In this case $S = \mathbb{C}\{X,Y\}$, $L$ is a vector space over the quotient field of $S$, and $K$ is a $V$-submodule of $L$, where $V$ is a DVR tightly dominating $S$. The derivation $D : S \to K$ is carefully chosen to force $R = D^{-1}(K) \cap S$ to be a Noetherian ring [13, Proposition 3.3]. The method of Ferrand and Raynaud was abstracted and improved upon by Goodearl and Lenagan in 1989 to obtain more examples in dimensions 1 and 2 (see also Section 3). A good bit of the method of Goodearl and Lenagan can be fit into the larger framework of twisted subrings discussed below. However, as discussed in Section 2, Goodearl and Lenagan also extended their method in dimension 1 to higher order derivations, and as a consequence were able to construct one-dimensional analytically ramified local Noetherian domains whose multiplicity and embedding dimension differ. By contrast, the method of strongly twisted subrings as outlined below and demonstrated in Section 3 produces stable domains in dimension 1, and multiplicity and embedding dimension coincide for such rings.

Goodearl and Lenagan also abstracted the method of Ferrand and Raynaud to dimension 2, and provided a wider class of examples of two-dimensional local Noetherian domains whose completions have an embedded prime. Their method produces the following example. Let $k$ be a field, let $X$ be an analytic indeterminate for $k$ and let $y, z \in Xk[[X]]$ be algebraically independent over $k(X)$. Let $U = k(X, z) \cap k[[X]]$, let $S$ be the localization of $U[y]$ at the maximal ideal $(X, y)$, and let $d : S \to k((X))$ be the restriction of the partial derivative $\partial / \partial z$ on $k(X, y, z)$ to $S$. Then $R = d^{-1}(k[[X]] \cap k(X, y, z))$ is a two-dimensional analytically ramified local Noetherian domain with normalization $S$ [14, p. 494]. This example is an instance of something more general:

**Proposition 6.1.** (Goodearl-Lenagan [14, p. 494]) If $X, Y, Z$ are indeterminates for the field $k$, then there exists a two-dimensional analytically ramified local Noetherian domain $R$ containing $k[X,Y,Z]$ with quotient field $k(X,Y,Z)$ whose normalization is a regular local ring.

In [43], derivations are used to construct large classes of Noetherian domains without finite normalization. The method allows for strong control over the constructed ring, and as with the methods discussed above, realizes the ring as a subring of a “standard” Noetherian domain $S$. The method can be arranged to produce rings of any Krull dimension, but the only way of which I am aware to use the method to produce rings of dimension $> 1$ is with rather specific assumptions on the quotient field of $S$, as evidenced by Theorem 6.3. In dimension 1, the method is much easier to implement. Ultimately the technical reason for this is that immediate extensions of DVRs
are more easily found in nature than are examples of the higher dimensional analogue of “strongly analytic” extensions. Strongly analytic extensions are discussed later in this section.

More formally, we begin with a domain $S$ having quotient field $F$ and a derivation $D$ from $F$ to a torsion-free divisible $S$-module $L$. As in the method of Ferrand and Raynaud, we choose a submodule $K$ of $L$ and define $R = S \cap D^{-1}(K)$. The ring $R$ then is the object of interest. To obtain control over the ideal-theoretic traits of $R$, the derivation $D$ needs to be somewhat special:

**Definition 6.2.** Let $S$ be a domain with quotient field $F$, and let $R$ be a subring of $S$. Let $K$ be a torsion-free $S$-module, and let $FK$ denote the divisible hull $F \otimes_S K$ of $K$. We say that $R$ is **strongly twisted by $K$** if there is a derivation $D : F \to FK$ such that:

(a) $R = S \cap D^{-1}(K)$,
(b) $D(F)$ generates $FK$ as an $F$-vector space,
(c) $S \subseteq \text{Ker } D + sS$ for all $0 \neq s \in S$.

The derivation $D$ **strongly twists $R$ by $K$**.

The reason for the adverb “strongly” here is that there is a weaker notion of a subring twisted along a multiplicatively closed subset of $S$; we discuss this later in the section. Parsing the above definition shows that the real demand on $D$ occurs in (c). This is a strong property, and it is what makes the requirements of the definition a challenge to satisfy. By contrast, (a) asserts nothing, since we may simply define $R$ to be $S \cap D^{-1}(K)$. (Recall from our earlier discussion that $D^{-1}(K)$ is always a ring.) The criterion (b) can also easily be arranged: If $D(F)$ falls short of generating all of $FK$, then, setting $L$ to be the $F$-subspace of $FK$ generated by $D(F)$, we may replace $K$ with $L \cap K$, and doing so will not change $R$.

It also follows from the definition that if there is some nonzero torsion-free module $K$ such that the subring $R$ of $S$ is strongly twisted by $K$, then there are many strongly twisted subrings of $S$. Indeed, it is easy to see that for every torsion-free $S$-module $J$ with rank$(J) \leq$ rank$(K)$, there exists a subring of $S$ that is strongly twisted by $J$ [43, Lemma 3.1]. Thus the issue for finding strongly twisted subrings of the Noetherian domain $S$ is whether there exists a derivation fulfilling the requirements of the definition. When $S$ is a DVR with quotient field $F$, then the existence of the derivation depends entirely on whether there is a DVR subring $U$ of $S$ with quotient field $Q$ such that $U \subseteq S$ is an immediate extension and $\Omega_{F/Q} \neq 0$; see Theorem 3.10 and Corollary 3.11. But when $S$ has dimension $>1$, then the only way I know to satisfy the requirements of the definition is via the following theorem, which guarantees the existence of a strongly twisted subring when the quotient field of $S$ is sufficiently large and of positive characteristic.

**Theorem 6.3.** [43, Lemma 3.4 and Theorem 3.5] Let $F/k$ be a field extension such that $k$ has positive characteristic and at most countably many elements. Suppose that the
cardinality of \( F \) and the dimension of the \( F \)-vector space \( \Omega_{F/k} \) are the same (which is the case if \( F/k \) is a separably generated extension of infinite transcendence degree). If \( S \) is a \( k \)-subalgebra of \( F \) with quotient field \( F \) and \( K \) is a torsion-free \( S \) module of at most countable rank, then there exists a subring \( R \) of \( S \) that is strongly twisted by \( K \).

The proof of the theorem reduces to proving that under the hypotheses on \( S \) and its quotient field, there exists a subring \( A \) of \( S \) such that the \( A \)-module \( S/A \) is torsion-free and divisible and the extension \( A \subseteq S \) has trivial generic fibers; i.e., \( P \cap A \neq 0 \) for all nonzero prime ideals \( P \) of \( S \). In [43], such an extension is termed a strongly analytic extension. It is the condition of having trivial generic fibers where positive characteristic is needed, so as to arrange \( S \) to be a purely inseparable extension of \( A \). Strongly analytic extensions of DVRs are exactly the immediate extensions, but in higher dimensions, the only strongly analytic extensions of Noetherian rings I am aware of are those constructed in the proof of the theorem.

Strongly analytic extensions provide an alternative, derivation-free way to view strongly twisted subrings: A subring \( R \) of the domain \( S \) is strongly twisted by an \( S \)-module if and only if there is a subring \( A \) of \( R \) such that \( A \subseteq S \) is a strongly analytic extension, \( R \) and \( S \) share the same quotient field and the extension \( R \subseteq S \) is quadratic, meaning that every \( R \)-submodule between \( R \) and \( S \) is a ring [43, Corollary 2.6].

Theorem 6.3 assures the existence of easy-to-locate strongly twisted subrings. For example, if \( k \) is a field of positive characteristic that is a separably generated extension of infinite transcendence degree over a countable subfield, and \( X_1, \ldots, X_n \) are indeterminates for \( k \), then every ring \( S \) between \( k[X_1, \ldots, X_n] \) and \( k((X_1, \ldots, X_n)) \) has strongly twisted subrings; in fact, for each such ring \( S \), there exists for each nonzero torsion-free \( S \)-module of at most countable rank a subring of \( S \) that is strongly twisted by \( K \).

Granted existence, the next theorems deal with the properties of strongly twisted subrings, and we see that these are determined by the choice of \( K \) and \( S \). Although we are mainly interested in the Noetherian case, a few general facts can be stated for strongly twisted subrings of a not-necessarily-Noetherian domain. The extension \( R \subseteq S \), where \( R \) is a strongly twisted subring of \( S \), proves to be a particularly strong sort of integral extension. It is, first of all, a quadratic extension, as defined above; that is, for all \( s, t \in S \), it is the case that \( st \in sR + tR + R \). A quadratic extension is clearly an integral extension. In our context these quadratic extensions are also subintegral, in the sense of Swan [60]: In addition to the extension \( R \subseteq S \) being integral, the contraction mapping \( \text{Spec}(S) \rightarrow \text{Spec}(R) \) is a bijection and the induced maps on residue field extensions are isomorphisms (so for every prime ideal \( P \) of \( S \), \( S_P = R_{P \cap R} + PS_P \)). This is included in the following theorem, which collects together a number of observations from [43].

**Theorem 6.4.** [43, Lemma 4.1 and Theorems 4.2 and 4.4] Let \( S \) be a domain with quotient field \( F \), let \( K \) be a nonzero torsion-free \( S \)-module and let \( FK = F \otimes_R K \).
Suppose that $R$ is a subring of $S$ that is strongly twisted by $K$, and let $D$ be the derivation that twists $R$. Then:

1. $R$ and $S$ share the same quotient field.
2. The extension $R \subseteq S$ is subintegral and quadratic.
3. The derivation $D$ induces an isomorphism of $R$-modules $S/R \rightarrow FK/K$.
4. Every ring $T$ between $R$ and $S$ must be of the form $T = S \cap D^{-1}(L)$ for some unique $S$-module $L$ with $K \subseteq L \subseteq FK$.
5. Given an $S$-submodule $L$ with $K \subseteq L \subseteq FK$, there is an intermediate ring $R \subseteq T \subseteq S$ such that $T$ is strongly twisted by $L$.
6. The ring $S$ is a finite $R$-module only when $R = S$.

Whether a strongly twisted subring $R$ of a Noetherian ring $S$ is Noetherian is determined by the module $K$. This is the content of the next theorem, which relies on the following idea. When $R$ is a subring of $S$ strongly twisted by $K$ and $D$ is the derivation that twists $R$, then there is an embedding $R \rightarrow S \star K : r \mapsto (r, D(r))$. This mapping is faithfully flat, and for each $0 \neq a \in S \cap \text{Ker } D$, it induces an isomorphism $R/aR \rightarrow S/aS \star K/aK$ [43, Theorem 4.6]. Thus the mapping $R \rightarrow S \star K$ is “locally” an isomorphism, a fact which is behind many of the results mentioned in this section. In this way, a strongly twisted subring of $S$ is a kind of inversion of idealization: Rather than ramify $S$ with $K$ to produce $S \star K$, we use $K$ to excavate a subring $R$ of $S$ which behaves in small enough neighborhoods like $S \star K$.

**Theorem 6.5.** [43, Theorem 5.2] Let $S$ be a domain, let $K$ be a torsion-free $S$-module, and suppose that $R$ is a subring of $S$ strongly twisted by $K$. Let $D$ be the derivation that strongly twists $R$. Then $R$ is a Noetherian domain if and only if $S$ is a Noetherian domain and for each $0 \neq a \in S \cap \text{Ker } D$, $K/aK$ is a finite $S$-module.

Thus if $S$ is a Noetherian domain and $K$ is a finitely generated $S$-module, then $R$ is a Noetherian domain. We will outline this case in Theorem 6.7. However, the theorem leaves just enough room for other, more subtle, choices of $K$; one such case is treated in Theorem 6.9. So we elaborate shortly on the structural relationship between $R$, $S$ and $K$ in two cases that depend on the choice of $K$. But both cases will force $R$ to be Noetherian, so we consider first in Theorem 6.6 the general situation in which $R$ is Noetherian, or, equivalently, by Theorem 6.5, the case in which $K/aK$ is a finite $S$-module for all $0 \neq a \in S \cap \text{Ker } D$.

To state Theorem 6.6, we recall the following standard notions. Let $I$ be an ideal of a ring $A$. Then an ideal $J \subseteq I$ is a reduction of $I$ if there exists $n > 0$ such that $I^{n+1} = JI^n$. The smallest number $n$ for which such an equation holds for $J$ and $I$ is the reduction number of $I$ with respect to $J$. The ideal $J$ is a minimal reduction of $I$ if $J$ itself has no proper reduction. If $I$ is an ideal of a local Noetherian ring, then minimal reductions must exist [61, Theorem 8.3.5]. The minimum of the reduction
numbers of the minimal reductions of the ideal $I$ of $A$ is denoted $r_A(I)$. The analytic spread of an ideal $I$ in a local Noetherian ring $(A, \mathfrak{m})$, denoted $\ell_A(I)$, which is useful in detecting minimal reductions (see for example [6] or [61]), is defined to be the Krull dimension of the fiber cone of $I$ with respect to $A$, where for an ideal $I$ and an $A$-module $L$, the fiber cone of $I$ with respect to $L$ is:

$$F_{I,L} := L[It]/\mathfrak{m}L[It] \cong \bigoplus_{n=0}^{\infty} I^nL/\mathfrak{m}I^nL.$$ 

In addition to facts about reductions of ideals in strongly twisted subrings, Theorem 6.6 calculates the local cohomology of such rings: If $A$ is a local ring and $I$ is an ideal of $A$, then $H^i_I(L)$ is the right derived functor of the $I$-torsion function $\Gamma_I$ defined for each $A$-module $L$ by $\Gamma_I(L) = \{ \ell \in L : I^k\ell = 0 \text{ for some } k > 0 \}$. If $I$ is an ideal of the local Noetherian ring $A$ and $L$ is an $A$-module, then $\text{depth}_I(L)$ is the greatest integer $i$ such that for all $j < i$, $H^j_I(L) = 0$. When $L$ is a finite $A$-module, then $\text{depth}_I(L)$ is the length of a maximal regular sequence on $L$ [5, Theorem 6.27].

The underlying theme of the next theorem, as well as Theorems 6.7 and 6.9, is that various properties of nonzero ideals $I$ of a strongly twisted subring $R$ of $S$ which are contracted from $S$ are determined by $IS$ and $K$. We use “contracted” here in the basic sense: The ideal $I$ of $R$ is contracted from $S$ if there is an ideal $J$ of $S$ such that $I = J \cap R$; equivalently, $I = IS \cap R$. Since $R \subseteq S$ is by Theorem 6.4 an integral extension, every integrally closed ideal of $R$ is contracted from an ideal of $S$. In particular, every prime ideal of $R$ is contracted from an ideal of $S$. Statement (7) of the theorem can be found in [43, Corollary 4.3]; (8) in [43, Corollary 4.7]; and the remaining statements are collected from [46, Sections 3 and 4].

**Theorem 6.6.** Let $S$ be a local Noetherian domain with maximal ideal $N$ and quotient field $F$, and let $K$ be a nonzero torsion-free $S$-module. Suppose that $R$ is a Noetherian subring of $S$ that is strongly twisted by $K$. Then statements (1)–(6) of Theorem 6.4 hold for $R$, as do all of the following statements.

(7) $R$ is a local ring; $R$ and $S$ have the same residue field; and $N = MS$, where $M$ is the maximal ideal of $R$.

(8) There is an isomorphism of rings: $\hat{R} \cong \hat{S} \ast \hat{K}$, where $\hat{R}$ is the completion of $R$ in the $M$-adic topology and $\hat{S}$ and $\hat{K}$ denote the completions of $S$ and $K$, respectively, in the $N$-adic topology.

(9) For each nonzero ideal $I$ of $R$ contracted from $S$, there is an isomorphism of rings:

$$F_{I,R} \cong F_{IS,S} \ast F_{IS,K}.$$ 

(10) For each nonzero ideal $I$ of $R$ contracted from $S$, $\ell_R(I) = \ell_S(IS)$.

(11) Suppose $R$ (equivalently, $S$) has infinite residue field. If $I$ is a nonzero ideal of $R$ contracted from $S$, then $r_S(IS) \leq r_R(I) \leq r_S(IS) + 1$. 

If $I$ is a nonzero ideal of $R$, then there is an isomorphism of $R$-modules:

$$H^1_i(R) \cong H^1_i(S) \oplus H^1_i(K).$$

If $I$ is an ideal of $R$, then $\text{depth}_I(R) = \min\{\text{depth}_{IS}(S), \text{depth}_{IS}(K)\}$.

Examples show the bound on $r_R(I)$ in (11) cannot be improved [46, Section 3]. Note also that (11) implies that if $I$ is an ideal of $R$ contracted from a principal ideal of $S$, then $I^2 = iI$ for some $i \in I$. Thus, in the terminology of Section 3, $I$ is a stable ideal of $R$. Therefore, if $S$ is a DVR, then $R$ is a stable domain. This is one reason for the emphasis on stable domains in Section 3, since the existence results in that section which involve stable domains rely on the method of strongly twisted subrings to produce these rings. More generally, strongly twisted subrings of Dedekind domains must be stable domains [43, Theorem 7.1].

We specialize next to the case where $K$ is a finitely generated $S$-module. Since by Theorem 6.5, a subring $R$ of a Noetherian domain $S$ strongly twisted by a finitely generated $S$-module $K$ is Noetherian, all the properties listed in Theorem 6.6 hold for $R$. But the fact that $K$ is finitely generated allows for the stronger conclusions of the next theorem, which utilizes the following standard terminology. When $(A, m)$ is a local Noetherian ring and $L$ is a finitely generated $A$-module, then $\mu(L)$ denotes the minimal number of generators needed to generate $L$; that is $\mu(L)$ is the dimension of $A/m$-vector space $L/mL$. The embedding dimension of $A$, denoted $\text{emb.dim}_A$, is $\mu(m)$. When $J$ is an $m$-primary ideal of $A$, the Hilbert function of the module $L$ with respect to $J$, denoted $H_{J,L}$, is given then by

$$H_{J,L}(n) = \text{length } J^n L / J^{n+1} L,$$

with the convention $J^0 = A$. For large enough $n$, the Hilbert function $H_{J,L}$ agrees with a polynomial having rational coefficients [12, Proposition 12.2 and Exercise 12.6]. The leading coefficient of this polynomial is of the form $e/(d-1)!$ for some positive integer $e$, which is designated the multiplicity of $J$ on $L$. We write $e(J, L)$ for $e$ to emphasize the dependence on $J$ and $L$. The multiplicity of the local ring $A$ is denoted $e(A)$, and is defined by $e(A) = e(m, A)$. Statements (14)–(17) of the next theorem can be found in [46, Section 6]; (18)–(23) can be deduced from the preceding statements and [43, Theorem 7.2]; (24) is proved in [43, Corollary 7.3]; (25) follows from [43, Proposition 5.3].

**Theorem 6.7.** Let $S$ be a local Noetherian domain with maximal ideal $N$ and quotient field $F$, and let $K$ be a nonzero finitely generated torsion-free $S$-module. Suppose that $R$ is a subring of $S$ strongly twisted by $K$. Then $R$ is a local Noetherian domain and statements (1)–(6) of Theorem 6.4 and statements (7) - (13) of Theorem 6.6 hold for $R$, as do all of the following statements.

- If $I$ is a nonzero ideal of $R$ contracted from $S$, then $\mu_R(I) = \mu_S(IS) + \mu_S(K)$. 


(15) \( \text{emb.dim } R = \text{emb.dim } S + \mu_S(K) \).

(16) If \( I \) and \( J \) are ideals of \( R \) contracted from \( S \), with \( J \) an \( M \)-primary ideal, then:

\[
\nu(I, J) = \nu(IS, S) \cdot (1 + \text{rank}(K)).
\]

(17) The multiplicity of \( R \) is \( \nu(R) = \nu(S) \cdot (1 + \text{rank}(K)) \).

(18) \( R \) is a Cohen-Macaulay ring if and only if \( S \) is a Cohen-Macaulay ring and \( K \) is a maximal Cohen-Macaulay \( S \)-module.

(19) \( R \) is a Gorenstein ring if and only if \( S \) is a Cohen-Macaulay ring that admits a canonical module \( \omega_S \) and \( K \cong \omega_S \). Moreover, if \( R \) is a Gorenstein ring, then \( \nu(R) = 2\nu(S) \).

(20) \( R \) is a complete intersection if and only if \( S \) is a complete intersection and \( K \cong S \).

(21) \( R \) is a hypersurface if and only if \( S \) is a regular local ring and \( K \cong S \).

(22) If \( R \) is a hypersurface, then \( R \) has minimal multiplicity, and in fact, \( \nu(R) = 2 \) and \( \text{emb.dim } R = d + 1 \), where \( d \) is the dimension of \( S \).

(23) If \( S \) is a regular local ring, and \( K \) is a finitely generated free \( S \)-module, then \( R \) is a Cohen-Macaulay ring of minimal multiplicity.

(24) The Cohen-Macaulay rings properly between \( R \) and \( S \) are in one-to-one correspondence (see Theorem 6.4(7) and (8)) with the maximal Cohen-Macaulay modules properly between \( K \) and \( FK \).

(25) If \( S \) has Krull dimension \( > 1 \), then there exists a non-Noetherian ring between \( R \) and \( S \).

The following example illustrates the theorem with a simple choice of \( S \) and \( K \).

**Example 6.8.** Let \( k \) be a field of postive characteristic that is a separably generated extension of infinite transcendence degree over a countable subfield. Let \( X_1, \ldots, X_d \) be indeterminates for \( K \), and define \( S = k[X_1, \ldots, X_d]((X_1, \ldots, X_d)) \). Let \( K \) be a free \( S \)-module of rank \( n > 0 \). Then by Theorems 6.3, 6.4 and 6.6, there is a local Noetherian subring \( R \) of \( S \) strongly twisted by \( K \) such that every nonzero integrally closed ideal of \( R \), in particular every prime ideal of \( R \), needs at least \( n + 1 \) generators. Moreover, \( R \subseteq S \) is a quadratic subintegral extension, and \( R \) and \( S \) both share the same quotient field. The embedding dimension of \( R \) is \( d + n \), its multiplicity is \( 1 + n \) and \( R \) is a Cohen-Macaulay ring. If \( n \) is chosen to be \( 1 \), then \( R \) is a hypersurface.

One of Nagata’s bad Noetherian rings is a two-dimensional local Noetherian domain \( R \) that has a non-Noetherian domain between \( R \) and its normalization [36, Example 4, p. 206]. Theorem 6.7(25) provides another source of such examples; e.g., set \( d = 2 \) in Example 6.8.
Next we consider a case in which $K$ is not finitely generated yet still produces a strongly twisted subring that is Noetherian. Though not finitely generated, the module $K$ has the property that $K/sK$ is finitely generated for all $0 \neq s \in S$, which in turn guarantees by Theorem 6.5 that $R$ is a Noetherian ring. In addition to being an $S$-module, $K$ is also a $V$-module, where $V$ is a DVR that finitely dominates $S$. Such DVRs are the subject of Theorem 4.3, and we recall from that theorem that if $S$ is essentially of finite type over a field, then such a DVR must exist. This assures that there are plenty of natural choices for $S$ and $K$ to which the construction can be applied. That the ring $R$ in the theorem is Noetherian follows from Theorem 6.5; for an explicit argument see [43, Example 5.4]. Statements (14) and (16) can be found in [46, Section 7]; (15) and (17) follow from (14) and (16), respectively; (18) is in [43, Proposition 6.4]; (19) follows from (18); (20) and (21) follow from (19); (22) is proved in [45, Proposition 6.6]; (23) can be found in [46, Section 7]; (24) follows from (23); and (25) is in [43, Proposition 5.6].

**Theorem 6.9.** Let $S$ be a local Noetherian domain with maximal ideal $N$ and quotient field $F$, and such that there exists a DVR $V$ that finitely dominates $S$. Let $m = \dim_{S/N} V/NV$, let $K$ be a nonzero torsion-free finite rank $V$-module with $K \neq FK$, and let $r_K = \dim_{V/\mathfrak{M}} K/\mathfrak{M}K$, where $\mathfrak{M}$ is the maximal ideal of $V$. Suppose that $R$ is a subring of $S$ that is strongly twisted by $K$. Then $R$ is a local Noetherian domain and statements (1)–(6) of Theorem 6.4 and statements (7)–(13) of Theorem 6.6 hold for $R$, as do all of the following statements.

14. For each nonzero ideal $I$ of $R$ contracted from $S$, $\mu_R(I) = \mu_S(IS) + m \cdot r_K$, and if $K$ is a free $V$-module, then $r_K = \text{rank}(K)$.

15. $\text{emb.dim } R = \text{emb.dim } S + m \cdot r_K$.

16. If $I$ and $J$ are ideals of $R$ contracted from $S$ and $J$ is $M$-primary, then:

$$e(J, I) = e(JS, S) + \begin{cases} r_K \cdot \text{length } V/JV & \text{if } S \text{ has Krull dimension } 1 \\ 0 & \text{if } S \text{ has Krull dimension } > 1. \end{cases}$$

17. The multiplicity of the local ring $R$ is

$$e(R) = e(S) + \begin{cases} m \cdot r_K & \text{if } S \text{ has Krull dimension } 1 \\ 0 & \text{if } S \text{ has Krull dimension } > 1. \end{cases}$$

18. For each nonmaximal prime ideal $P$ of $R$, $R_P = S_{P'}$, where $P'$ is the unique prime ideal of $S$ lying over $R$.

19. For each $i$ less than the Krull dimension of $S$, Serre’s regularity condition $R_i$ holds for $S$ if and only if it holds for $R$.

20. If $S$ is integrally closed and has Krull dimension $> 1$, then for each height 1 prime ideal $P$ of $R$, $R_P$ is a DVR.
(21) If $S$ is a regular local ring, then $R$ has an isolated singularity.

(22) The maximal ideal $M$ of $R$ is the associated prime of a nonzero principal ideal.

(23) The local cohomology modules for a nonzero ideal $I$ of $R$ are given by

$$H^i_I(R) \cong \begin{cases} 
0 & \text{if } i = 0 \\
H^1_I(S) \oplus FK/K & \text{if } i = 1 \\
H^1_I(S) & \text{if } i > 1.
\end{cases}$$

(24) $\text{depth}_M(R) = 1$.

(25) If $V$ tightly dominates $R$, then every intermediate ring $T$, $R \subseteq T \subsetneq S$, is a local Noetherian ring that is strongly twisted by some $V$-module $L$ with $K \subseteq L \subsetneq FK$.

Here is a simple example:

**Example 6.10.** Let $k$ be a field of positive characteristic, and suppose that $k$ is separably generated and of infinite transcendence degree over a countable subfield. Let $S = k[X_1, \ldots, X_d]_{(X_1, \ldots, X_d)}$, where $d > 1$. Then by Theorem 4.3, there exists a DVR $V$ that meets the requirements of Theorem 6.9. Let $K$ be a torsion-free $V$-module of finite rank $n > 0$. Then by Theorem 6.3, there exists a local Noetherian subring $R$ of $S$ strongly twisted by $K$ and satisfying all the statements of Theorems 6.4, 6.6 and 6.9. Moreover, as discussed at the beginning of Section 4, the DVR $V$ can be chosen to tightly dominate $S$, so that $m = 1$ in Theorem 6.9.

Nagata constructs an example of a local Noetherian domain of multiplicity 1 that is not a regular local ring [36, Example 2, p. 203]. Example 6.10, with $K = S$, is another such example when the DVR $V$ is chosen so that it tightly dominates $S$.

Abyhankar constructs in [1] for each pair of integers $n > d > 1$ a local ring of embedding dimension $n$, Krull dimension $d$ and multiplicity 2. Example 6.10 accomplishes something similar, but with multiplicity 1 and all the rings $R$ occur in subintegral extensions $R \subseteq S$, with $S$ fixed: Choose $V$ to be a DVR tightly dominating $S$ and for $n > d$, choose $K$ to be a free $V$-module of rank $n - d$.

We conclude with a note on generality. To simplify the presentation, we have restricted everything to strongly twisted subrings, but there is a weaker notion of twisted subring for which suitable variations on many of the preceding ideas apply. Let $S$ be a ring, let $K$ be an $S$-module, and let $C$ be a multiplicatively closed subset of nonzerodivisors of $S$ that are also nonzerodivisors on $K$. Then a subring $R$ of $S$ is **twisted by $K$ along $C$** if there is a $C$-linear derivation $D : S_C \to K_C$ such that:

(a) $R = S \cap D^{-1}(K)$,

(b) $D(S_C)$ generates $K_C$ as an $S_C$-module, and

(c) $S \subseteq \text{Ker } D + cS$ for all $c \in C$. 

A strongly twisted subring of a domain $S$ is twisted along $C = S \setminus \{0\}$, but the converse need not be true; more precisely, being twisted along $C = S \setminus \{0\}$ is not sufficient to guarantee condition (c) in the definition of strongly twisted subring.

Relativized versions of many of the properties discussed in this section hold for a subring $R$ of $S$ twisted along $C$. Instead of working with nonzero ideals of $R$ contracted from $S$, one usually must restrict to ideals of $R$ contracted from $S$ and meeting $C$. For example, when $S$ is Noetherian and $K$ is a finitely generated $S$-module, then it is in general possible to conclude only that every ideal of $R$ meeting $C$ is finitely generated. Thus Noetherianness of $R$ is no longer guaranteed; instead, there is a filter of ideals which behave like ideals in a Noetherian ring. However, in the special circumstance when $S$ is a two-dimensional local UFD finitely dominated by a DVR $V$, then a subring $R$ of $S$ twisted by a torsion-free finite rank $V$-module along any multiplicatively closed subset containing a nonunit of $S$ is Noetherian [43, Theorem 5.7]. This is an abstract version of the example of Ferrand and Raynaud of a two-dimensional local Noetherian domain whose completion has embedded primes [13, Proposition 3.1].

**Bibliography**


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