ONE-POINT ORDER-COMPACTIFICATIONS

GURAM BEZHANISHVILI AND PATRICK J. MORANDI

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Abstract. We classify all one-point order-compactifications of a noncompact locally compact order-Hausdorff ordered topological space \( X \). We give a necessary and sufficient condition for a one-point order-compactification of \( X \) to be a Priestley space. We show that although among the one-point order-compactifications of \( X \) there may not be a least one, there always is a largest one, which coincides with the one-point order-compactification of \( X \) described by McCallion [7]. In fact, we prove that whenever \( X \) satisfies the condition given in McCartan [8], then the largest one-point order-compactification of \( X \) coincides with the one described by McCartan [8].

1. Introduction

We recall that an ordered topological space is a triple \( (X, \tau, \leq) \), where \( (X, \tau) \) is a topological space and \( \leq \) is a partial order on \( X \). The study of ordered topological spaces was initiated by Leopoldo Nachbin back in the 1940s, and his main contributions have been collected in [9]. In particular, Nachbin generalized the notion of a completely regular space to that of a completely order-regular space; he also generalized the notion of a compactification to that of an order-compactification, and proved that an ordered topological space has an order-compactification iff it is completely order-regular. This is a generalization of the well-known result of Tychonoff that a topological space has a compactification iff it is completely regular. Nachbin also showed that among the order-compactifications of a completely order-regular space \( X \), there always exists a largest one, which became known as the Nachbin order-compactification of \( X \). We denote it by \( n(X) \). It generalizes the concept of the Stone-\v{C}ech compactification.

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\( \beta(X) \) of a completely regular space \( X \). The least order-compactification, on the other hand, may or may not exist. It is a well-known theorem in topology, which goes back to Alexandroff, that a completely regular noncompact space \( X \) has a least compactification iff the one-point compactification of \( X \) exists, which in turn is equivalent to \( X \) being locally compact (see, e.g., [4, Thms. 3.5.11 and 3.5.12]).

The situation is more complicated for ordered topological spaces. We show that an ordered topological space \( X \) may have no one-point order-compactification, but may have a least order-compactification nevertheless. We give a necessary and sufficient condition for a one-point order-compactification of \( X \) to exist, and give a classification of all one-point order-compactifications of \( X \), which we consider the main contribution of this article. We also provide a necessary and sufficient condition for a one-point order-compactification of \( X \) to be a Priestley space. As a consequence, we obtain that the one-point compactification of a noncompact locally compact space \( X \) is a Stone space iff \( X \) is zero-dimensional. We show that although among the one-point order-compactifications of a noncompact locally compact order-Hausdorff space \( X \) there may not exist a least one, there always exists a largest one, which coincides with the one-point order-compactification described by McCallion [7]. In fact, we prove that if \( X \) satisfies the condition given in McCartan [8], which we call the McCartan condition, then the largest one-point order-compactification of \( X \) coincides with the one described by McCartan [8].

2. ORDER-COMPACTIFICATIONS

Let \( X \) be an ordered topological space and \( A \subseteq X \). We recall that \( \uparrow A = \{x \in X : \exists a \in A \text{ with } a \leq x\} \), that \( \downarrow A = \{x \in X : \exists a \in A \text{ with } x \leq a\} \), that \( A \) is an upset if \( A = \uparrow A \), and that \( A \) is a downset if \( A = \downarrow A \). For \( x \in X \), we also recall that \( N \subseteq X \) is a neighborhood of \( x \) if there exists an open subset \( U \) of \( X \) such that \( x \in U \subseteq N \).

**Definition 2.1.** Let \( X \) be an ordered topological space.

1. [8, p. 966] We call \( X \) order-Hausdorff if for each \( x, y \in X \) with \( x \not\leq y \), there exists an upset neighborhood \( N \) of \( x \) and a downset neighborhood \( M \) of \( y \) such that \( N \cap M = \emptyset \).

2. [9, p. 54] We call \( X \) completely order-regular if (i) for each \( x, y \in X \) with \( x \not\leq y \), there exists a continuous order-preserving \( f : X \rightarrow [0,1] \) such that \( f(x) > f(y) \), and (ii) for each \( x \in X \) and a closed set \( F \) with \( x \notin F \), there exist a continuous order-preserving \( f : X \rightarrow [0,1] \) and a continuous order-reversing \( g : X \rightarrow [0,1] \) such that \( f(x) = 1 = g(x) \) and \( F \subseteq f^{-1}(0) \cup g^{-1}(0) \).
It is well known that the notion of order-Hausdorff generalizes that of a Hausdorff space to ordered topological spaces, and that $X$ is order-Hausdorff iff $\leq$ is closed in $X^2$ (see [9, Prop. 1] or [8, Thm. 2]). It is also clear that the notion of completely order-regular generalizes that of a completely regular space to ordered topological spaces.

**Definition 2.2.** [9, p. 103] Suppose $X$ and $Y$ are ordered topological spaces. We call $Y$ an order-compactification of $X$ if $Y$ is compact order-Hausdorff and $X$ is order-homeomorphic to a dense subspace of $Y$.

It is a well-known theorem of Nachbin that an ordered topological space $X$ has an order-compactification iff $X$ is completely order-regular, which generalizes a similar theorem about completely regular spaces and compactifications to the setting of ordered topological spaces. Let $X$ be completely order-regular, and let $Y$ and $Z$ be two order-compactifications of $X$. We define $Y \leq Z$ if there exists a continuous order-preserving $f : Z \to Y$ such that $f \circ e_Z = e_Y$, where $e_Y : X \to Y$ and $e_Z : X \to Z$ are the order-homeomorphic embeddings. It is easy to verify that $\leq$ is reflexive and transitive. We say that $Y$ is equivalent to $Z$ if $Y \leq Z$ and $Z \leq Y$. It follows from [9, pp. 103–104] that $Y$ is equivalent to $Z$ iff there exists an order-homeomorphism $f : Y \to Z$ such that $f \circ e_Y = e_Z$. Then $\leq$ induces a partial order on the equivalence classes of order-compactifications of $X$. Let $(\mathcal{OC}(X), \leq)$ denote the poset of inequivalent order-compactifications of $X$. One of the fundamental results of Nachbin states that the Nachbin order-compactification $n(X)$ is the largest element of $(\mathcal{OC}(X), \leq)$. We refer to [9, 7, 3, 6, 12] for different constructions of $n(X)$. In the next section we discuss the issue whether or not $(\mathcal{OC}(X), \leq)$ has a least element.

Among order-compactifications of $X$, an important role is played by Priestley order-compactifications of $X$. We recall that an ordered topological space $Y$ is a Priestley space if it is compact and satisfies the Priestley separation axiom: $x \nleq y$ implies there exists a clopen upset $U$ of $X$ such that $x \in U$ and $y \notin U$. Here by clopen we mean a closed and open subset of $X$. Priestley spaces play a fundamental role in the study of distributive lattices, since they serve as dual objects of the category of bounded distributive lattices and bounded lattice homomorphisms [10, 11].

**Definition 2.3.** [2, Def. 3.4] Let $X$ and $Y$ be ordered topological spaces. We call $Y$ a Priestley order-compactification of $X$ if $Y$ is a Priestley space which is an order-compactification of $X$. 
Priestley order-compactifications generalize the concept of zero-dimensional compactifications to the setting of ordered topological spaces. Let \((\mathcal{POC}(X), \leq)\) denote the poset of inequivalent Priestley order-compactifications of \(X\). Clearly \((\mathcal{POC}(X), \leq)\) is a subposet of \((\mathcal{OC}(X), \leq)\). A description of \((\mathcal{POC}(X), \leq)\) is given in [2, Thm. 5.2] by means of Priestley rings of upsets of \(X\). In the next section we give a necessary and sufficient condition for a one-point order-compactification of \(X\) to be a Priestley order-compactification.

### 3. One-point order-compactifications

Let \(X\) be a completely order-regular space. As we already saw, \(n(X)\) is the largest element of \((\mathcal{OC}(X), \leq)\). But does \((\mathcal{OC}(X), \leq)\) have a least element? And if it does, does it have to be obtained from \(X\) by adding a single point to \(X\), like in the case of topological spaces? As we will see shortly, the answer to both questions is negative. We start off by giving an example of a completely order-regular space \(X\) which has a unique (up to equivalence) order-compactification, but it is not a one-point order-compactification. We need the following auxiliary lemma.

**Lemma 3.1.** Let \(X\) be a linearly ordered topological space, and let \(Y\) be an order-compactification of \(X\). Then \(Y\) is also linearly ordered, with a greatest and a least element.

**Proof.** We identify \(X\) with a dense subspace of \(Y\). Because \(X\) is linearly ordered, for each \(x \in X\) we have \(X \subseteq \uparrow x \cup \downarrow x\), and as \(Y\) is order-Hausdorff, \(\uparrow x \cup \downarrow x\) is closed, by [9, Prop. 1] or [8, Thm. 2]. Therefore, since \(X\) is dense in \(Y\), we have \(Y = \uparrow x \cup \downarrow x\). Thus, for each \(x \in X\) and each \(y \in Y - X\) we have \(x \leq y\) or \(y \leq x\). Consequently, for each \(y \in Y\) we have \(Y = \uparrow y \cup \downarrow y\), and so \(Y\) is linearly ordered. If there is no greatest element in \(Y\), then \(\{\uparrow y\}_{y \in Y}\) is a family of closed subsets of \(Y\) with the finite intersection property such that \(\bigcap_{y \in Y} \{\uparrow y\} = \emptyset\), which is impossible since \(Y\) is compact. Therefore, \(Y\) has a greatest element. A similar argument shows that \(Y\) has a least element as well. \(\square\)

**Remark 3.2.** The proof of Lemma 3.1 is essentially the same as the proof of Theorem 2.9 in Esakia [5], although the statement of Lemma 3.1 does not appear in [5].

Now we are ready to give an example of a completely order-regular space \(X\) such that \((\mathcal{OC}(X), \leq)\) has a least element, but it is not a one-point order-compactification of \(X\).
Example 3.3. Consider $(0, 1)$ with its usual order and topology and let $X$ be an order-compactification of $(0, 1)$. It is well-known (and easy to see) that $(0, 1)$ is completely order-regular. By Lemma 3.1, $X$ has a largest element $x$ and a least element $y$. We show that $(0, 1) = X - \{x, y\}$. Since $(0, 1)$ is dense in $X$, there exist $\alpha, \beta \in (0, 1)$ such that $\alpha < z < \beta$. Therefore, either $z = \inf(\uparrow z \cap (0, 1))$ or $z = \sup(\downarrow z \cap (0, 1))$, which by completeness implies that $z \in (0, 1)$. Thus, $X$ is order-homeomorphic to $[0, 1]$, and so $[0, 1]$ is a unique (up to equivalence) order-compactification of $(0, 1)$. In particular, $[0, 1]$ is order-homeomorphic to the Nachbin order-compactification of $(0, 1)$. Consequently, $(0, 1)$ has a smallest order-compactification, but it is not a one-point order-compactification.

Remark 3.4. Since $(0, 1)$ is a noncompact locally compact space, the one-point compactification $\alpha(0, 1)$ of $(0, 1)$ exists. In fact, $\alpha(0, 1)$ is homeomorphic to the circle. Nevertheless, it is impossible to define a closed order on $\alpha(0, 1)$ extending the order on $(0, 1)$.

On the other hand, the next example shows that there are many inequivalent ways to define an order $\leq'$ on the one-point compactification $\alpha(X)$ of a noncompact locally compact ordered topological space $X$ so that $(\alpha(X), \leq')$ is an order-compactification of $X$.

Example 3.5. Let $X = \mathbb{N}$ with trivial order and discrete topology. Clearly $X$ is noncompact and locally compact. Thus, the one-point compactification $\alpha(X) = X \cup \{\infty\}$ of $X$ exists. However, there are uncountably many ways to define an order on $\alpha(X)$ that extends the order on $X$. Here we only indicate two extreme cases. Later on we will classify all possible one-point order-compactifications of $X$. We extend the trivial order on $X$ to two orders $\leq_1$ and $\leq_2$ on $\alpha(X)$ by setting $x \leq_1 \infty$ and $\infty \leq_2 x$, respectively, for all $x \in X$.

Then it is routine to verify that both $(\alpha(X), \leq_1)$ and $(\alpha(X), \leq_2)$ are order-compactifications of $X$, and that $(\alpha(X), \leq_1)$ is not order-homeomorphic to $(\alpha(X), \leq_2)$. 

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\begin{itemize}
\item Example 3.3
\item Remark 3.4
\item Example 3.5
\end{itemize}
Our aim is to classify all such orders on $\alpha(X)$ in general. Let $X$ be a noncompact locally compact order-Hausdorff space, and let $\alpha(X)$ be the one-point compactification of $X$. Let $\mathcal{PO}(\alpha(X))$ denote the set of partial orders $\leq'$ on $\alpha(X)$ such that $(\alpha(X), \leq')$ is an order-compactification of $X$. We order $\mathcal{PO}(\alpha(X))$ by inclusion.

In order to motivate the next definition, for a one-point order-compactification $(\alpha(X), \leq')$ of $X$, we set $A = \uparrow \infty \cap X$ and $B = \downarrow \infty \cap X$. We will see that $(\alpha(X), \leq')$ is characterized by the pair $(A, B)$. In fact, we will show that all one-point order-compactifications of $X$ are characterized by the pairs $(A, B)$ that satisfy the conditions of the next definition.

**Definition 3.6.** For a noncompact locally compact order-Hausdorff space $X$, let $\Pi(X)$ denote the set of pairs $(A, B)$ such that

(i) $A$ is an upset and $B$ is a downset of $X$;
(ii) $A \cap B = \emptyset$;
(iii) For each $a \in A$ and each $b \in B$, we have $b \leq a$;
(iv) If $x \notin A$, then there is an open downset $V$ of $X$ containing $x$ with $\overline{V}$ compact and $V \cap A = \emptyset$;
(v) If $x \notin B$, then there is an open upset $U$ of $X$ containing $x$ with $\overline{U}$ compact and $U \cap B = \emptyset$.

We define $\leq$ on $\Pi(X)$ by $(A, B) \leq (C, D)$ if $A \subseteq C$ and $B \subseteq D$. Then $(\Pi(X), \leq)$ is a poset.

**Remark 3.7.** Clearly Conditions (iv) and (v) imply that $A$ and $B$ are closed. Conversely, if we assume that $A$ is a closed upset, then we can drop the assumption $V \cap A = \emptyset$ in Condition (iv). To see this, let $V' = V \cap A^c$. Then $x \in V'$. Since $A$ is a closed upset, $V'$ is an open downset. Moreover, $\overline{V'} \subseteq \overline{V}$, so $\overline{V'}$ is compact. Finally, it follows from the definition of $V'$ that $V' \cap A = \emptyset$. A similar argument shows that if we assume that $B$ is a closed downset, then we can drop the assumption that $U \cap B = \emptyset$ in Condition (v).

**Remark 3.8.** We also note that in Condition (iv) the open downset $V$ can be selected so that $\overline{V} \cap A = \emptyset$. To see this, suppose $V$ is an open downset of $X$ such that $x \in V$, $\overline{V}$ is compact, and $V \cap A = \emptyset$. Let $\overline{V} \cap A \neq \emptyset$. We consider the space $X' = \overline{V}$ and its closed subset $A' = \overline{V} \cap A$. For each $y \in A'$ we have $y \not\leq x$. Since $X'$ is compact and order-Hausdorff, by [9, Prop. 1] and [1, Prop. 2.3], for each $y \in A'$ there exist an open upset $U_y$ in $X'$ and an open downset $V_y$ in $X'$ such that $y \in U_y$, $x \in V_y$, and $U_y \cap V_y = \emptyset$. Because $A'$ is compact and $\{U_y : y \in A'\}$ is an open cover of $A'$, there exists a finite subcover $\{U_1, \ldots, U_n\}$. 
Theorem 3.9. (Main Theorem) The poset $(\mathfrak{D}(\alpha(X)), \subseteq)$ is isomorphic to the poset $(\Pi(X), \leq)$.

Proof. Define $\varphi: \mathfrak{D}(\alpha(X)) \to \Pi(X)$ as follows. If $\leq \in \mathfrak{D}(\alpha(X))$, then let

$$A = \uparrow \infty \cap X = \{ x \in X : x \leq' x \},$$

$$B = \downarrow \infty \cap X = \{ x \in X : x \leq' \infty \},$$

and set $\varphi(\leq)' = (A, B)$. That the pair $(A, B)$ satisfies the first three conditions of Definition 3.6 is obvious. To see that $(A, B)$ satisfies the fourth, take $x \notin A$. Then $\infty \notin' x$. Since $(\alpha(X), \leq')$ is compact and order-Hausdorff, by [9, Prop. 1] and [1, Prop. 2.3], there is an open upset $U$ and an open downset $V$ of $\alpha(X)$ such that $\infty \in U$, $x \in V$, and $U \cap V = \varnothing$. Because $\infty \notin V$, we see that $V$ is an open downset of $X$. Let $W$ be the closure of $V$ in $\alpha(X)$ and let $V$ be the closure of $V$ in $X$. We have $W \subseteq \alpha(X) - U \subseteq X$. Therefore, $\overline{V} = W$, and $\overline{V}$ is then compact since $W$ is a closed subset of the compact space $\alpha(X)$. Finally, if $y \in V \cap A$, then $\infty \leq y$, which forces $\infty \in V$, as $V$ is a downset. This contradiction implies that $V \cap A = \varnothing$. A similar argument shows that $(A, B)$ also satisfies the fifth condition of Definition 3.6. Thus, $(A, B) \in \Pi(X)$, and so $\varphi$ is well defined.

Let $\leq_1, \leq_2 \in \mathfrak{D}(\alpha(X))$ with $\leq_1 \subseteq \leq_2$. Then $A_1 = \{ x \in X : \infty \leq_1 x \} \subseteq \{ x \in X : \infty \leq_2 x \} = A_2$ and $B_1 = \{ x \in X : x \leq_1 \infty \} \subseteq \{ x \in X : x \leq_2 \infty \} = B_2$. Thus, $\varphi(\leq_1) = (A_1, B_1) \leq (A_2, B_2) = \varphi(\leq_2)$, and so $\varphi$ is order-preserving.

Now define $\psi: \Pi(X) \to \mathfrak{D}(\alpha(X))$ as follows. If $(A, B) \in \Pi(X)$, then set

$$\leq' = \leq \cup \{ (\infty) \times A \} \cup (B \times \{ \infty \}) \cup \{ (\infty, \infty) \},$$

and set $\psi(A, B) = \leq'$. It is clear that $\leq'$ is a binary relation on $\alpha(X)$ with $\leq' \cap X^2 = \leq$. It is reflexive since it contains $(\infty, \infty)$ and $\leq$. Next, suppose that $x \leq y$ and $y \leq' x$. If $x, y \in X$, then $x = y$ since $\leq$ is anti-symmetric. If $x = \infty$ and $y \in X$, then $y \in A \cap B$, a contradiction to Condition (ii) of Definition 3.6. Thus, if $x = \infty$, then $y = \infty$. Similarly, if $y = \infty$, then $x = \infty$. Hence, $\leq'$ is anti-symmetric. Finally, suppose that $x \leq y$ and $y \leq z$. If $x, y, z \in X$, then $x \leq' z$ since $\leq$ is transitive. If at least two of $x, y, z$ are equal to $\infty$, then $x \leq' z$.
trivially. Therefore, we may assume that only one of \(x, y, z\) is equal to \(\infty\). If \(x = \infty\), then \(y \in A\). As \(z \in X\), by Condition (i) of Definition 3.6, \(z \in A\). Thus, \(\infty \leq z\). If \(y = \infty\), then \(x \in B\) and \(z \in A\). By Condition (iii) of Definition 3.6, we get \(x \leq z\), so \(x \leq' z\). Finally, suppose that \(z = \infty\). Then \(y \in B\) and \(x \in X\), so by Condition (i) of Definition 3.6, \(x \in B\), and so \(x \leq' \infty\). This shows that \(\leq'\) is transitive. Therefore, \(\leq'\) is a partial order on \(\alpha(X)\).

To see that \((\alpha(X), \leq')\) is order-Hausdorff, let \(x, y \in \alpha(X)\) with \(x \ngeq' y\). If \(x = \infty\), then \(y \notin A\). By Condition (iv) of Definition 3.6, there exists an open downset \(V\) of \(X\) with \(y \in V\), \(\overline{V}\) compact in \(X\), and \(V \cap A = \emptyset\). But then \(\alpha(X) - V\) is an open neighborhood of \(\infty\) in \(\alpha(X)\). Also, as \(X\) is open in \(\alpha(X)\), \(V\) is an open neighborhood of \(y\) in \(\alpha(X)\). Thus, \(\alpha(X) - V\) is an upset of \(\alpha(X)\) and a neighborhood of \(\infty\), \(V\) is a downset of \(\alpha(X)\) and a neighborhood of \(y\), and \((\alpha(X) - V) \cap V = \emptyset\). A similar argument shows that if \(y = \infty\), then there exist an upset \(U\) and a downset \(V\) of \(\alpha(X)\) such that \(U\) is a neighborhood of \(x\), \(V\) is a neighborhood of \(\infty\), and \(U \cap V = \emptyset\). Finally, suppose \(x, y \in X\). Since \(X\) is order-Hausdorff, there exist an upset \(U\) of \(X\) and a downset \(V\) of \(X\) such that \(U\) is a neighborhood of \(x\), \(V\) is a neighborhood of \(y\), and \(U \cap V = \emptyset\). Because \(x \ngeq' y\), we have that either \(x \ngeq' \infty\) or \(\infty \ngeq' y\). If \(\infty \ngeq' \infty\), then \(x \notin B\). By Condition (v) of Definition 3.6, there exists an open upset \(U_1\) of \(X\) such that \(x \in U_1\), \(\overline{U_1}\) is compact, and \(U_1 \cap B = \emptyset\). Let \(U' = U \cap U_1\) and \(V' = (\alpha(X) - U_1) \cup V\). Then \(U'\) is an upset and \(V'\) is a downset of \(\alpha(X)\), \(U'\) is a neighborhood of \(x\), \(V'\) is a neighborhood of \(y\), and \(U' \cap V' = \emptyset\). The case when \(\infty \ngeq' y\) is treated similarly. Thus, \((\alpha(X), \leq')\) is order-Hausdorff. It follows that \((\alpha(X), \leq')\) is an order-compactification of \(X\). Consequently, \(\leq' \in \mathcal{PO}(\alpha(X))\), and so \(\psi\) is well defined.

Let \((A_1, B_1), (A_2, B_2) \in \Pi(X)\) with \((A_1, B_1) \leq (A_2, B_2)\). Then \(\{\infty\} \times A_1 \subseteq \{\infty\} \times A_2\) and \(B_1 \times \{\infty\} \subseteq B_2 \times \{\infty\}\). Thus, \(\psi(A_1, B_1) = \leq_1 \subseteq \leq_2 = \psi(A_2, B_2)\), and so \(\psi\) is order-preserving.

Finally, for \(\leq' \in \mathcal{PO}(\alpha(X))\) and \((A, B) \in \Pi(X)\), it is clear that \(\psi(\varphi(\leq')) = \leq'\) and that \(\varphi(\psi(A, B)) = (A, B)\). Thus, the posets \((\mathcal{PO}(\alpha(X)), \subseteq)\) and \((\Pi(X), \leq)\) are isomorphic. \(\square\)

**Remark 3.10.** Theorem 3.9 gives another explanation to why \(X = (0, 1)\) has no one-point order-compactifications: for each \(x \in X\) there is no open upset or downset that contains \(x\) and whose closure is compact in \(X\). Thus, Conditions (iv) and (v) of Definition 3.6 are not satisfied.

Let \((\mathcal{OC}_\alpha(X), \leq)\) denote the subposet of \((\mathcal{OC}(X), \leq)\) consisting of all inequivalent one-point order-compactifications of \(X\).
Proposition 3.11. The posets \((\mathcal{O}C_\alpha(X), \leq)\) and \((\mathcal{P}D(\alpha(X)), \subseteq)\) are dually isomorphic.

Proof. Obviously each member of \(\mathcal{O}C_\alpha(X)\) has the form \((\alpha(X), \leq')\) for some \(\leq' \in \mathcal{P}D(\alpha(X))\). Therefore, for \(\leq_1, \leq_2 \in \mathcal{P}D(\alpha(X))\), we have \(\leq_1 \subseteq \leq_2\) iff the identity map \(\text{id} : (\alpha(X), \leq_1) \rightarrow (\alpha(X), \leq_2)\) is continuous and order-preserving, which means that \((\alpha(X), \leq_2) \leq (\alpha(X), \leq_1)\). Thus, \((\mathcal{P}D(\alpha(X)), \subseteq)\) is dually isomorphic to \((\mathcal{O}C_\alpha(X), \leq)\). \(\square\)

This together with Theorem 3.9 immediately give us:

Corollary 3.12. The posets \((\mathcal{O}C_\alpha(X), \leq)\) and \((\Pi(X), \leq)\) are dually isomorphic.

Another corollary of Theorem 3.9 is that if \((\mathcal{O}C_\alpha(X), \leq)\) is nonempty, then it has a greatest element.

Corollary 3.13. If \((\mathcal{O}C_\alpha(X), \leq)\) is nonempty, then it has a greatest element.

Proof. Let \((A_0, B_0) = (\bigcap_{(A,B) \in \Pi(X)} A, \bigcap_{(A,B) \in \Pi(X)} B)\). It is easy to verify that \((A_0, B_0)\) satisfies the five conditions of Definition 3.6, thus \((A_0, B_0) \in \Pi(X)\). Moreover, \((A_0, B_0)\) is clearly the least element of \(\Pi(X)\). Let \(\leq_0\) be the corresponding element of \(\mathcal{P}D(\alpha(X))\). Then \(\leq_0\) is the smallest element of \(\mathcal{P}D(\alpha(X))\), and so \((\alpha(X), \leq_0)\) is the greatest (up to equivalence) one-point order-compactification of \(X\). \(\square\)

On the other hand, \(\mathcal{P}D(\alpha(X))\) may not have a greatest element, and so \((\mathcal{O}C_\alpha(X), \leq)\) may not have a least element, as the next example shows.

Example 3.14. Let \(X = \mathbb{N}\) be the space considered in Example 3.5. Then \(\Pi(X) = \{(C, \emptyset), (\emptyset, C)\}_{C \subseteq X}\). To see this, it is clear that \((C, \emptyset), (\emptyset, C) \in \Pi(X)\) for each \(C \subseteq X\); conversely, if \((A, B) \in \Pi(X)\) with both \(A, B \neq \emptyset\), then by Condition (iii) of Definition 3.6, \(X\) has a nontrivial order, which is a contradiction. Clearly \((\emptyset, \emptyset)\) is the smallest element of \(\Pi(X)\), thus the corresponding one-point order-compactification \((\alpha(X), \emptyset)\) is the greatest (up to equivalence) one-point order-compactification of \(X\). On the other hand, \((X, \emptyset)\) and \((\emptyset, X)\) are the only two maximal elements of \(\Pi(X)\), but \(\Pi(X)\) has no largest element. Consequently, there are uncountably many inequivalent one-point order-compactifications, but no least one-point order-compactification of \(X\). In fact, the two one-point order-compactifications of \(X\) shown in Fig. 1 are (up to equivalence) the only two minimal one-point order-compactifications of \(X\), corresponding to \((X, \emptyset)\) and \((\emptyset, X)\), respectively.
Let \( X \) be an ordered topological space. We let \( \mathcal{CpUp}(X) \) denote the set of clopen upsets of \( X \) and recall from [2, Def. 3.7] that \( X \) is order-zero-dimensional if \( X \) satisfies the Priestley separation axiom and the set \( \{ U - V : U, V \in \mathcal{CpUp}(X) \} \) forms a basis for the topology. Let \( X \) be a noncompact locally compact order-Hausdorff space, and let \( (\alpha(X), \leq') \) be a one-point order-compactification of \( X \). We show that \( (\alpha(X), \leq') \) is a Priestley space iff \( X \) is order-zero-dimensional.

**Theorem 3.15.** Let \( X \) be a noncompact locally compact order-Hausdorff space, and let \( (\alpha(X), \leq') \) be a one-point order-compactification of \( X \). Then \( (\alpha(X), \leq') \) is a Priestley space iff \( X \) is order-zero-dimensional.

**Proof.** If \( (\alpha(X), \leq') \) is a Priestley space, then \( X \) has a Priestley order-compactification. Thus, by [2, Cor. 3.8], \( X \) is order-zero-dimensional. Conversely, suppose that \( X \) is order-zero-dimensional. We need to show that \( (\alpha(X), \leq') \) satisfies the Priestley separation axiom. Let \( x \not< y \). Since \( (\alpha(X), \leq') \) is compact and order-Hausdorff, by [9, Prop. 1] and [1, Prop. 2.3], there exist an open upset \( U' \) and an open downset \( V' \) of \( \alpha(X) \) such that \( x \in U', y \in V', \) and \( U' \cap V' = \emptyset \). First suppose that \( x = \infty \). Then \( \alpha(X) - U' \) is a compact downset of \( X \) containing \( V' \). Therefore, \( V := \overline{V'} \subseteq \alpha(X) - U' \) is a compact subset of \( X \). Moreover, since \( X \) satisfies the Priestley separation axiom, so does \( V \). Thus, \( V \) is a Priestley space and \( V' \) is an open downset of \( V \). Therefore, \( V' \) is a union of clopen downsets of \( V \). Let \( W \) be a clopen downset of \( V \) such that \( y \in W \subseteq V' \). Since \( V' \) is an open downset of \( X \) and \( W \) is a clopen downset of \( V \) contained in \( V' \), then \( W \) is a compact open downset of \( X \). Let \( U = \alpha(X) - W \). Then \( U \) is a clopen upset of \( \alpha(X) \), \( \infty \in U \), and \( y \notin U \). A similar argument shows that if \( y = \infty \), then there is a clopen upset \( U \) of \( \alpha(X) \) such that \( x \in U \) and \( \infty \notin U \). Finally, suppose \( x, y \in X \). Because \( x \not< y \), we have \( x \not< y \), and since \( X \) satisfies the Priestley separation axiom, there is a clopen upset \( U' \) of \( X \) such that \( x \in U' \) and \( y \notin U' \). From \( x \not< y \) it follows that either \( x \not< \infty \) or \( \infty \not< y \). Suppose \( x \not< \infty \). Since \( (\alpha(X), \leq') \) is compact and order-Hausdorff, by [9, Prop. 1] and [1, Prop. 2.3], there exist an open upset \( U_1 \) and an open downset \( V_1 \) of \( \alpha(X) \) such that \( x \in U_1 \), \( \infty \in V_1 \), and \( U_1 \cap V_1 = \emptyset \). Therefore, \( \alpha(X) - V_1 \) is a compact subset of \( X \). Thus, \( U' \cap U_1 \subseteq \alpha(X) - V_1 \) is a compact subset of \( X \), and since \( U' \cap U_1 \) satisfies the Priestley separation axiom, \( U' \cap U_1 \) is a Priestley space. Moreover, \( U' \cap U_1 \) is an open upset of \( \overline{U' \cap U_1} \). Therefore, there exists a clopen upset \( U \) of \( \overline{U' \cap U_1} \) such that \( x \in U \subseteq U' \cap U_1 \). But then \( U \) is a clopen upset of \( \alpha(X) \) with \( x \in U \) and \( y \notin U \). The case when \( \infty \not< y \) is similar. Thus, \( (\alpha X, \leq') \) is a Priestley space. \( \square \)
As an immediate consequence of Theorem 3.15 we obtain that the one-point compactification $\alpha(X)$ of a noncompact locally compact Hausdorff space $X$ is a Stone space iff $X$ is zero-dimensional. Here we recall that $X$ is zero-dimensional if clopen subsets of $X$ form a basis for the topology, that $X$ is a Stone space if it is compact Hausdorff zero-dimensional, and that Stone spaces are fundamental in the study of Boolean algebras, as they serve as dual objects of the category of Boolean algebras and Boolean algebra homomorphisms [13]. We also recall that $X$ is totally disconnected if each pair of distinct points of $X$ can be separated by a clopen set. These spaces are also called totally separated in the literature. It is well known that for compact Hausdorff spaces the notions of zero-dimensional and totally disconnected are equivalent.

**Corollary 3.16.** Let $X$ be a noncompact locally compact Hausdorff space, and let $\alpha(X)$ be the one-point compactification of $X$. Then the following conditions are equivalent:

1. $X$ is zero-dimensional.
2. $\alpha(X)$ is totally disconnected.
3. $\alpha(X)$ is zero-dimensional.
4. $\alpha(X)$ is a Stone space.

**Proof.** Let $X$ be a noncompact locally compact Hausdorff space, and let $\alpha(X)$ be the one-point compactification of $X$. We view both $X$ and $\alpha(X)$ as ordered topological spaces with trivial orders $\leq$ and $\leq'$, respectively. Then both $(X, \leq)$ and $(\alpha(X), \leq')$ are order-Hausdorff. Moreover, by Theorem 3.15 and compactness of $\alpha(X)$ we have that $X$ is zero-dimensional iff $(X, \leq)$ is order-zero-dimensional, iff $(\alpha(X), \leq')$ is a Priestley space, iff $\alpha(X)$ is totally disconnected, iff $\alpha(X)$ is zero-dimensional, iff $\alpha(X)$ is a Stone space. □

**Remark 3.17.** Theorem 3.15 and Corollary 3.16 are in pleasant contrast with the fact that the Nachbin order-compactification of an order-zero-dimensional space may not be a Priestley space, and that the Stone-Čech compactification of a zero-dimensional space may not be totally disconnected; see, e.g., [2, Sec. 4] and [4, Ex. 6.2.20], respectively.

### 4. McCartan and McCallion order-compactifications

We recall that McCartan [8] and McCallion [7] each described a one-point order-compactification of a noncompact locally compact order-Hausdorff space, and gave necessary and sufficient conditions for their existence. We conclude the
paper by characterizing the McCartan and McCallion order-compactifications in terms of our \((A, B)\) pairs. We start with the McCartan order-compactification.

Let \(X\) be a noncompact locally compact order-Hausdorff space, and let \(\alpha(X)\) be the one-point compactification of \(X\). McCartan \cite{8} considers the partial order \(\leq'\) on \(\alpha(X)\) extending \(\leq\) such that \(\infty\) is not related to any point in \(X\). He proves that \((\alpha(X), \leq')\) is a one-point order-compactification of \(X\) iff for each compact subset \(A\) of \(X\), both \(\uparrow A\) and \(\downarrow A\) are compact. We call this the McCartan Condition, the order \(\leq'\) the McCartan order, and the order-compactification \((\alpha(X), \leq')\) the McCartan order-compactification. (We note that McCartan calls spaces satisfying his condition order-compactible.) We show that \(X\) satisfies the McCartan condition iff \((\emptyset, \emptyset) \in \Pi(X)\). Consequently, whenever \(X\) satisfies the McCartan condition, then the McCartan order-compactification is the largest (up to equivalence) one-point order-compactification of \(X\).

**Lemma 4.1.** If \(X\) is a noncompact locally compact order-Hausdorff space that satisfies the McCartan condition, then for each closed subset \(F\) of \(X\), both \(\uparrow F\) and \(\downarrow F\) are closed.

**Proof.** Let \(X\) be a noncompact locally compact order-Hausdorff space satisfying the McCartan condition, and let \(F\) be a closed subset of \(X\). We show that \(\downarrow F\) is closed. The proof that \(\uparrow F\) is closed is similar. Suppose \(x \notin \downarrow F\). Since \(X\) is locally compact, there exists an open neighborhood \(U\) of \(x\) such that \(\overline{U}\) is compact. Let \(A = \uparrow U\). By the McCartan condition, \(A\) is compact. Moreover, \(A \cap F\) is closed in \(A\), and since \(A\) is an upset, \(A \cap \downarrow F = \downarrow (A \cap F)\), where we define \(\downarrow B = A \cap \downarrow B\) for each subset \(B\) of \(A\). Because \(A\) is a compact order-Hausdorff space and \(A \cap F\) is closed in \(A\), by \cite[Prop. 2.3]{1}, so is \(\downarrow (A \cap F)\). Thus, \(A \cap \downarrow F\) is closed in \(A\) and \(x \notin A \cap \downarrow F\). Since \(A\) is a regular space, there exist \(V, W\) open in \(A\) such that \(x \in V\), \(A \cap \downarrow F \subseteq W\) and \(V \cap W = \emptyset\). Because \(A\) is a subspace of \(X\), there exist open subsets \(V_1\) and \(W_1\) of \(X\) such that \(V = V_1 \cap A\) and \(W = W_1 \cap A\). Consider \(V_2 = U \cap V_1\) and \(W_2 = A^c \cup W_1\). We have that both \(V_2\) and \(W_2\) are open in \(X\), that \(x \in V_2\), that \(\downarrow F \subseteq W_2\), and that \(V_2 \cap W_2 = \emptyset\). Thus, there exists an open neighborhood \(V_2\) of \(x\) missing \(\downarrow F\), and so \(\downarrow F\) is closed. \(\square\)

**Proposition 4.2.** Let \(X\) be a noncompact locally compact order-Hausdorff space, and let \(\alpha(X)\) be the one-point compactification of \(X\). Then \(X\) satisfies the McCartan condition iff \((\emptyset, \emptyset) \in \Pi(X)\). Consequently, when \(X\) satisfies the McCartan condition, then the McCartan order corresponds to \((\emptyset, \emptyset)\) under the isomorphism of Theorem 3.9, and the McCartan order-compactification is the largest (up to equivalence) one-point order-compactification of \(X\).
Proof. Let $X$ satisfy the McCartan condition. The pair $(\emptyset, \emptyset)$ automatically satisfies the first three conditions of Definition 3.6. We show that $(\emptyset, \emptyset)$ satisfies Condition (iv). The proof that $(\emptyset, \emptyset)$ satisfies Condition (v) is similar. Suppose $x \in X$. We need to find an open downset $V$ of $X$ such that $x \in V$ and $\overline{V}$ is compact. By the McCartan condition, $\downarrow x$ is compact. Since $X$ is locally compact and $\downarrow x$ is compact, there exists an open set $U$ such that $\downarrow x \subseteq U$ and $\overline{U}$ is compact. Let $V = (\overline{U^c})^c$. Then $V$ is a downset and $\downarrow x \subseteq V$. Since $V \subseteq U$ and $\overline{U}$ is compact, so is $\overline{V}$. By Lemma 4.1, $V$ is open. Thus, there exists an open downset $V$ of $X$ such that $x \in V$ and $\overline{V}$ is compact. Consequently, $(\emptyset, \emptyset) \in \Pi(X)$.

Now let $(\emptyset, \emptyset) \in \Pi(X)$. Suppose $F$ is a compact subset of $X$. We show that $\downarrow F$ is compact. The proof that $\uparrow F$ is compact is similar. By Condition (iv) of Definition 3.6, for each $x \in F$ there exists an open downset $V_x$ such that $x \in V_x$ and $\overline{V_x}$ is compact. Since $F$ is compact and $F \subseteq \bigcup_{x \in F} V_x$, there exist finitely many, and hence one $V$ such that $V$ is an open downset, $F \subseteq V$ and $\overline{V}$ is compact. Because $V$ is a downset, $F \subseteq V$ implies $\downarrow F \subseteq V$. Now, since $F$ is a closed subset of $\overline{V}$ and $\overline{V}$ is a compact order-Hausdorff space, by [1, Prop. 2.3], we have that $\downarrow F$ is also a closed subset of $\overline{V}$, and so is compact. Thus, the McCartan condition is satisfied.

Finally, whenever $X$ satisfies the McCartan condition, it follows from the definition of the partial order corresponding to $(\emptyset, \emptyset)$ that it is the McCartan order, and so the McCartan order-compactification is the largest (up to equivalence) one-point order-compactification of $X$. □

Of course, in many cases the McCartan condition is not satisfied, and so the McCartan order-compactification does not exist. McCallion [7] generalized the McCartan condition, and showed that whenever his condition is satisfied, then the ordered space under consideration has a one-point order-compactification. We call this order-compactification the McCallion order-compactification, and the corresponding order the McCallion order. Let $X$ be a noncompact locally compact order-Hausdorff space, and let $\alpha(X)$ be the one-point compactification of $X$. For $x, y \in X$, by $x < y$ we mean that $x \leq y$ and $x \neq y$. We say that $X$ satisfies the McCallion condition if for each $x, y \in X$ with $x \neq y$, either there is an open upset $U$ containing $x$ with $\overline{U}$ compact or an open downset $V$ containing $y$ with $\overline{V}$ compact. (Note that McCallion calls spaces satisfying his condition strongly locally compact.) We show that $X$ satisfies the McCallion condition iff $\Pi(X)$ is nonempty, and that whenever the McCallion condition is satisfied, then the McCallion order-compactification is the largest one-point order-compactification of $X$. 


Proposition 4.3. Let $X$ be a noncompact locally compact order-Hausdorff space, and let $\alpha(X)$ be the one-point compactification of $X$. Then $X$ satisfies the McCallion condition iff $\Pi(X)$ is not empty. Moreover, whenever $X$ satisfies the McCallion condition, then the McCallion order on $\alpha(X)$ is the smallest element of $\mathfrak{PO}(\alpha(X))$, and so the McCallion order-compactification is the largest (up to equivalence) one-point order-compactification of $X$.

Proof. Let $(A, B) \in \Pi(X)$ and let $x \not< y$. If $x \not< \infty$, then $x \not\in B$, and so, by Condition (v) of Definition 3.6, there is an open upset $U$ containing $x$ with $U$ compact. If $x \leq \infty$, because $x \not< y$, we see that $\infty \not< y$, so $y \not\in A$. Therefore, by Condition (iv) of Definition 3.6, there is an open downset $V$ containing $y$ with $V$ compact. Thus, $X$ satisfies the McCallion condition.

Conversely, suppose $X$ satisfies the McCallion condition. Set

$$A_0 = \left\{ x \in X : \text{there is an open upset } U \text{ with } x \in U \text{ and } U \text{ compact, and } \right.$$ 

$$\left. \text{for all open downsets } V \text{ with } x \in V \text{ we have } V \text{ is not compact} \right\},$$

and

$$B_0 = \left\{ x \in X : \text{there is an open downset } V \text{ with } x \in V \text{ and } V \text{ compact, and } \right.$$ 

$$\left. \text{for all open upsets } U \text{ with } x \in U \text{ we have } U \text{ is not compact} \right\}.$$

We show that $(A_0, B_0) \in \Pi(X)$. If $x \in A_0$ and $x \leq y$, then there exists an open upset $U$ containing $x$ with $U$ compact. So $y \in U$, and so $U$ is an open upset containing $y$ with compact closure. If $V$ is an open downset containing $y$, then $x \in V$, and since $x \in A_0$, we see that $V$ is not compact. Thus, $y \in A_0$. This proves that $A_0$ is an upset. Similarly, $B_0$ is a downset, and so Condition (i) is satisfied.

It is clear that $A_0 \cap B_0 = \emptyset$, hence Condition (ii) is satisfied. Next, let $b \in B_0$ and $a \in A_0$. Suppose that $b \not\leq a$. By the definition of $A_0$ and $B_0$, there is no open upset containing $b$ with compact closure, and no open downset containing $a$ with compact closure. This violates the McCallion condition. Therefore, $b \leq a$, and so Condition (iii) is satisfied. To check Condition (iv), suppose that $x \not\in A_0$. Because $x \not< x$, by the McCallion condition, either there is an open upset $U$ containing $x$ with $U$ compact or an open downset $V$ containing $x$ with $V$ compact. This together with $x \not\in A_0$ imply that there is an open downset $V$ containing $x$ with $V$ compact. Therefore, Condition (iv) is satisfied. That Condition (v) is also satisfied is proved similarly. Thus, $(A_0, B_0) \in \Pi(X)$.

We show that $(A_0, B_0)$ is the least element of $\Pi(X)$. Let $(A, B) \in \Pi(X)$. If $x \not\in A$, then there is an open downset $V$ such that $x \in V$, $V$ is compact, and $V \cap A = \emptyset$. Therefore, $x \not\in A_0$, and so $A_0 \subseteq A$. A similar argument shows that $B_0 \subseteq B$. Thus, $(A_0, B_0) \leq (A, B)$, and so $(A_0, B_0)$ is the least element of $\Pi(X)$. 

Finally, we recall [7, p. 472] that the McCallion order $\leq_0$ is defined on $\alpha(X)$ by
\[ \leq_0 = \leq \cup \{(\infty) \times A_0\} \cup (B_0 \times \{\infty\}) \cup \{(\infty, \infty)\}, \]
But $\leq_0$ is exactly the partial order corresponding to the pair $(A_0, B_0) \in \Pi(X)$. Thus, the McCallion order is the smallest element of $\mathcal{PO}(\alpha(X))$, and so the McCallion order-compactification is the largest (up to equivalence) one-point order-compactification of $X$. □

References


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(Guram Bezhanishvili) Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003-8001
E-mail address: gbezhani@nmsu.edu

(Patrick J. Morandi) Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003-8001
E-mail address: pmorandi@nmsu.edu