Completeness of S4 with respect to the real line: revisited

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Abstract

We prove that S4 is complete with respect to Boolean combinations of countable unions of convex subsets of the real line, thus strengthening a 1944 result of McKinsey and Tarski (Ann. of Math. (2) 45 (1944) 141). We also prove that the same result holds for the bimodal system S4 + S5 + C, which is a strengthening of a 1999 result of Shehtman (J. Appl. Non-Classical Logics 9 (1999) 369).

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1. Introduction

It was shown in McKinsey and Tarski [8] that every finite well-connected topological space is an open image of a metric separable dense-in-itself space. This together with the finite model property of S4 implies that S4 is complete with respect to any metric separable dense-in-itself space. Most importantly, it implies that S4 is complete with respect to the real line R. Shehtman [13] strengthened the McKinsey and Tarski result by showing that

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every finite connected space is an open image of a (connected) metric separable dense-in-itself space. (That every finite connected space is an open image of a Euclidean space was first established in Puckett [11].) As a result, Shehtman obtained that in the language enriched with the universal modality \( \forall \) the complete logic of a connected metric separable dense-in-itself space is the logic \( \mathbf{S4} + \mathbf{S5} + \mathbf{C} \), where \( \mathbf{S4} + \mathbf{S5} \) is Bennett’s logic [2] (being \( \mathbf{S4} \) for \( \Box \), \( \mathbf{S5} \) for \( \forall \), plus the bridge axiom \( \forall \varphi \rightarrow \Box \varphi \)) and \( \mathbf{C} \) is the connectedness axiom \( \forall(\Box \varphi \rightarrow \varphi) \rightarrow (\forall \varphi \lor \forall \neg \varphi) \).

The original proof of McKinsey and Tarski was quite complicated. The later version in Rasiowa and Sikorski [12] was not much more accessible. Recently Mints [10] and Aiello et al. [1] obtained simpler model-theoretic proofs of completeness of \( \mathbf{S4} \) with respect to the Cantor space \( \mathbf{C} \) and the real line \( \mathbf{R} \). In this paper we give yet another, more topological, proof of completeness of \( \mathbf{S4} \) with respect to \( \mathbf{R} \). It is not only more accessible than the original proof, but also strengthens both the McKinsey and Tarski, and Shehtman results.

The paper is organized as follows. In Section 2 we recall a one-to-one correspondence between Alexandroff spaces and quasi-ordered sets; we also recall the modal systems \( \mathbf{S4} \), \( \mathbf{S4} + \mathbf{S5} \) and \( \mathbf{S4} + \mathbf{S5} + \mathbf{C} \), and their algebraic semantics. In Section 3 we give a simplified proof that a finite well-connected topological space is an open image of \( \mathbf{R} \). It follows that \( \mathbf{S4} \) is complete with respect to Boolean combinations of countable unions of convex subsets of \( \mathbf{R} \), which is a strengthening of the McKinsey and Tarski result. As a by-product, we obtain a new proof of completeness of the intuitionistic propositional logic \( \mathbf{Int} \) with respect to open subsets of \( \mathbf{R} \), and completeness of the Grzegorczyk logic \( \mathbf{Grz} \) with respect to Boolean combinations of open subsets of \( \mathbf{R} \). In Section 4 we give a simplified proof that a finite topological space is an open image of \( \mathbf{R} \) iff it is connected. Consequently, we obtain that \( \mathbf{S4} + \mathbf{S5} + \mathbf{C} \) is complete with respect to Boolean combinations of countable unions of convex subsets of \( \mathbf{R} \), which is a strengthening of the Shehtman result. We conclude the paper by mentioning several open problems.

2. Preliminaries

2.1. Topology and order

Suppose \( X \) is a topological space. For \( A \subseteq X \) we denote by \( \overline{A} \) the closure of \( A \), and by \( \text{Int}(A) \) the interior of \( A \). We recall that \( A \) is dense if \( \overline{A} = X \), and that \( A \) is nowhere dense or boundary if \( \text{Int}(A) = \emptyset \). The definition of closed and open subsets of \( X \) is usual. We call a subset of \( X \) clopen if it is simultaneously closed and open. The space \( X \) is called connected if \( \emptyset \) and \( X \) are the only clopen subsets of \( X \); it is called well-connected if there exists a least nonempty closed subset of \( X \). It is obvious that every well-connected space is connected, but the converse is not necessarily true. We call \( X \) an Alexandroff space if the intersection of any family of open subsets of \( X \) is open. Obviously every finite space is an Alexandroff space. For two topological spaces \( X \) and \( Y \), a continuous map \( f : X \rightarrow Y \) is called open if the \( f \)-image of every open subset of \( X \) is an open subset of \( Y \). Thus, \( f \) is an open map iff it preserves and reflects opens.

Suppose \( X \) is a nonempty set. A binary relation \( \leq \) on \( X \) is called a quasi-order if \( \leq \) is reflexive and transitive; if in addition \( \leq \) is antisymmetric, then \( \leq \) is called a partial order. If \( \leq \) is a quasi-order on \( X \), then \( X \) is called a quasi-ordered set or simply a qoset; if \( \leq \) is
a partial order, then $X$ is called a partially ordered set or simply a poset. For two qosets $X$ and $Y$, an order-preserving map $f : X \rightarrow Y$ is called a $p$-morphism if for every $x \in X$ and $y \in Y$, from $f(x) \leq y$ it follows that there exists $z \in X$ such that $x \leq z$ and $f(z) = y$.

Suppose $X$ is a qoset. For $A \subseteq X$ let $\uparrow A = \{ x \in X : \exists a \in A \text{ with } a \leq x \}$ and $\downarrow A = \{ x \in X : \exists a \in A \text{ with } x \leq a \}$. We call $A \subseteq X$ an upset if $A = \uparrow A$, and a downset if $A = \downarrow A$. For $x \in X$ let $C[x] = \{ y \in X : x \leq y \text{ and } y \leq x \}$. We call $C \subseteq X$ a cluster if there is $x \in X$ such that $C = C[x]$. We call $x \in X$ maximal if $x \leq y$ implies $x = y$, and quasi-maximal if $x \leq y$ implies $y \leq x$; similarly, we call $x \in X$ minimal if $y \leq x$ implies $y = x$, and quasi-minimal if $y \leq x$ implies $x \leq y$. If $X$ is a poset, then it is obvious that the notions of maximal and quasi-maximal points, as well as the notions of minimal and quasi-minimal points coincide. We call a cluster $C$ maximal if $C = C[x]$ for some quasi-maximal $x \in X$; a cluster $C$ is called minimal if $C = C[x]$ for some quasi-minimal $x \in X$. We call $r \in X$ a root of $X$ if $r \leq x$ for every $x \in X$; a qoset $X$ is called rooted if it has a root $r$; note that $r$ is not unique: every element of $C[r]$ serves as a root of $X$. We say that there exists a $\leq$-path between two points $x, y$ of $X$ if there exists a sequence $w_1, \ldots, w_n$ of points of $X$ such that $w_1 = x, w_n = y$, and either $w_i \leq w_{i+1}$ or $w_{i+1} \leq w_i$ for any $1 \leq i \leq n - 1$. We call $X$ a connected component if there is a $\leq$-path between any two points of $X$. Note that every rooted qoset is a connected component, but not vice versa.

For a qoset $X$ let $\tau_{\leq}$ denote the set of upsets of $X$. It is easy to verify that $\tau_{\leq}$ is an Alexandroff topology on $X$. Conversely, if $X$ is a topological space, then we define the specialization order $\leq_{\tau}$ on $X$ by putting $x \leq_{\tau} y$ iff $x \in \overline{\{y\}}$. It is routine to check that $\leq_{\tau}$ is a quasi-order on $X$. Moreover, $\leq_{\tau}$ is a partial order iff $X$ is a $T_0$-space. Now a standard argument shows that $\leq = \leq_{\tau}$ and that $\tau \subseteq \tau_{\leq}$, Furthermore, $\tau = \tau_{\leq}$ iff $\tau$ is an Alexandroff topology. This establishes a one-to-one correspondence between qosets and Alexandroff spaces, and between posets and Alexandroff $T_0$-spaces. In particular, we obtain a one-to-one correspondence between finite qosets and finite topological spaces, and between finite posets and finite $T_0$-spaces. We note that under this correspondence order-preserving maps correspond to continuous maps, and $p$-morphisms correspond to open maps. Moreover, connected spaces correspond to connected components and well-connected spaces correspond to rooted qosets (see, e.g., Aiello et al. [1] for details).

Subsequently, we will not distinguish between Alexandroff spaces and qosets, and between Alexandroff $T_0$-spaces and posets. For these spaces we will use interchangeably the notions of open maps and $p$-morphisms, connected spaces and connected components, and well-connected spaces and rooted qosets.

2.2. $\textbf{S4}, \textbf{S4 + S5}, \text{and} \textbf{S4 + S5 + C}$

We recall that $\textbf{S4}$ is the least set of formulae of the propositional modal language $\mathcal{L}$ containing the axioms $\Box \varphi \rightarrow \varphi$, $\Box \varphi \rightarrow \Box \Box \varphi$, $\Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$, and closed under modus ponens ($\varphi, \varphi \rightarrow \psi \rightarrow \psi$), substitution ($\varphi(p_1, \ldots, p_n)/\psi(p_1, \ldots, p_n)$), and necessitation ($\varphi/\Box \varphi$).

It was shown in McKinsey and Tarski [9] that algebraic models of $\textbf{S4}$ are closure algebras. We recall that a closure algebra is a pair $(B, C)$, where $B$ is a Boolean algebra and
\( C : B \to B \) is a function satisfying the following identities: (i) \( a \leq Ca \), (ii) \( CCa = Ca \), (iii) \( C(a \lor b) = Ca \lor Cb \), and (iv) \( C0 = 0 \). We call \( C \) a closure operator on \( B \).

To give an example of a closure algebra, let \( X \) be a qoset and let \( \mathcal{P}(X) \) denote the powerset of \( X \). It is easy to check that \( \downarrow \) is a closure operator on \( \mathcal{P}(X) \). Hence, \((\mathcal{P}(X), \downarrow)\) is a closure algebra. We call \((\mathcal{P}(X), \downarrow)\) the closure algebra over the qoset \( X \). More generally, if \( X \) is a topological space, then it is routine to verify that \((\mathcal{P}(X), \neg)\) is a closure algebra. We call \((\mathcal{P}(X), \neg)\) the closure algebra over the topological space \( X \).

Suppose \( X \) and \( Y \) are topological spaces and \( f : X \to Y \) is an open map. Then it is easy to verify that for \( A \subseteq Y \) we have \( f^{-1}(A) = \overline{f^{-1}(A)} \). Therefore, \( f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X) \) is a closure algebra homomorphism. Moreover, if \( f \) is onto, then \( f^{-1} \) is one-to-one, and hence \((\mathcal{P}(Y), \neg)\) is isomorphic to a subalgebra of \((\mathcal{P}(X), \neg)\).

**Theorem 1.** (a) Every closure algebra can be represented as a subalgebra of the closure algebra over a topological space. In fact, every closure algebra can be represented as a subalgebra of the closure algebra over an Alexandroff space, or equivalently, over a qoset.

(b) If a closure algebra is finite, then it is isomorphic to the closure algebra over a finite space, or equivalently, over a finite qoset.

(c) A finite closure algebra is subdirectly irreducible iff it is isomorphic to the closure algebra over a finite well-connected space, or equivalently, over a finite rooted qoset.

(d) \( S4 \) is complete with respect to finite subdirectly irreducible closure algebras. Hence, \( S4 \) is complete with respect to the closure algebras over finite well-connected spaces, or equivalently, over finite rooted qosets.

**Proof.** In the light of the above correspondence between Alexandroff spaces and qosets, (a) follows from [8, Theorem 2.4] and [6, Theorem 3.14]; (b) follows from [3, Lemma 1]; (c) follows from [4, the paragraph after the Theorem of Duality]; and finally, (d) follows from [8, Theorem 4.16]. □

Let \( \mathcal{L}(V) \) denote the enrichment of \( \mathcal{L} \) by the universal modality \( V \). As usual, the existential modality \( \exists \) is the abbreviation of \( \neg V \neg \). We recall that Bennett’s logic \( S4 + S5 \) is the least set of formulae of \( \mathcal{L}(V) \) containing the \( \square \)-axioms for \( S4 \), the \( \forall \)-axioms for \( S5 \) (that is \( \forall \)-axioms for \( S4 \) plus the axiom \( \exists \phi \to \forall \exists \phi \)), the bridge axiom \( \forall \phi \to \square \phi \), and closed under modus ponens, substitution, \( \square \)-necessitation, and \( \forall \)-necessitation (\( \phi / \forall \phi \)).

Algebraic models of \( S4 + S5 \) are the triples \((B, C, \exists)\), where (i) \((B, C)\) is a closure algebra, (ii) \((B, \exists)\) is a monadic algebra (that is \((B, \exists)\) is a closure algebra satisfying the identity \( \exists a = \exists a \)), and (iii) \( Ca \leq \exists a \). We call \((B, C, \exists)\) an \( (S4 + S5) \)-algebra.

Examples of \((S4 + S5)\)-algebras can be obtained from the closure algebras over topological spaces. Let \( X \) be a topological space. We define \( \exists \) on \( \mathcal{P}(X) \) by setting

\[
\exists A = \begin{cases} 
\emptyset, & \text{if } A = \emptyset \\
X, & \text{otherwise.}
\end{cases}
\]

Then \((\mathcal{P}(X), \neg, \exists)\) is an \( (S4 + S5) \)-algebra, called the \( (S4 + S5) \)-algebra over the topological space \( X \). In particular, if \( X \) is a qoset, then \((\mathcal{P}(X), \downarrow, \exists)\) is an \( (S4 + S5) \)-algebra, called the \( (S4 + S5) \)-algebra over the qoset \( X \).
Theorem 2. (a) Every \((S4 + S5)\)-algebra over a topological space is simple (has no proper congruences).
(b) Every simple \((S4 + S5)\)-algebra can be represented as a subalgebra of the \((S4 + S5)\)-algebra over some (Alexandroff) space.
(c) If a simple \((S4 + S5)\)-algebra is finite, then it is isomorphic to the \((S4 + S5)\)-algebra over a finite space, or equivalently, over a finite poset.
(d) \(S4 + S5\) is complete with respect to finite simple \((S4 + S5)\)-algebras. Hence, \(S4 + S5\) is complete with respect to the \((S4 + S5)\)-algebras over finite topological spaces, or equivalently, over finite posets.

Proof. For (a) see [5, Lemma 3.1]. For (b) observe that a \((S4 + S5)\)-algebra \((B, C, \exists)\) is simple iff for every \(a \in B\) we have \(a \neq 0\) implies \(\exists a = 1\). Now apply Theorem 1(a). (c) follows from (b) and Theorem 1(b). For (d) see [13, Theorem 7] or [5, Theorem 5.9]. □

It was proved in [13, Lemma 8] that the connectedness axiom

\[ C = \forall(\Diamond\varphi \rightarrow \Box\varphi) \rightarrow (\forall\varphi \vee \forall\neg\varphi) \]

is valid in the \((S4 + S5)\)-algebra over a topological space \(X\) iff \(X\) is connected. In particular, \(C\) is valid in the \((S4 + S5)\)-algebra over a poset \(X\) iff \(X\) is a connected component.
Let \(S4 + S5 + C\) denote the normal extension of \((S4 + S5)\) by the connectedness axiom. We call an \((S4 + S5)\)-algebra \((B, C, \exists)\) a \((S4 + S5 + C)\)-algebra if the connectedness axiom is valid in \((B, C, \exists)\).

Theorem 3. \(S4 + S5 + C\) is complete with respect to finite simple \((S4 + S5 + C)\)-algebras. Hence, \(S4 + S5 + C\) is complete with respect to the \((S4 + S5 + C)\)-algebras over finite connected spaces, or equivalently, over finite connected components.

Proof. See [13, Theorem 10]. □

3. Completeness of \(S4\)

We recall that a subset \(A\) of \(\mathbb{R}\) is said to be convex if \(x, y \in A\) and \(x \leq z \leq y\) imply that \(z \in A\). We denote by \(C(\mathbb{R})\) the set of convex subsets of \(\mathbb{R}\), and by \(C^\infty(\mathbb{R})\) the set of countable unions of convex subsets of \(\mathbb{R}\). We also let \(B(C^\infty(\mathbb{R}))\) denote the Boolean algebra generated by \(C^\infty(\mathbb{R})\). It is obvious that every open interval of \(\mathbb{R}\) belongs to \(C(\mathbb{R})\). Now since every open subset of \(\mathbb{R}\) is a countable union of open intervals of \(\mathbb{R}\), it follows that every open subset of \(\mathbb{R}\), and hence every closed subset of \(\mathbb{R}\), belongs to \(B(C^\infty(\mathbb{R}))\). Therefore, \((B(C^\infty(\mathbb{R})), \neg)\) is a closure algebra. In fact, \((B(C^\infty(\mathbb{R})), \neg)\) is a proper subalgebra of \((\mathcal{P}(\mathbb{R}), \neg)\). Our goal is to show that \(S4\) is complete with respect to \((B(C^\infty(\mathbb{R})), \neg)\). For this, as follows from Theorem 1, it is sufficient to show that every closure algebra over a finite rooted poset is isomorphic to a subalgebra of \((B(C^\infty(\mathbb{R})), \neg)\).

Suppose \(X\) is a finite poset. We call \(Y \subseteq X\) a chain if for every \(x, y \in Y\) we have \(x \leq y\) or \(y \leq x\). For \(x \in X\) let \(d(x)\) be the number of elements of a maximal chain with the root \(x\); we call \(d(x)\) the depth of \(x\). Let also \(d(X) = \sup\{d(x) : x \in X\}\); we call \(d(X)\) the depth of \(X\). For \(x, y \in X\) let \(x < y\) mean that \(x \leq y\) and \(x \neq y\). We call \(y\) an immediate successor of \(x\) if \(x < y\) and there is no \(z\) such that \(x < z < y\). For \(x \in X\) let \(b(x)\) be the number of immediate successors of \(x\); we call \(b(x)\) the branching of \(x\). Let also
\( b(X) = \sup \{ b(x) : x \in X \} \); we call \( b(X) \) the branching of \( X \). A finite poset \( X \) is called a tree if \( \downarrow x \) is a chain for every \( x \in X \); if in the tree \( X \) we have \( b(x) = n \) for every \( x \in X \), then we call \( X \) an \( n \)-tree.

**Lemma 4.** (a) Every finite rooted poset is a \( p \)-morphic image of a finite tree.
(b) Every tree of branching \( n \) and depth \( m \) is a \( p \)-morphic image of the \( n \)-tree of depth \( m \).
(c) For every finite rooted poset \( X \) there exists \( n \) such that \( X \) is a \( p \)-morphic image of a finite \( n \)-tree.

**Proof.** For (a) see [7, Proposition 2]; (b) follows from [7, Theorem 1]; finally, (c) follows from (a) and (b). \( \square \)

We call a finite poset \( X \) \( q \)-regular if every cluster of \( X \) consists of exactly \( q \) elements. We define an equivalence relation \( \sim \) on \( X \) by putting \( x \sim y \) iff \( C[x] = C[y] \). Let \( X/\sim \) denote the quotient of \( X \) under \( \sim \), where \( [x] \leq [y] \) if there exist \( x' \in [x] \) and \( y' \in [y] \) such that \( x' \leq y' \). Obviously \( X/\sim \) is a finite poset, called the skeleton of \( X \). We call \( X \) a quasi-tree if \( X/\sim \) is a tree; we call \( X \) a quasi-\( n \)-tree if \( X/\sim \) is an \( n \)-tree; finally, we call \( X \) a quasi-\((q, n)\)-tree if \( X \) is a \( q \)-regular quasi-\( n \)-tree. The following lemma is an easy generalization of Lemma 4 to qosets.

**Lemma 5.** For every finite rooted poset \( X \) there exist \( q, n \) such that \( X \) is a \( p \)-morphic image of a finite quasi-\((q, n)\)-tree.

**Proof (Sketch).** Let \( q = \sup \{ |C[x]| : x \in X \} \). Then replacing every cluster of \( X \) by a \( q \)-element cluster, we get a new \( q \)-regular poset \( Y \). Obviously \( X \) is a \( p \)-morphic image of \( Y \) and \( X/\sim \) is isomorphic to \( Y/\sim \). From the previous lemma we know that there exist an \( n \)-tree \( T_n \) and a \( p \)-morphism \( f \) from \( T_n \) onto \( Y/\sim \). We denote by \( T_{q,n} \) the quasi-tree obtained from \( T_n \) by replacing every node \( t \) of \( T_n \) by a \( q \)-element cluster \( [t] = \{ t_1, \ldots, t_q \} \). Obviously \( T_{q,n} \) is a finite quasi-\((q, n)\)-tree and \( T_n \) is (isomorphic to) \( T_{q,n}/\sim \). Suppose \( [y] = \{ y_1, \ldots, y_q \} \) is an element of \( Y/\sim \) and \( [t] = \{ t_1, \ldots, t_q \} \) is an element of \( T_{q,n}/\sim = T_n \). We define \( h : T_{q,n} \to Y \) by putting \( h(t_i) = y_i \) if \( f([t]) = [y] \), \( t_i \in [t] \), and \( y_i \in [y] \) for \( 1 \leq i \leq q \). Since \( [h(t_i)] = f([t]) \) and \( f \) is an onto \( p \)-morphism, so is \( h \). So \( Y \) is a \( p \)-morphic image of \( T_{q,n} \), and since \( X \) is a \( p \)-morphic image of \( Y \), it is also a \( p \)-morphic image of \( T_{q,n} \). \( \square \)

**Corollary 6.** \( S_4 \) is complete with respect to the closure algebras over finite quasi-trees.

**Proof.** It follows from Theorem 1(d) that \( S_4 \) is complete with respect to the closure algebras over finite rooted qosets. From Lemma 5 it follows that the closure algebra over a finite rooted qoset is isomorphic to a subalgebra of the closure algebra over some finite quasi-tree. Thus, \( S_4 \) is complete with respect to the closure algebras over finite quasi-trees. \( \square \)

Now we are in a position to show that finite rooted qosets are open images of \( \mathbb{R} \). We first show that every finite rooted poset is an open image of \( \mathbb{R} \), and then extend this result to finite qosets. Let us start by showing that the \( n \)-tree \( T \) of depth 2 shown in Fig. 1 is an open image of any bounded interval \( I \subseteq \mathbb{R} \).
Suppose $a, b \in \mathbb{R}, a < b, I = (a, b), I = [a, b), I = (a, b], or I = [a, b]$. We recall that the Cantor set $C$ is constructed inside $I$ by taking out open intervals from $I$ infinitely many times. More precisely, in step 1 of the construction the open interval

$$I_1^1 = \left( a + \frac{b - a}{3}, a + \frac{2(b - a)}{3} \right)$$

is taken out. We denote the remaining closed intervals by $J_1^1$. In step 2 the open intervals

$$I_1^2 = \left( a + \frac{b - a}{3}, a + \frac{2(b - a)}{3} \right) \quad \text{and} \quad I_2^1 = \left( a + \frac{7(b - a)}{32}, a + \frac{8(b - a)}{32} \right)$$

are taken out. We denote the remaining closed intervals by $J_1^2, J_2^1, J_2^2, \text{ and } J_2^3$. In general, in step $m$ the open intervals $I_m^1, \ldots, I_m^{2^m - 1}$ are taken out, and the closed intervals $J_m^1, \ldots, J_m^{2^m}$ remain. We will use the construction of $C$ to obtain $T$ as an open image of $I$.

**Lemma 7.** $T$ is an open image of $I$.

**Proof.** Define $f_T^I : I \to T$ by putting

$$f_T^I(x) = \begin{cases} t_k, & \text{if } x \in \bigcup_{m \equiv k \text{(mod n)}} \bigcup_{p=1}^{2^{m-1}} I_p^m, \\ r, & \text{otherwise} \end{cases}$$

Obviously, $f_T^I$ is a well-defined onto map. Moreover,

$$(f_T^I)^{-1}(t_k) = \bigcup_{m \equiv k \text{(mod n)}} \bigcup_{p=1}^{2^{m-1}} I_p^m \quad \text{and} \quad (f_T^I)^{-1}(r) = C.$$
The construction of $\mathcal{C}$. Thus, $f^T_d(U) \supseteq \{t_1, \ldots, t_m\}$ and $f^T_d(U) = T$. Hence, $f^T_d(U)$ is open for any open interval $U$ of $I$. It follows that $f^T_d$ is an onto open map. \hfill \Box

**Theorem 8.** Every finite $n$-tree is an open image of $I$.

**Proof.** For an arbitrary finite $n$-tree $T$ we define a map $f_I : I \rightarrow T$ by induction on the depth of $T$. If the depth of $T$ is 1, then $T$ is a 1-tree consisting of a single element $t$, and for every $x \in I$ we set $f_I(x) = t$. Then it is obvious that $f_I$ is onto and open. If the depth of $T$ is 2, then for every $x \in I$ we define $f_I(x) = f^1_I(x)$. Then the previous lemma guarantees that $f_I$ is onto and open. Now suppose the depth of $T$ is $d + 1$, $d \geq 2$. Let $t_1, \ldots, t_m (m = n^d)$ be the elements of $T$ of depth 2, and let $T_d$ be the subtree of $T$ of all elements of $T$ of depth $\geq 2$ (see Fig. 2).

We note that for each $k \in \{1, \ldots, m\}$ the upset $\uparrow t_k$ is isomorphic to the $n$-tree of depth 2, and that $T_d$ is the $n$-tree of depth $d$. So by the induction hypothesis there exists an onto open map $f^d_I : I \rightarrow T_d$. We use $f^d_I$ to define $f_I : I \rightarrow T$ as follows. For each $k \in \{1, \ldots, m\}$ and $x \in (f^d_I)^{-1}(t_k)$ let $I_x$ denote the connected component of $(f^d_I)^{-1}(t_k)$ containing $x$. We set

$$f_I(x) = \begin{cases} f^d_I(x), & \text{if } f^d_I(x) \notin \{t_1, \ldots, t_m\} \\ f^d_{I_x}(x), & \text{if } f^d_I(x) = t_k. \end{cases}$$

It is clear that $f_I$ is a well-defined onto map. To show that $f_I$ is continuous observe that for $t \in T - T_d$ there is a unique $t_k$ such that $t_k < t$. Hence, we have

$$f_I^{-1}(t) = \bigcup \{(f^d_I)^{-1}(t') : I' \text{ is a connected component of } (f^d_I)^{-1}(t_k)\}.$$ 

Also for $t \in T_d$ we have

$$f_I^{-1}(\uparrow t) = (f^d_I)^{-1}(\uparrow t).$$

Now since the family $\{\emptyset\} \cup \{\{t\} : t \in T - T_d\} \cup \{\uparrow t : t \in T_d\}$ forms a basis for $T$, we have that $f_I$ is continuous.

To show that $f_I$ is open, let $U = (c, d)$ be an open interval in $I$. If $U \subsetneq I'$ where $I'$ is a connected component of $(f^d_I)^{-1}(t_k)$ for some $k$, then $f_I(U) = (f^d_I)^{-1}(U)$. Therefore, $f_I(U)$ is open by the previous lemma. Assume $U \not\subset I'$ for any $k$ and $I'$. We want to show that $f_I(U) = \uparrow f^d_I(U)$. If $t \in T - (f^d_I)^{-1}(U)$, then $f_I^{-1}(t) = (f^d_I)^{-1}(t)$, and thus $t \in f_I(U)$.
iff \( t \in f_I^d(U) \). So we can assume that \( t \in \uparrow t_k \) for some \( k \). Then if \( t \in f_I(U) \), there is \( x \in U \) with \( f_I(x) = t \). Hence, by the definition of \( f_I \), there exists a connected component \( I' \) of \((f_I^d)^{-1}(t_k)\) with \( x \in I' \) and \( f_I(x) = f_I^{t_k}(x) \). Therefore, \( x \in U \cap (f_I^d)^{-1}(t_k) \), which implies that \( t_k \in f_I^d(U) \). Hence, \( t \in \uparrow t_k \subseteq \uparrow f_I^d(U) \). Conversely, if \( t \in \uparrow f_I^d(U) \), then there exist \( k \in \{1, \ldots, m\} \) and \( x \in U \) with \( f_I^d(x) = t_k \leq t \). Hence, \( x \in (f_I^d)^{-1}(t_k) \), and there is a connected component \( I' = (p, q) \) of \((f_I^d)^{-1}(t_k)\) containing \( x \). Since \( U \cap I' \neq \emptyset \) and by assumption \( U \subseteq I' \), we have that \( U \cap I' \) is either \((p, d)\) or \((c, q)\). As both \((p, d)\) and \((c, q)\) must intersect the Cantor set constructed in \( I' \) and \( f_I^{t_k} \) is open, we have \( f_I(U) \supseteq f_I(U \cap I') = f_I^{t_k}(U \cap I') = \uparrow t_k \). It follows that \( t \in \uparrow t_k \subseteq f_I(U) \). Therefore, \( f_I(U) = \uparrow f_I^d(U) \), and so \( f_I(U) \) is open. Thus, \( f_I \) is an onto open map, implying that \( T \) is an open image of \( I \). \( \square \)

**Corollary 9.** Every finite rooted poset, or equivalently, every finite well-connected \( T_0 \)-space is an open image of \( \mathbb{R} \).

**Proof.** It follows from Lemma 4 and Theorem 8 that every finite rooted poset is an open image of any bounded interval \( I \subseteq \mathbb{R} \). In particular, if \( I \) is open, then \( I \) is homeomorphic to \( \mathbb{R} \), and so the corollary follows. \( \square \)

**Remark 10.** It follows from Corollary 9 that the Heyting algebra of upsets of a finite rooted poset is isomorphic to a subalgebra of the Heyting algebra \( \mathcal{O}(\mathbb{R}) \) of open subsets of \( \mathbb{R} \). Hence, every finite subdirectly irreducible Heyting algebra is isomorphic to a subalgebra of \( \mathcal{O}(\mathbb{R}) \). This together with the finite model property of the intuitionistic propositional logic \textbf{Int} gives a new proof of completeness of \textbf{Int} with respect to \( \mathcal{O}(\mathbb{R}) \), a fact first established by Tarski [14] back in 1938. Now, applying the Blok–Esakia theorem, we obtain that the Grzegorczyk modal system \( \textbf{Grz} = \textbf{S4} + \Box(\varphi \rightarrow \Box \varphi) \rightarrow \varphi \rightarrow \varphi \) is complete with respect to the Boolean closure \( B(\mathcal{O}(\mathbb{R})) \) of \( \mathcal{O}(\mathbb{R}) \).

We are now in a position to expand on Corollary 9 and show that finite rooted qosets are open images of \( \mathbb{R} \). We start by showing that the quasi-\((q, n)\)-tree \( Q \) of depth 2 shown in Fig. 3 is an open image of \( I \).

**Lemma 11.** If \( X \) has a countable basis and every countable subset of \( X \) is boundary, then for any natural number \( n \) there exist disjoint dense boundary subsets \( A_1, \ldots, A_n \) of \( X \) such that \( X = \bigcup_{i=1}^n A_i \).
Proof. Suppose $\{B_i\}_{i=1}^\infty$ is a countable basis of $X$. Since every countable subset of $X$ is boundary, each $B_i$ is uncountable. We pick from each $B_i$ a point $x_i^1$ and set $A_1 = \{x_i^1\}_{i=1}^\infty$. Since $A_1$ is countable, each $B_i - A_1$ is uncountable. So we pick from each $B_i - A_1$ a point $x_i^2$ and set $A_2 = \{x_i^2\}_{i=1}^\infty$. We repeat the same construction for each $B_i - (A_1 \cup A_2)$ to obtain $A_3$. After repeating the construction $n - 1$ times we obtain $n - 1$ many sets $A_1, \ldots, A_{n-1}$. Finally, we set $A_n = X - \bigcup_{i=1}^{n-1} A_i$. It is clear that different $A_i$'s are disjoint from each other and that $X = \bigcup_{i=1}^{n} A_i$. Moreover, each $A_i$ contains at least one point from every basic open set. Hence, each $A_i$ is dense. Furthermore, no basic open set is a subset of any $A_i$. Therefore, every $A_i$ is boundary. \qed

Lemma 12. $Q$ is an open image of $I$.

Proof. We denote the least cluster of $Q$ by $r$ and its elements by $r_1, \ldots, r_q$. Also for $1 \leq i \leq n$ we denote the $i$-th maximal cluster of $Q$ by $t_i$ and its elements by $t_i^1, \ldots, t_i^q$.

Since the Cantor set $C$ satisfies the conditions of Lemma 11, it can be divided into $q$-many disjoint dense boundary subsets $C_1, \ldots, C_q$. Also each $I_p^m (1 \leq p \leq 2^{m-1}, m \in \omega)$ satisfies the conditions of Lemma 11, and so each $I_p^m$ can be divided into $q$-many disjoint dense boundary subsets $(I_p^m)^1, \ldots, (I_p^m)^q$. Suppose $1 \leq k \leq q$. We define $f_I^Q : I \rightarrow Q$ by putting

$$f_I^Q(x) = \begin{cases} r_i^k & \text{if } x \in \bigcup_{m=i \pmod n} I_p^m (I_p^m)^k \\ r_k & \text{if } x \in C_k \end{cases}.$$  

It is clear that $f_I^Q$ is a well-defined onto map. Similar to Lemma 7 we have

$$(f_I^Q)^{-1}(t_i^1) = \bigcup_{m=i \pmod n} \bigcup_{p=1}^{2^{m-1}} I_p^m \text{ and } (f_I^Q)^{-1}(r) = C.$$  

Hence, $f_I^Q$ is continuous. To show that $f_I^Q$ is open let $U$ be an open interval in $I$. If $U \cap C = \emptyset$, then $f_I^Q(U) \subseteq \bigcup_{i=1}^q t_i^1$. Moreover, since $(I_p^m)^1, \ldots, (I_p^m)^q$ partition $I_p^m$ into $q$-many disjoint dense boundary subsets, $U \cap I_p^m \neq \emptyset$ implies $U \cap (I_p^m)^k \neq \emptyset$ for every $k \in \{1, \ldots, q\}$. Hence, if $f_I^Q(U)$ contains an element of a cluster $t_i^1$, it contains the whole cluster. Thus, $f_I^Q(U)$ is open. Now suppose $U \cap C \neq \emptyset$. Since $C_1, \ldots, C_q$ partition $C$ into $q$-many disjoint dense boundary subsets, $U \cap C_k \neq \emptyset$ for every $k \in \{1, \ldots, q\}$. Hence, $r \subseteq f_I^Q(U)$. Moreover, the same argument as in the proof of Lemma 7 guarantees that every point greater than points in $r$ also belongs to $f_I^Q(U)$. Thus $f_I^Q(U) = Q$, implying that $f_I^Q$ is an onto open map. \qed

Theorem 13. Every finite quasi-$(q, n)$-tree is an open image of $I$.

Proof. This follows along the same lines as the proof of Theorem 8 but is based on Lemma 12 instead of Lemma 7. \qed

Corollary 14. Every finite rooted qoset, or equivalently, every finite well-connected space is an open image of $\mathbb{R}$. 
Theorem 15. $S_4$ is complete with respect to $\langle B(C^\infty(\mathbb{R})), \text{--} \rangle$.

Proof. It is sufficient to show that the closure algebra over a quasi-$(q, n)$-tree is isomorphic to a subalgebra of $\langle B(C^\infty(\mathbb{R})), \text{--} \rangle$. So let $X$ be a quasi-$(q, n)$-tree and $I$ be a bounded interval of $\mathbb{R}$. We denote by $\mathcal{C}$ the Cantor set constructed inside $I$, and by $\mathcal{C}_1, \ldots, \mathcal{C}_q$ disjoint dense boundary subsets of $\mathcal{C}$ constructed in Lemma 11. By Theorem 13 there exists an onto open map $f_I : I \rightarrow X$. We show that for every $x \in X$ we have $(f_I)^{-1}(x) \in B(C^\infty(I))$. If $x$ is a quasi-minimal point of $X$, then by Lemma 12 $(f_I)^{-1}(x) = \mathcal{C}_k$ for some $k \in \{1, \ldots, q\}$. From the proof of Lemma 11 it follows that either $\mathcal{C}_k$ or $\mathcal{C} - \mathcal{C}_k$ is a countable subset of $I$. In either case we have $(f_I)^{-1}(x) \in B(C^\infty(I))$. Now suppose $x$ is neither a quasi-minimal nor a quasi-maximal point of $X$. Then by the proof of Theorem 13, which follows along the same lines as the proof of Theorem 8, $(f_I)^{-1}(x)$ is a countable union of the sets $\mathcal{C}_k^n$, where each $\mathcal{C}_k^n$ is a dense boundary subset of the Cantor set $\mathcal{C}$ constructed inside some open interval $I'$ of $I$. Let $U$ denote the (countable) union of these open intervals. Then by Lemma 11 $(f_I)^{-1}(x) \cup U = (f_I)^{-1}(x)$ is countable. Thus, $(f_I)^{-1}(x) \in B(C^\infty(I))$. Finally, if $x$ is a quasi-maximal point of $X$, then $(f_I)^{-1}(x) = \bigcup_{m=i \mod n} \bigcup_{p=1}^{\infty} (I^n_p)^k$ for some $k \in \{1, \ldots, q\}$, where each $(I^n_p)^k$ is a dense boundary subset of the interval $I^n_p$ constructed inside some open interval of $I$. Let $U$ denote the (countable) union of these open intervals. Then the same argument as above guarantees that $(f_I)^{-1}(x) \cup U = (f_I)^{-1}(x)$ is countable. Therefore, $(f_I)^{-1}(x) \in B(C^\infty(I))$. Thus, the closure algebra over a quasi-$(q, n)$-tree is isomorphic to a subalgebra of $\langle B(C^\infty(I)), \text{--} \rangle$.

Now if $I$ is an open interval, then $I$ is homeomorphic to $\mathbb{R}$. Hence, the closure algebra over a quasi-$(q, n)$-tree is isomorphic to a subalgebra of $\langle B(C^\infty(\mathbb{R})), \text{--} \rangle$, and so $S_4$ is complete with respect to $\langle B(C^\infty(\mathbb{R})), \text{--} \rangle$. \hfill \Box

4. Completeness of $S_4 + S_5 + C$

In this section we show that $S_4 + S_5 + \mathcal{C}$ is complete with respect to the algebra $\langle B(C^\infty(\mathbb{R})), \text{--}, \exists \rangle$. For this, by Theorem 3, it is sufficient to construct an open map from $\mathbb{R}$ onto every finite connected component $X$ such that for every $x \in X$ we have $f^{-1}(x) \in B(C^\infty(\mathbb{R}))$.

Suppose $T_1, \ldots, T_n$ are finite trees (of branching $\geq 2$). Let $T'_i$ and $T''_i$ denote two distinct maximal nodes of $T_i$. Consider the disjoint union $\bigsqcup_{i=1}^n T_i$, and identify $T'_{i-1}$ with $T'_i$ and $T''_{i-1}$ with $T''_i$ with $T'_{i+1}$. We call this construction the tree sum of $T_1, \ldots, T_n$ and denote it by $\bigsqcup_{i=1}^n T_i$ (see Fig. 4).

We can generalize this construction to quasi-trees. Suppose $Q_1, \ldots, Q_n$ are finite $q$-regular quasi-trees (of branching $\geq 2$). Let $C_i^1$ and $C_i^q$ denote two distinct maximal clusters of $Q_i$. Consider the disjoint union $\bigsqcup_{i=1}^n Q_i$, and identify $C_{i-1}^1$ with $C_i^1$ and $C_{i-1}^q$ with $C_i^q$. We call this construction the regular quasi-tree sum of $Q_1, \ldots, Q_n$ and denote it by $\bigsqcup_{i=1}^n Q_i$.

Lemma 16 (Compare with [13, Lemma 13]).
For every finite partially ordered connected component $X$ there exist trees $T_1, \ldots, T_n$ such that $X$ is a $p$-morphic image of $\bigoplus_{i=1}^n T_i$.

For every finite connected component $X$ there exist $q$-regular quasi-trees $Q_1, \ldots, Q_n$ such that $X$ is a $p$-morphic image of $\bigoplus_{i=1}^n Q_i$.

**Proof.** (a) follows from (b) and the fact that the regular quasi-tree sum of trees is in fact their tree sum.

(b) Suppose $X$ is a finite connected component. Let $C_1, \ldots, C_n$ denote minimal clusters of $X$. Consider $(\uparrow C_1, \leq_1), \ldots, (\uparrow C_n, \leq_n)$, where $\leq_i$ is the restriction of $\leq$ to $\uparrow C_i$. Obviously each $(\uparrow C_i, \leq_i)$ is a finite rooted qoset and $\bigcup_{i=1}^n C_i = X$. As follows from Lemma 5, for each $(\uparrow C_i, \leq_i)$ there exist $q_i, m_i$ such that $(\uparrow C_i, \leq_i)$ is a $p$-morphic image of a finite quasi-$(q_i, m_i)$-tree. Let $q = \sup\{q_1, \ldots, q_n\}$, and consider quasi-$(q, m_i)$-trees $Q_1, \ldots, Q_n$. Obviously for each $i$ there exists a $p$-morphism $f_i$ from $Q_i$ onto $(\uparrow C_i, \leq_i)$.

Also note that for each $i$ there exists a maximal cluster $C$ of $X$ such that $C$ is a subset of both $\uparrow C_{i-1}$ and $\uparrow C_i$. Since $f_{i-1}$ is a $p$-morphism, there exists a maximal cluster $D_{i-1}$ of $\bigoplus_{i=1}^n C_{i-1} = (\uparrow C_{i-1}, \leq_{i-1})$ such that $f_{i-1}(D_{i-1}) = C$. Similarly there exists a maximal cluster $D_i$ of $\bigoplus_{i=1}^n Q_i$ such that $f_i(D_i) = C$. We form $\bigoplus_{i=1}^n Q_i$ by identifying $D_{i-1}$ with $D_i$ and $D_i$ with $D_{i+1}$. Now define $f : \bigoplus_{i=1}^n Q_i \to X$ by putting $f(t) = f_i(t)$ for $t \in Q_i$. It is routine to check that $f$ is well defined and that it is an onto $p$-morphism.

**Theorem 17.** The tree sum of finitely many finite trees is an open image of $\mathbb{R}$.

**Proof.** Suppose $T_1, \ldots, T_n$ are finite trees. Consider $\bigoplus_{k=1}^n T_k$. For $2 \leq k \leq n - 1$ let $t^l_k$ and $t^r_k$ denote the maximal nodes of $T_k$ which got identified with the corresponding nodes.

![Diagram](image_url)
observe that if $t \in k$, there exists an onto open map \( f : I_k \to T_k \). Hence, if $t \in k$, then $n \in k$. If $t \in k$, then \( f := \bigcup_{k=1}^{n} T_k \) by putting
\[
f(x) = \begin{cases} f_I(x), & \text{if } x \in I_k \\ t_k, & \text{if } x \in (2k-1, 2k) \\ f_k(x), & \text{if } x \in I_n \end{cases}
\]
where $k \in \{1, \ldots, n-1\}$. It is obvious that $f$ is a well-defined onto map. For $t \in T_k$ observe that if $t_k, t_k' \notin t$, then
\[
f^{-1}(\uparrow t) = f_{k-1}^{-1}(\uparrow t),
\]
if $t_k' \in \uparrow t$ and $t_k' \notin t$, then
\[
f^{-1}(\uparrow t) = f_{k-1}^{-1}(t_k') \cup (2k - 3, 2k - 2) \cup f_k^{-1}(\uparrow t),
\]
if $t_k' \notin \uparrow t$ and $t_k' \notin \uparrow t$, then
\[
f^{-1}(\uparrow t) = f_k^{-1}(\uparrow t) \cup (2k - 1, 2k) \cup f_{k+1}^{-1}(t_k'),
\]
and finally, if $t_k, t_k' \in \uparrow t$, then
\[
f^{-1}(\uparrow t) = f_{k-1}^{-1}(t_k') \cup (2k - 3, 2k - 2) \cup f_k^{-1}(\uparrow t) \cup (2k - 1, 2k) \cup f_{k+1}^{-1}(t_k').
\]
Hence, $f$ is continuous. Moreover, for an open interval $U \subseteq (0, 2n - 1)$, if $U \subseteq I_k$, then $f(U) = f_k(U)$; and if $U \subseteq (2k - 1, 2k)$, then $f(U) = \{t_k\}$. In either case $f(U)$ is open in $\bigcup_{k=1}^{n} T_k$. Now every open interval $U \subseteq (0, 2n - 1)$ is the union $U = U_1 \cup \ldots \cup U_{2n-1}$, where $U_{2k} = U \cap (2k - 1, 2k)$ for $k = 1, \ldots, n - 1$, and $U_{2k+1} = U \cap I_{k+1}$ for

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 2n - 4 & 2n - 3 & 2n - 2 & 2n - 1 \\
\hline
& & & \uparrow & & & & \\
\end{array}
\]
$$k = 0, \ldots, n - 1. \text{ Thus, } f(U) = f(U_1) \cup \ldots \cup f(U_{2n-1}), \text{ and so } f(U) \text{ is an open set in } \bigoplus_{k=1}^{n} T_k. \text{ Hence, } f \text{ is an onto open map, implying that } \bigoplus_{k=1}^{n} T_k \text{ is an open image of } (0, 2n - 1). \text{ Since } (0, 2n - 1) \text{ is homeomorphic to } \mathbb{R}, \text{ we obtain that } \bigoplus_{k=1}^{n} T_k \text{ is an open image of } \mathbb{R}. \quad \square$$

**Corollary 18.** A finite $T_0$-space is an open image of $\mathbb{R}$ iff it is connected.

**Proof.** Since finite connected $T_0$-spaces correspond to finite connected partially ordered components, it follows from Lemma 16 and Theorem 17 that every finite connected $T_0$-space is an open image of $\mathbb{R}$. Conversely, since $\mathbb{R}$ is connected and open (even continuous) images of connected spaces are connected, finite $T_0$ images of $\mathbb{R}$ are connected. \hfill \square

**Theorem 19.** The regular quasi-tree sum of finitely many finite $q$-regular quasi-trees is an open image of $\mathbb{R}$.

**Proof.** This follows along the same lines as the proof of Theorem 17 but is based on Theorem 13 instead of Theorem 8. In addition, according to Lemma 11, for $k = 1, \ldots, n - 1$ we divide each interval $(2k - 1, 2k)$ into $q$-many disjoint dense boundary subsets $A^k_1, \ldots, A^k_r$ and define $f : (0, 2n - 1) \to \bigoplus_{k=1}^{n} Q_k$ by putting

$$f(x) = \begin{cases} f_k(x), & \text{if } x \in I_k \\ (t^x_k)_i, & \text{if } x \in A^k_i \\ f_{\ell_0}(x), & \text{if } x \in I_n \end{cases}$$

where $(t^x_k)_i$ is the $i$-th element of $C^*_{k_i}$ and $k \in \{1, \ldots, n - 1\}$. As a result we obtain that $\bigoplus_{k=1}^{n} Q_k$ is an open image of $(0, 2n - 1)$, and so $\bigoplus_{k=1}^{n} Q_k$ is an open image of $\mathbb{R}$. \hfill \square

**Corollary 20.** A finite topological space is an open image of $\mathbb{R}$ iff it is connected.

**Proof.** This follows along the same lines as the proof of Corollary 18 but is based on Theorem 19 instead of Theorem 17. \hfill \square

**Theorem 21.** $S_4 + S_5 + C$ is complete with respect to $(B(C(\mathbb{R})), \bar{\land}, \exists)$. 

**Proof.** Suppose $Q_1, \ldots, Q_r$ are arbitrary $q$-regular quasi-trees. It is sufficient to show that the $(S_4 + S_5 + C)$-algebra over the regular quasi-tree sum $\bigoplus_{k=1}^{n} Q_k$ is isomorphic to a subalgebra of $(B(C(\mathbb{R})), \bar{\land}, \exists)$. The proof of Theorem 15 implies that for each $Q_k$ there exists $I_k = [2k - 2, 2k - 1]$ and an onto open map $f_k : I_k \to Q_k$ such that for every $t \in Q_k$ we have $f^{-1}_k(t) \in B(C(\mathbb{R}))$. It follows from the proof of Theorem 19 that there exists an onto open map $f : (0, 2n - 1) \to \bigoplus_{k=1}^{n} Q_k$. If $t \in Q_k$ does not belong to either $C^*_1$ or $C^*_r$, then $f^{-1}(t) = f^{-1}_k(t)$, and so $f^{-1}(t) \in B(C(\mathbb{R}(0, 2n - 1)))$.

If $t \in C^*_1$, then $f^{-1}(t)$ is the union of $f^{-1}_k(t) \cup f^{-1}_{k-1}(t)$ with a disjoint dense boundary subset of $(2k - 3, 2k - 2)$ constructed in Theorem 19; and if $t \in C^*_r$, then $f^{-1}(t)$ is the union of $f^{-1}_k(t) \cup f^{-1}_{k+1}(t)$ with a disjoint dense boundary subset of $(2k - 1, 2k)$ constructed in the same theorem. In either case $f^{-1}(t) \in B(C(\mathbb{R}(0, 2n - 1)))$. Therefore, $f^{-1}(t) \in B(C(\mathbb{R}(0, 2n - 1)))$ for every $t \in \bigoplus_{k=1}^{n} Q_k$. Thus, the $(S_4 + S_5 + C)$-algebra over $\bigoplus_{k=1}^{n} Q_k$ is isomorphic to a subalgebra of $(B(C(\mathbb{R}(0, 2n - 1))), \bar{\land}, \exists)$, and so it is isomorphic to a subalgebra of $(B(C(\mathbb{R})), \bar{\land}, \exists)$. It follows that $S_4 + S_5 + C$ is complete with respect to $(B(C(\mathbb{R})), \bar{\land}, \exists)$. \hfill \square
5. Conclusions

In this paper we proved that $S_4$ is complete with respect to the closure algebra $(B(C^∞(R)), \neg)$. It follows that $S_4$ is complete with respect to any closure algebra containing $(B(C^∞(R)), \neg)$ and contained in $(P(R)), \neg)$. One closure algebra in the interval $[B(C^∞(R)), \neg), (P(R)), \neg)]$ deserves special mention. Let $\mathfrak{B}(R)$ denote the Boolean algebra of Borel sets over open subsets of $R$; that is $\mathfrak{B}(R)$ is the countably complete Boolean algebra countably generated by $O(R)$. It is obvious that $B(C^∞(R)) \subseteq \mathfrak{B}(R) \subseteq P(R)$. In fact, both of the inclusions are proper. As a result we obtain that $S_4$ is complete with respect to the closure algebra $(\mathfrak{B}(R), \neg)$.

In Remark 10 we pointed out that the modal system $\textbf{Grz}$ is complete with respect to the closure algebra $(B(O(R)), \neg)$. It still remains an open problem to classify the complete logics of the closure algebras in between $B(O(R)), \neg)$ and $B(C^∞(R)), \neg)$.

In the language $L(\forall)$ a natural extension of $\textbf{Grz}$ is the bimodal system $\textbf{Grz} + S_5 + C$. However, it remains an open problem whether $\textbf{Grz} + S_5 + C$ has the finite model property. Therefore, it is still an open problem whether $\textbf{Grz} + S_5 + C$ is complete with respect to $(B(O(R)), \neg), \exists)$. Let $B(C(R))$ denote the Boolean algebra generated by $C(R)$. It was proved in Aiello et al. [1] that the complete logic of $(B(C(R)), \neg)$ is the complete logic of the closure algebra over the 2-tree of depth 2. This result was extended to the bimodal language $L(\forall)$ in van Benthem et al. [15]. It still remains an open problem to classify the complete logics of the closure algebras in the interval $[(B(C(R)), \neg), (B(O(R)), \neg)]$, as well as the bimodal logics of the $(S_4 + S_5 + C)$-algebras in the interval $[(B(C(R)), \neg), \exists), (B(C^∞(R)), \neg), \exists)]$.

References