Zero-dimensional proximities and zero-dimensional compactifications

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We introduce zero-dimensional proximities and show that the poset \( \langle Z(X), \leq \rangle \) of inequivalent zero-dimensional compactifications of a zero-dimensional Hausdorff space \( X \) is isomorphic to the poset \( \langle \mathcal{P}(X), \leq \rangle \) of zero-dimensional proximities on \( X \) that induce the topology on \( X \). This solves a problem posed by Leo Esakia. We also show that \( \langle \mathcal{P}(X), \leq \rangle \) is isomorphic to the poset \( \langle \mathcal{B}(X), \subseteq \rangle \) of Boolean bases of \( X \), and derive Dwinger's theorem that \( \langle Z(X), \leq \rangle \) is isomorphic to \( \langle \mathcal{B}(X), \subseteq \rangle \) as a corollary. As another corollary, we obtain that for a regular extremally disconnected space \( X \), the Stone–Čech compactification of \( X \) is a unique up to equivalence extremally disconnected compactification of \( X \).

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1. Introduction

The notion of proximity on a set \( X \) was introduced by Efremovič [4]. It was shown by Smirnov [14] that the poset \( \langle C(X), \leq \rangle \) of inequivalent compactifications of a completely regular space \( X \) is isomorphic to the poset \( \langle \Sigma(X), \leq \rangle \) of proximities on \( X \) that induce the topology on \( X \). We refer to this as the Smirnov theorem. It provides a characterization of the structure of compactifications of a completely regular space \( X \) by means of proximities on \( X \) that induce the topology on \( X \). Put differently, compactifications of \( X \) give a characterization of the structure of proximities on \( X \) that are compatible with the topology on \( X \). An excellent account of the theory of proximity spaces in general, and of the Smirnov theorem in particular can be found in Naimpally and Warrack [13], which is our primary source of reference.

Compact Hausdorff spaces that in addition are zero-dimensional play a fundamental role in the theory of Boolean algebras and related structures. The celebrated Stone theorem [15] asserts that the category of Boolean algebras and Boolean algebra homomorphisms is dually equivalent to the category of compact Hausdorff zero-dimensional spaces and continuous maps. Because of this fundamental result, compact Hausdorff zero-dimensional spaces are often called Stone spaces. It is only natural to seek a characterization of Stone compactifications of completely regular spaces. That is, we are interested in zero-dimensional compactifications of a completely regular space \( X \). Since a subspace of a zero-dimensional space is also zero-dimensional, \( X \) has to be zero-dimensional as well. Moreover, because in the realm of zero-dimensional spaces, completely regular becomes simply equivalent to Hausdorff (and even to \( T_0 \)), we are interested in a characterization of zero-dimensional compactifications of a zero-dimensional Hausdorff space. Such a characterization was given by Dwinger [3] by means of Boolean bases (see also Magill and Glasenapp [12], where the authors prove the Dwinger theorem independently, apparently being unaware of Dwinger’s result). On the other hand, by the Smirnov theorem, zero-dimensional compacti-
2. Compactifications and proximities

We start by recalling the basic facts about compactifications of completely regular spaces. All the necessary information can be found in Engelking’s excellent textbook [5].

Let $X$ be a topological space. We recall that a compactification of $X$ is a pair $(Y, e)$, where $Y$ is a compact Hausdorff space and $e : X \to Y$ is a homeomorphic embedding such that $e[X]$ is dense in $Y$. It is a well-known theorem of Tychonoff that $X$ has a compactification iff $X$ is completely regular. Because of this, we restrict ourselves to completely regular spaces throughout the paper.

Let $X$ be a completely regular space. If $(Y, e)$ and $(Z, k)$ are two compactifications of $X$, then we order them by $(Y, e) \leq (Z, k)$ iff there is a continuous $f : Z \to Y$ such that $f \circ k = e$. Then it is well known that (i) $f$ is actually onto, that (ii) $\leq$ is reflexive and transitive, but not antisymmetric, and that (iii) from $(Y, e) \leq (Z, k)$ and $(Z, k) \leq (Y, e)$ it follows that there is a homeomorphism $f : Y \to Z$ such that $f \circ k = e$. Thus, $\leq$ is a partial order on the equivalence classes of $\leq$.

Let $C(X)$ be the set of inequivalent compactifications of $X$. We can view $(C(X), \leq)$ as a poset. It is well known that the Stone–Čech compactification $\beta(X)$ of $X$ is the largest element of $(C(X), \leq)$. However, in general $(C(X), \leq)$ may not have a least element. In fact, $(C(X), \leq)$ has a least element iff $X$ is locally compact. If $X$ is a noncompact locally compact space, then it is well known that the least element of $(C(X), \leq)$ is the one-point compactification $\alpha(X)$ of $X$ constructed by Alexandroff (see, e.g., [5, Sec. 3.5]).

The structure of $(C(X), \leq)$ is rather complicated. Several interesting results on the structure of $(C(X), \leq)$ can be found in the work of Magill [9–11]. One way to characterize $(C(X), \leq)$ is through proximities on $X$. A proximity on $X$ is a binary relation $\delta$ on the power set of $X$ that captures the idea of two subsets of $X$ to be close to each other. There are several equivalent axiomatizations of proximities. We have chosen one given in [13].

**Definition 2.1. ([13, p. 7])** A proximity on a set $X$ is a binary relation $\delta$ on the power set of $X$ satisfying the following six conditions:

1. $\delta$ is symmetric;
2. $A \delta (B \cup C)$ iff $A \delta B$ or $A \delta C$;
3. $A \delta B$ implies $A, B \not\equiv \emptyset$;
4. $A \not\delta B$ implies there is a $C \subseteq X$ such that $A \not\delta C$ and $(X - C) \not\delta B$;
5. $A \cap B \not\equiv \emptyset$ implies $A \delta B$;
6. $x \not\equiv y$ implies $x \not\delta y$.

Note that we should really write $\langle x \rangle \not\delta \langle y \rangle$ (and $\langle x \rangle \delta \langle y \rangle$), but we slightly abuse the notation and write $x \not\delta y$ (and $x \delta y$) instead. We will also write $x \delta A$ and $x \not\delta A$ instead of $\langle x \rangle \delta A$ and $\langle x \rangle \not\delta A$.

Axiom (P4) is the most fundamental among the above six axioms. Because of its importance, some authors call it the strong axiom (see, e.g., [13, p. 9]). In the next section we will strengthen it even further in order to axiomatize zero-dimensional proximities.

Note that some authors (including Naimpally and Warrack [13]) do not assume (P6); when a proximity satisfies (P6), they call it a separated (or a Hausdorff) proximity. In this paper we are interested only in separated proximities, thus we assume from the outset that $\delta$ satisfies (P1)–(P6). If $\delta$ is a proximity on $X$, then we call the pair $(X, \delta)$ a proximity space.

The following properties of proximities are easy to verify and are useful in calculations:

1. $A \delta B$, $A \subseteq C$, and $B \subseteq D$ imply $C \delta D$;
2. $\emptyset \not\delta A$ for each $A \subseteq X$;
3. $A \not\delta B$ implies there are $C, D \subseteq X$ such that $A \not\delta C$, $B \not\delta D$, and $C \cup D = X$;
4. $A \not\delta B$ implies there is an $x \in X$ such that $A \not\delta x$ and $x \not\delta B$.

Given two proximities $\delta_1$ and $\delta_2$ on $X$, we set $\delta_1 \leq \delta_2$ iff $A \delta_2 B$ implies $A \delta_1 B$ for each $A, B \subseteq X$. Obviously $\leq$ is a partial order on the set $\Sigma(X)$ of all proximities of $X$. fictions of $X$ correspond to special proximities on $X$ compatible with the topology on $X$. Back in the 1960s, Leo Esakia posed an open problem to axiomatize/characterize such proximities. As far as we know, this problem remained unsolved until now. In this paper we solve this problem by providing a natural axiomatization of the proximities corresponding to the zero-dimensional compactifications of a zero-dimensional Hausdorff space by strengthening the most nontrivial axiom defining a proximity space, known as the strong axiom. We call the proximities satisfying this new axiom zero-dimensional proximities and show that the poset $\langle Z(X), \leq \rangle$ of inequivalent zero-dimensional compactifications of a zero-dimensional Hausdorff space $X$ is isomorphic to the poset $\langle \Pi(X), \leq \rangle$ of zero-dimensional proximities on $X$ that induce the topology on $X$. We also show that $\langle \Pi(X), \leq \rangle$ is isomorphic to the poset $\langle \mathcal{B}(X), \subseteq \rangle$ of Boolean bases of $X$, and obtain Dwinger’s theorem as a corollary. As another corollary, we show that for a regular extremally disconnected space $X$, the Stone–Čech compactification of $X$ is a unique up to equivalence extremally disconnected compactification of $X$. (P3)}

A (P6)}

(x, y) \not\in \delta
Let \( (X, \delta) \) be a proximity space. We set
\[
\text{cl}(A) = \{ x \in X : x \delta A \}.
\]

It is well known (see, e.g., [13, Thm. 2.3]) that cl satisfies the four Kuratowski axioms of a closure operator, hence induces a topology on \( X \), called the \textit{topology induced by the proximity} \( \delta \). When \( \delta \) induces a topology on \( X \), we say that \( \delta \) is \textit{compatible} with the topology on \( X \).

It is a fundamental result of Smirnov [14] (see also [13, Ch. II]) that for a completely regular space \( X \), the poset \( \langle \mathcal{C}(X), \subseteq \rangle \) of inequivalent compactifications of \( X \) is isomorphic to the poset \( \langle \Sigma(X), \subseteq \rangle \) of proximities on \( X \) that induce the topology on \( X \). To make the paper self-contained, we give a brief outline of the proof.

\[ \text{Theorem 2.2} \quad \text{(Smirnov theorem). Let } X \text{ be a completely regular space. Then the poset } \langle \mathcal{C}(X), \subseteq \rangle \text{ of inequivalent compactifications of } X \text{ is isomorphic to the poset } \langle \Sigma(X), \subseteq \rangle \text{ of proximities on } X \text{ that induce the topology on } X. \]

\[ \text{Proof. (Sketch) Let } (Y, e) \text{ be a compactification of } X \text{ and let } \text{cl}_Y \text{ denote the closure operator of } Y. \text{ Define } \delta_Y \text{ on } X \text{ by} \]
\[
A \delta_Y B \quad \text{iff} \quad \text{cl}_Y(e[A]) \cap \text{cl}_Y(e[B]) \neq \emptyset.
\]

It can be verified that \( \delta_Y \) is a proximity on \( X \) that induces the topology on \( X \).

Conversely, let \( \delta \) be a proximity on \( X \) that induces the topology on \( X \). We recall (see, e.g., [13, Def. 5.4]) that a subset \( \sigma \) of the powerset of \( X \) is a \textit{cluster} if:

\[ \begin{align*}
(\text{C1}) & \quad A, B \in \sigma \text{ implies } A \delta B; \\
(\text{C2}) & \quad \text{if } A \delta B \text{ for each } B \in \sigma, \text{ then } A \in \sigma; \\
(\text{C3}) & \quad A \cup B \in \sigma \text{ implies } A \in \sigma \text{ or } B \in \sigma.
\end{align*} \]

It follows from (P4) that each \( \sigma_X = \{ A : x \delta A \} \) is a cluster, called a \textit{point cluster}. Also, it is easy to verify that for each cluster \( \sigma \) we have (i) \( X \in \sigma \), (ii) \( A \in \sigma \) and \( A \subseteq \mathcal{B} \) imply \( B \in \sigma \), and (iii) \( A \in \sigma \) or \( X - A \in \sigma \).

Note that clusters are not in general closed under finite intersections, hence are not necessarily (ultra)filters. Nevertheless, there is a close connection between ultrafilters and clusters on \( (X, \delta) \). Indeed, as follows from [13, Thm. 5.8], \( \sigma \) is a cluster on \( (X, \delta) \) iff there is an ultrafilter \( \mathcal{V} \) on \( X \) such that
\[ \sigma = \{ A : A \delta B \text{ for each } B \in \mathcal{V} \}. \]

However, there may exist different ultrafilters \( \mathcal{V}_1 \neq \mathcal{V}_2 \) determining the same cluster \( \sigma \). Thus, we do not have a 1–1 correspondence between ultrafilters and clusters.

Now we construct the compactification \( (Y_\delta, e_\delta) \) of \( X \) using an adaptation of the Stone construction: Let \( Y_\delta \) be the set of all clusters of \( (X, \delta) \), and let \( e_\delta(x) = \sigma_X \). Then it is easy to see that \( e_\delta : X \to Y_\delta \) is a well-defined 1–1 map. Let \( \varphi : \mathcal{P}(X) \to \mathcal{P}(Y_\delta) \) be the Stone map:
\[ \varphi(A) = \{ \sigma \in \mathcal{P} : A \in \sigma \}. \]

Then it is easy to see that the following hold:
\[ \begin{align*}
(1) & \quad \varphi(\emptyset) = \emptyset \text{ and } \varphi(X) = Y_\delta; \\
(2) & \quad \varphi(A \cup B) = \varphi(A) \cup \varphi(B); \\
(3) & \quad \varphi(A \cap B) \subseteq \varphi(A) \cap \varphi(B); \\
(4) & \quad Y_\delta - \varphi(A) \subseteq X - A.
\end{align*} \]

But in general \( \varphi(A \cap B) \neq \varphi(A) \cap \varphi(B) \) and \( \varphi(X - A) \neq Y_\delta - \varphi(A) \). Define \( \delta^* \) on \( Y_\delta \) by
\[
\mathcal{P} \delta^* \mathcal{Q} \quad \text{iff for each } A, B \subseteq X \text{ from } \mathcal{P} \subseteq \varphi(A) \text{ and } \mathcal{Q} \subseteq \varphi(B) \text{ it follows that } A \delta B.
\]

Then \( \delta^* \) is a proximity on \( Y_\delta \), hence it induces a completely regular topology on \( Y_\delta \). In fact, \( Y_\delta \) is compact because each cluster on \( Y_\delta \) is a point cluster [13, Lem. 7.5]. In addition, the following identity holds:
\[ \varphi(A) = \text{cl}_{Y_\delta}(e_\delta[A]). \]

Consequently, \( \varphi(A) \) is closed for each \( A \), and the set \( \{ \varphi(A) : A \subseteq X \} \) forms a basis for the closed subsets of \( Y_\delta \). Since \( Y_\delta = \text{cl}_{Y_\delta}(e_\delta[X]) \), we obtain that \( e_\delta[X] \) is dense in \( Y_\delta \). Moreover, \( A \delta B \) iff \( \varphi(A) \delta^* \varphi(B) \). Therefore, \( e_\delta \) is a proximity isomorphism, and hence a homeomorphism from \( X \) onto \( e_\delta[X] \). It follows that \( (Y_\delta, e_\delta) \) is a compactification of \( X \).

Since \( A \delta B \) iff \( \varphi(A) \delta^* \varphi(B) \) iff \( \text{cl}_{Y_\delta}(e_\delta[A]) \delta^* \text{cl}_{Y_\delta}(e_\delta[B]) \), we obtain that the proximity \( \delta_{Y_\delta} \) on \( X \) corresponding to the compactification \( (Y_\delta, e_\delta) \) is exactly \( \delta \). The converse is also true: Given a compactification \( (Y, e) \) of \( X \), the compactification \( (Y^*_\delta, e^*_\delta) \) corresponding to the proximity \( \delta_Y \) is equivalent to the compactification \( (Y, e) \). Thus, we obtain a 1–1 correspondence between proximities on \( X \) that induce the topology on \( X \) and inequivalent compactifications of \( X \). In fact, this 1–1 correspondence is an isomorphism between the poset \( \langle \Sigma(X), \subseteq \rangle \) of proximities on \( X \) that induce the topology on \( X \) and the poset \( \langle \mathcal{C}(X), \subseteq \rangle \) of inequivalent compactifications of \( X \), which concludes (the sketch of the) proof. □
Remark 2.3. As we pointed out earlier, the Stone–Čech compactification $\beta(X)$ is the largest element of $<C(X), \leq>$. Consequently, the Smirnov theorem implies that $<\Sigma(X), \leq>$ also has a largest element. It follows from [13, Rem. 3.15] that the largest element $\delta^\beta_\beta$ of $<\Sigma(X), \leq>$ is given by
\[
A \delta^\beta_\beta B \quad \text{iff} \quad \text{there is a continuous } f: X \to [0,1] \text{ such that } A \subseteq f^{-1}(0) \text{ and } B \subseteq f^{-1}(1).
\]
Moreover, whenever $X$ is noncompact locally compact, the one-point compactification $\alpha(X)$ is the least element of $<C(X), \leq>$. As follows from [13, p. 45], the corresponding least element $\delta^\alpha_\alpha$ of $<\Sigma(X), \leq>$ is given by
\[
A \delta^\alpha_\alpha B \quad \text{iff} \quad \text{cl}(A) \cap \text{cl}(B) = \emptyset \text{ and either cl}(A) \text{ or cl}(B) \text{ is compact}.
\]

Remark 2.4. There is an alternative way to develop the theory of proximity spaces. Let $<X, \delta>$ be a proximity space and let $A, B \subseteq X$. Following [13, Sec. 3], we call $B$ a $\delta$-neighborhood of $A$, and write $A \prec B$, if $A \not\subsetneq (X - B)$. Then $\prec$ is a binary relation on the powerset of $X$. By [13, Thm. 3.9], $\prec$ can be axiomatized as follows:

(W1) $X \prec X$;
(W2) $A \prec B$ implies $A \subseteq B$;
(W3) $A \subseteq B \prec C \subseteq D$ implies $A \prec D$;
(W4) $A \prec B, C$ implies $A \prec B \cap C$;
(W5) $A \prec B$ implies $X - B \prec X - A$;
(W6) $A \prec B$ implies there is a $C \subseteq X$ such that $A \prec C < B$;
(W7) $x \not\in y$ implies $x \prec X - y$.

The relation $\prec$ captures the idea of two subsets of $X$ to be far away from each other; that is, $A \prec B$ means that $A$ is far away from $X - B$. Engelking [5, Sec. 8.4] calls the relation $\prec$ a strong inclusion. We prefer to call it a way below relation. It follows that each proximity $\delta$ on $X$ gives rise to the way below relation $\prec_\delta$ on $X$. In fact, the converse is also true: given a way below relation $\prec$ on $X$, define $\delta_\prec$ by
\[
A \delta_\prec B \quad \text{iff} \quad A \not\subsetneq (X - B).
\]
By [13, Thm. 3.11], $\delta_\prec$ is a proximity on $X$. Moreover, in the topology induced by $\delta_\prec$, a set $U$ is open if and only if $U = \{x \in X: x \prec U\}$. In addition, we have $\delta = \delta_\prec$ and $\prec = \prec_\delta$. Thus, there is a 1–1 correspondence between proximities and way below relations on $X$. This 1–1 correspondence is in fact an isomorphism between the respective posets. Consequently, we can develop the theory of compactifications of a completely regular space $X$ by means of way below relations on $X$. If we do so, then instead of working with clusters of $X$, which become the points of the compactification of $X$ corresponding to the proximity $\delta$, it is more convenient to work with ends. The notion of an end is dual to that of a cluster. We recall (see, e.g., [13, Def. 6.4]) that a subset $\eta$ of the powerset of $X$ is an end if:

(E1) $B, C \in \eta$ implies there is a nonempty subset $A \in \eta$ such that $A \prec B$ and $A \prec C$;
(E2) $A \prec B$ implies that either $X - A \in \eta$ or $B \in \eta$.

That an end is the dual notion of a cluster follows from the following observation: $\eta$ is an end if and only if $\eta^* = \{A: X - A \notin \eta\}$ is a cluster (see, e.g., [13, Thm. 6.11]). Consequently, given a way below relation $\prec$ on $X$ that is compatible with the topology on $X$, we can take ends as the points of the compactification of $X$ corresponding to $\prec$. In fact, this is the technique Smirnov used in proving his theorem [14]. That the notion of an end is dual to that of a cluster, and that compactifications of $X$ can be constructed by means of clusters was discovered later by Leader [7,8] (see also [13, Ch. II]).

3. Zero-dimensional compactifications and zero-dimensional proximities

We recall that a subset of a topological space $X$ is clopen if it is both closed and open, and that $X$ is zero-dimensional if the set of clopen subsets of $X$ forms a basis for the topology. Clearly the notions of Hausdorff and $T_0$ coincide in the realm of zero-dimensional spaces. But more is true: each zero-dimensional $T_0$-space is actually completely regular. This is easy to see. First observe that since $X$ is zero-dimensional and $T_0$, it is Hausdorff. Now let $x \notin F$, where $F$ is a closed subset of $X$. Then $X - F$ is an open neighborhood of $x$. Since $X$ is zero-dimensional, there is a clopen subset $A$ of $X$ such that $x \in A \subseteq X - F$. Therefore, $x \in A$ and $A \cap F = \emptyset$. We view $[0,1]$ as a discrete space and define $f: X \to [0,1]$ by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in X - A$. Since $A$ is clopen, we obtain that $f$ is continuous. Moreover, $f(x) = 0$ and $F \subseteq X - A = f^{-1}(1)$. Therefore, $f$ is a continuous function from $X$ onto $[0,1]$ separating $x$ from $F$. By composing $f$ with the inclusion $[0,1] \hookrightarrow [0,1]$, we obtain a continuous function $g: X \to [0,1]$ separating $x$ from $F$. Thus, $X$ is completely regular.

It follows from the Smirnov theorem that the poset of inequivalent compactifications of a completely regular space $X$ is isomorphic to the poset of proximities on $X$ that induce the topology on $X$. If $X$ is in addition zero-dimensional, then the Smirnov theorem implies that the zero-dimensional compactifications of $X$ are characterized by special proximities on $X$ that induce the topology on $X$. We call such proximities zero-dimensional. They are axiomatized by the following strengthening of axiom (P4).
Definition 3.1. Let $\delta$ be a proximity on $X$. We call $\delta$ a zero-dimensional proximity if it satisfies the following strong version of axiom (P4):

\[(SP4)\quad A \not\mathrel{\delta} B \text{ implies there is a } C \subseteq X \text{ such that } C \not\mathrel{\delta} (X - C), \ A \not\mathrel{\delta} C, \text{ and } (X - C) \not\mathrel{\delta} B.\]

Whenever $\delta$ is a zero-dimensional proximity on $X$, we call the pair $(X, \delta)$ a zero-dimensional proximity space.

Remark 3.2. Obviously axiom (SP4) can be rewritten by means of the way below relation $\prec_\delta$ as follows:

\[A \prec_\delta (X - B) \text{ implies there is a } C \subseteq X \text{ such that } C \prec_\delta C, \ A \prec_\delta (X - C), \text{ and } B \prec_\delta C.\]

Therefore, if one prefers to work with way below relations instead of proximities, then since axiom (P4) is equivalent to axiom (W6), an equivalent way to define a zero-dimensional way below relation is by postulating axioms (W1)-(W5) and (W7), and strengthening axiom (W6) as follows:

\[(SW6)\quad A \prec B \text{ implies there is a } C \subseteq X \text{ such that } C \prec C \text{ and } A \prec C \prec B.\]

Lemma 3.3. Let $X$ be a zero-dimensional Hausdorff space and let $(Y, e)$ be a zero-dimensional compactification of $X$. Then $\delta_Y$ is a zero-dimensional proximity on $X$.

Proof. Recall that $A \delta_Y B$ iff $cl_Y(e[A]) \cap cl_Y(e[B]) \neq \emptyset$. It follows from the Smirnov theorem that $\delta_Y$ is a proximity on $X$. We show that $\delta_Y$ satisfies axiom (SP4). Let $A \not\mathrel{\delta_Y} B$. Then $cl_Y(e[A]) \cap cl_Y(e[B]) = \emptyset$. Since $Y$ is compact Hausdorff zero-dimensional and both $cl_Y(e[A])$ and $cl_Y(e[B])$ are closed in $Y$, there is a clopen subset $U$ of $Y$ such that $cl_Y(e[A]) \subseteq U$ and $U \cap cl_Y(e[B]) = \emptyset$. Let $C = X - e^{-1}(U)$. Then

\[cl_Y(e[A]) \cap cl_Y(e[C]) \subseteq U \cap cl_Y(e[X - e^{-1}(U)]) \subseteq U \cap (Y - U) = \emptyset.\]

Therefore, $A \not\mathrel{\delta_Y} C$. Also, $X - C = e^{-1}(U)$, and so

\[cl_Y(e[C]) \cap cl_Y(e[X - C]) = cl_Y(e[X - e^{-1}(U)]) \cap cl_Y(e[e^{-1}(U)]) \subseteq (Y - U) \cap U = \emptyset,\]

and so $C \not\mathrel{\delta_Y} (X - C)$. It follows that there is a $C \subseteq X$ such that $C \not\mathrel{\delta_Y} (X - C)$, $A \not\mathrel{\delta_Y} C$, and $(X - C) \not\mathrel{\delta_Y} B$. Consequently, $\delta_Y$ satisfies axiom (SP4), and so it is a zero-dimensional proximity on $X$. □

Let $\delta$ be a zero-dimensional proximity on $X$ and let $(Y_\delta, e_\delta)$ be the corresponding compactification of $X$. It is our goal to show that $(Y_\delta, e_\delta)$ is zero-dimensional.

Lemma 3.4. Let $(X, \delta)$ be a proximity space. For $A \subseteq X$, the following conditions are equivalent:

1. $A \not\mathrel{\delta} (X - A)$.
2. $\varphi(A) \cap \varphi(X - A) = \emptyset$.
3. $\varphi(X - A) = Y_\delta - \varphi(A)$.

Moreover, if one of the above three conditions is satisfied, then $\varphi(A)$ is clopen in $Y_\delta$.

Proof. (1) implies (2): Let $\sigma \in \varphi(A) \cap \varphi(X - A)$. Then $\sigma \in \varphi(A)$ and $\sigma \in \varphi(X - A)$. Therefore, $A \in \sigma$ and $X - A \in \sigma$. Thus, $A \not\mathrel{\delta} (X - A)$, which contradicts (1). Consequently, $\varphi(A) \cap \varphi(X - A) = \emptyset$.

(2) implies (3): The inclusion $Y_\delta - \varphi(A) \subseteq \varphi(X - A)$ holds always. If $\varphi(A) \cap \varphi(X - A) = \emptyset$, then $\varphi(X - A) \subseteq Y_\delta - \varphi(A)$. Therefore, $\varphi(X - A) = Y_\delta - \varphi(A)$.

(3) implies (2): If $\varphi(X - A) = Y_\delta - \varphi(A)$, then $\varphi(A) \cap \varphi(X - A) = \varphi(A) \cap (Y_\delta - \varphi(A)) = \emptyset$.

(2) implies (1): Let $A \not\mathrel{\delta} (X - A)$. By [13, Thm. 5.14], there is a cluster $\sigma$ of $X$ such that $A, X - A \in \sigma$. Therefore, $\sigma \in \varphi(A) \cap \varphi(X - A)$, which contradicts (2). Thus, $A \not\mathrel{\delta} (X - A)$.

Moreover, since each set of the form $\varphi(E), E \subseteq X$, is closed in $Y_\delta$, we have that $\varphi(A)$ is closed in $Y_\delta$. In addition, $\varphi(X - A) = Y_\delta - \varphi(A)$ implies $Y_\delta - \varphi(A)$ is closed in $Y_\delta$. Therefore, $\varphi(A)$ is open in $Y_\delta$, and so the three equivalent conditions imply that $\varphi(A)$ is clopen in $Y_\delta$. □

However, there exist a zero-dimensional proximity space $(X, \delta)$ and a subset $A$ of $X$ such that $A \not\mathrel{\delta} (X - A)$, but nevertheless $\varphi(A)$ is clopen in $Y_\delta$, as the following example shows.
Example 3.5. Consider the ordinal $\omega \cdot 2 + 1$. Then $\omega \cdot 2 + 1$ is compact Hausdorff zero-dimensional in its interval topology. Moreover, $\omega \cdot 2$ is a noncompact zero-dimensional Hausdorff subspace of $\omega \cdot 2 + 1$, which is dense in $\omega \cdot 2 + 1$. Therefore, we can think of $\omega \cdot 2 + 1$ as the one-point compactification of $\omega \cdot 2$. For $A, B \subseteq \omega \cdot 2$ set $A \delta B$ iff the closures of $A$ and $B$ in $\omega \cdot 2 + 1$ intersect. Let $E = \omega \cdot 2 - \{\omega\}$. Then $E$ is dense in $\omega \cdot 2 + 1$. Therefore, $\varphi(E) = \omega \cdot 2 + 1$ is clopen in $\omega \cdot 2 + 1$.

Moreover, let $Y$ be a (zero-dimensional) compactification of a completely regular (zero-dimensional) space $X$. If there exists a proper dense subset $E$ of $X$, then $\varphi(A) = Y$ is clopen in $Y$, but the closures of $E$ and $X - E$ in $Y$ intersect, and so $E \delta (X - E)$.

Lemma 3.6. If $\delta$ is a zero-dimensional proximity on $X$, then $(Y_\delta, e_\delta)$ is zero-dimensional.

Proof. Let $\sigma, \rho$ be two distinct points of $Y_\delta$. By (C2), there exist $A \in \sigma$ and $B \in \rho$ such that $A \not\in \rho$. By (SP4), there is a $C \subseteq X$ such that $C \not\in (X - C), A \not\in C$, and $(X - C) \not\in B$. By Lemma 3.4, $C \not\in (X - C)$ implies $\varphi(X - C) = Y_\delta - \varphi(C)$, and so $\varphi(C)$ and $\varphi(X - C)$ are complementary clopen subsets of $Y_\delta$. By (P5), $A \not\in C$ implies $A \cap C = \emptyset$. Therefore, $A \subseteq X - C$, so $X - C \in \sigma$, and so $\sigma \not\in \varphi(X - C) = Y_\delta - \varphi(C)$. Similarly, $(X - C) \not\in B$ implies $(X - C) \cap B = \emptyset$. Thus, $B \subseteq C$, so $C \in \rho$, and so $\rho \not\in \varphi(C)$. Consequently, $\sigma \not\in \varphi(C)$ and $\rho \not\in \varphi(C)$, and so we found a clopen subset $\varphi(C)$ of $Y_\delta$ separating $\sigma$ and $\rho$. This means that $Y_\delta$ is zero-dimensional. □

Now, by putting Lemmas 3.3 and 3.6 together with the Smirnov theorem, we clinch our desired result:

Theorem 3.7. For a zero-dimensional Hausdorff space $X$, the poset $\langle Z(X), \subseteq \rangle$ of inequivalent zero-dimensional compactifications of $X$ is isomorphic to the poset $\langle IT(X), \subseteq \rangle$ of zero-dimensional proximities on $X$ that induce the topology on $X$.

Remark 3.8. The poset $\langle Z(X), \subseteq \rangle$ also has a largest element, we denote by $\gamma(X)$, but it may not coincide with the Stone–Čech compactification $\beta(X)$, because $\beta(X)$ may not be zero-dimensional, hence may not belong to $Z(X)$. Nevertheless, the construction of the largest zero-dimensional proximity $\delta_\gamma$ corresponding to $\gamma(X)$ is similar to that of $\delta_\rho$ (see Remark 2.3): we view $[0, 1]$ as a discrete space and set

$A \delta_\gamma B$ iff there is a continuous $f : X \to [0, 1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

On the other hand, whenever a zero-dimensional space $X$ is noncompact locally compact, then the one-point compactification $\alpha(X)$ of $X$ is also zero-dimensional [1, Cor. 3.16]. In this case $\langle Z(X), \subseteq \rangle$ has a least element, and the corresponding least element $b_\alpha$ of $\langle IT(X), \subseteq \rangle$ is defined the same way as in Remark 2.3.

4. Zero-dimensional proximities and Boolean bases

Let $X$ be a zero-dimensional Hausdorff space. The first characterization of the poset $\langle Z(X), \subseteq \rangle$ of inequivalent zero-dimensional compactifications of $X$ was given by Dwinger [3] by means of Boolean bases of $X$. Several interesting results about the structure of $\langle Z(X), \subseteq \rangle$ were obtained by Magill and Glasenapp [12]. Theorem 3.7 provides another characterization of $\langle Z(X), \subseteq \rangle$ by means of zero-dimensional proximities on $X$ compatible with the topology on $X$. The goal of this section is to show that the Dwinger theorem is a corollary of Theorem 3.7.

Lemma 4.1. Let $(X, \delta)$ be a proximity space.

1. If $A \delta (X - A)$, then $A$ is clopen in $X$.

2. Set $B_\delta = \{A : A \delta (X - A)\}$. Then $B_\delta$ is a Boolean algebra. Moreover, for $A, B \in B_\delta$ we have $A \delta B$ iff $A \cap B \neq \emptyset$.

Proof. (1) Let $x \in A$. If $x \delta (X - A)$, then, by (P2), $A \delta (X - A)$, a contradiction. Therefore, for each $x \in A$ we have $x \not\delta (X - A)$. This, by [13, Cor. 2.4], means that $A$ is open. To see that $A$ is closed, let $x \not\delta A$. From $A \not\delta (X - A)$, by (P4), it follows that there is a $C \subseteq X$ such that $A \not\in C$ and $(X - C) \not\in (X - A)$. If $x \in C$, then $x \not\delta A$ implies $C \delta A$, a contradiction. Therefore, $x \in X - C$. If $x \in X - A$, then $(X - C) \delta (X - A)$, a contradiction. Thus, $x \in A$, and so we obtain that $x \not\delta A$ implies $x \in A$. Consequently, $A$ is closed. This together with $A$ being open imply that $A$ is clopen.

(2) It is obvious that $\emptyset, X \in B_\delta$ and that $B_\delta$ is closed under set-theoretic complement. We show that $B_\delta$ is closed under set-theoretic union. Let $A, B \in B_\delta$. Then $A \not\delta (X - A)$ and $B \not\delta (X - B)$. If $(A \cup B) \delta (X - (A \cup B))$, then $(A \cup B) \delta (X - A) \cap (X - B)$. By (P2), this implies $A \delta (X - A) \cap (X - B)$ or $B \delta (X - A) \cap (X - B)$. Therefore, $A \delta (X - A)$ or $B \delta (X - B)$, which is a contradiction. Thus, $(A \cup B) \not\delta (X - (A \cup B))$, so $B_\delta$ is closed under set-theoretic union, and so $B_\delta$ is a Boolean algebra.

Let $A, B \in B_\delta$. If $A \cap B \neq \emptyset$, then it follows from (P5) that $A \delta B$. Conversely, suppose that $A \cap B = \emptyset$. Then $A \subseteq X - B$. If $A \delta B$, then the last inclusion implies $(X - B) \delta B$, which contradicts to $B \in B_\delta$. Therefore, $A \not\delta B$, and so $A \delta B$ iff $A \cap B \neq \emptyset$. □

Let $B$ be a basis of $X$. We recall that $B$ is a Boolean basis if $B$ is a Boolean algebra of clopen subsets of $X$. Let $B(X)$ denote the set of Boolean bases of $X$. Obviously $B(X)$ forms a poset with respect to set-theoretic inclusion.
Lemma 4.2. Let \((X, \delta)\) be a zero-dimensional proximity space. Then \(B_\delta\) is a Boolean basis for the topology on \(X\) induced by \(\delta\).

Proof. It follows from Lemma 4.1 that \(B_\delta\) is a Boolean algebra of clopen subsets of \(X\). We show that \(B_\delta\) is a basis for the topology on \(X\) induced by \(\delta\). Let \(U\) be an open subset in the topology on \(X\) induced by \(\delta\) and let \(x \in U\). We show that there is an \(A \in B_\delta\) such that \(x \in A \subseteq U\). Since \(U\) is open and \(x \in U\), we have \(x \notin (X - U)\). By (SP4), there is a \(C \subseteq X\) such that \(C \notin (X - C)\), and \((X - C) \notin (X - U)\). Let \(A = X - C\). From \(C \notin (X - C)\) it follows that \(A \notin (X - A)\), and so \(A \in B_\delta\). Since \(x \notin C\), we have \(x \notin C\), and so \(x \in A\). Lastly, \((X - C) \notin (X - U)\) implies \((X - C) \cap (X - U) = \emptyset\), and so \(A \subseteq U\). Consequently, \(B_\delta\) is a Boolean algebra and a basis of clopen subsets of \(X\), which means that \(B_\delta\) is a Boolean basis for the topology on \(X\) induced by \(\delta\). \(\square\)

Lemma 4.3. Let \(X\) be a zero-dimensional Hausdorff space and let \(B\) be a Boolean basis of \(X\). Set

\[ A \delta B \iff \text{ for each } C \in B \text{ from } A \subseteq C \text{ it follows that } B \nsubseteq X - C. \]

Then \(B_\delta\) is a zero-dimensional proximity on \(X\) that induces the topology on \(X\).

Proof. First we show that \(\delta_B\) is a zero-dimensional proximity on \(X\).

(P1) Let \(A \not\subseteq B\). Then there is an \(A \in B\) such that \(A \subseteq U\) and \(B \subseteq X - U\). Let \(V = X - U\). Then \(V \in B\), \(B \subseteq V\), and \(A \subseteq V\). Therefore, there is a \(V \in B\) such that \(B \subseteq V\) and \(A \subseteq X - V\). Thus, \(B \delta B\), and so \(\delta_B\) is symmetric.

(P2) Let \(A \delta B\). Then for each \(U \in B\) we have \(A \subseteq U\) implies \(B \subseteq U\) or \(C \subseteq X - U\). Therefore, for each \(U \in B\) we have \(A \subseteq U\) implies \(B \subseteq U\) or \(C \subseteq X - U\). Thus, \(A \delta B\) or \(A \delta B\). Conversely, if \(A \delta B\), then there is an \(U \in B\) such that \(A \subseteq U\) and \(B \subseteq U\). Therefore, there is an \(U \in B\) such that \(A \subseteq U\) and \(B \subseteq U\), and \(C \subseteq X - U\). Thus, \(A \delta B\) and \(A \delta B\).

(P3) Let \(A \delta B\). If \(A = \emptyset\), then \(A \subseteq \emptyset\) and \(B \subseteq X - \emptyset\). Therefore, as \(\emptyset \in B\), we obtain \(A \delta B\), which is a contradiction. Thus, \(A \not\subseteq \emptyset\). A similar argument shows that \(B \not\subseteq \emptyset\).

(P4) Let \(A \not\subseteq B\). Then there is an \(A \in B\) such that \(A \subseteq U\) and \(B \subseteq X - U\). Let \(C = X - U\). From \(B \subseteq U\) it follows that \(C \subseteq B\), which implies that \(C \notin (X - C)\). Since \(A \subseteq U\) and \(C \subseteq X - U\), we have \(A \notin B\). Because \(X - C \subseteq U\) and \(B \subseteq X - U\), we have \((X - C) \notin B\). Thus, there is a \(C \subseteq X\) such that \(C \notin (X - C)\), \(A \notin B\), and \((X - C) \notin B\).

(P5) Let \(A \delta B\). Then there is an \(U \in B\) such that \(A \subseteq U\) and \(B \subseteq X - U\). Therefore, \(A \cap B \subseteq U \cap (X - U) = \emptyset\), and so \(A \cap B \not\subseteq \emptyset\) implies \(A \delta B\).

(P6) Let \(x \notin y\). Since \(X\) is Hausdorff and \(B\) is a basis for the topology on \(X\), there is an \(U \in B\) such that \(x \in U\) and \(y \notin U\). Thus, \(|x| \subseteq U\) and \(|y| \subseteq U\), and so \(x \delta y\).

This shows that \(\delta_B\) is a zero-dimensional proximity on \(X\). Moreover, since \(B\) is a basis for the topology on \(X\), for a subset \(U\) of \(X\) we have that \(U\) is open if \(x \in U\) implies there is an \(A \in B\) such that \(x \in A\). The last condition is obviously equivalent to \(x \in U\) implies \(x \notin (X - U)\). Therefore, \(U\) is open iff \(x \in U\) implies \(x \not\subseteq (X - U)\). This, by [13, Cor. 2.4], means that \(\delta_B\) induces the topology on \(X\). \(\square\)

Lemma 4.4. For a zero-dimensional proximity \(\delta\) and a Boolean basis \(B\), we have \(\delta = \delta_{B_1} + B = B_{\delta_B}\).

Proof. First suppose that \(A \delta B\). Then there is an \(U \in B\) such that \(A \subseteq U\) and \(B \subseteq X - U\). Since \(U \in B\), \(U \delta X - U\). Let \(V = X - U\). From \(U \in B\) it follows that \(V \in B\). Therefore, \(V \delta X - U\). Assume that \(A \delta V\). This together with \(A \subseteq U\) imply \(U \delta V\), hence \(U \delta (X - U)\), a contradiction. Therefore, \(A \delta V\). Similarly, \((X - V) \delta B\) and \(B \subseteq X - U\) imply \((X - V) \delta (X - U)\), hence \(U \delta (X - U)\), a contradiction. Thus, \((X - V) \delta B\). Consequently, there is a \(V \subseteq X\) such that \(V \delta (X - V)\), \(A \delta V\), and \((X - V) \delta B\). This implies that \(A \delta B\).

Next suppose that \(A \delta B\). Then there is an \(U \subseteq X\) such that \(U \delta (X - U)\), \(A \delta U\), and \((X - U) \delta B\). From \(U \delta (X - U)\) it follows that \(U \subseteq B\). Let \(V = X - U\). Then \(V \in B\). Also, \(A \delta U\) implies \(A \subseteq V\), and so \(A \subseteq X - V\). Lastly, \((X - U) \delta B\) implies \((X - U) \cap B = \emptyset\), and so \(B \subseteq X - V\). Thus, there is a \(V \in B\) such that \(A \subseteq V\) and \(B \subseteq X - V\), which implies that \(A \delta B\). Consequently, \(A \delta B\) iff \(A \delta B\), and so \(\delta = \delta_{B_1}\).

Finally, \(A \delta B\) iff \(A \delta_B (X - A)\) iff there is an \(U \in B\) such that \(A \subseteq U\) and \(X - A \subseteq X - U\) iff there is an \(U \in B\) such that \(A \subseteq U\) and \(U \subseteq A\) iff there is an \(U \in B\) such that \(A = U\) iff \(A \in B\). Thus, \(B = B_{\delta_B}\). \(\square\)

It is also easy to see that for two zero-dimensional proximities \(\delta_1\) and \(\delta_2\) we have \(\delta_1 \leq \delta_2\) iff \(B_{\delta_1} \subseteq B_{\delta_2}\), and that for two Boolean bases \(B_1\) and \(B_2\) we have \(B_1 \subseteq B_2\) iff \(\delta_{B_1} \subseteq \delta_{B_2}\). This together with Lemmas 4.2–4.4 immediately give us:

Theorem 4.5. For a zero-dimensional Hausdorff space \(X\), the poset \((\Pi(X), \leq)\) of zero-dimensional proximities on \(X\) that induce the topology on \(X\) is isomorphic to the poset \((\mathfrak{B}(X), \subseteq)\) of Boolean bases for the topology on \(X\).

Remark 4.6. Obviously the largest element of \((\mathfrak{B}(X), \subseteq)\), corresponding to the largest element \(\delta_p\) of \((\Pi(X), \leq)\), is the Boolean basis of all clopen subsets of \(X\). Moreover, whenever \(X\) is noncompact locally compact, then the least Boolean basis, corresponding to the least element \(\delta_p\) of \((\Pi(X), \leq)\), consists of the clopen subset \(A\) of \(X\) such that either \(A\) or \(X - A\) is compact.
Theorem 4.7 (Dwinger theorem). For a zero-dimensional Hausdorff space $X$, the poset $\langle Z(X), \subseteq \rangle$ of inequivalent zero-dimensional compactifications of $X$ is isomorphic to the poset $\langle \mathcal{B}(X), \subseteq \rangle$ of Boolean bases of $X$.

Remark 4.8. For a zero-dimensional Hausdorff space $X$, the Dwinger theorem implies that the zero-dimensional compactification of $X$ corresponding to a Boolean basis $B$ of $X$ is the Stone space of ultrafilters of $B$. Therefore, the zero-dimensional compactification of $X$ corresponding to a zero-dimensional proximity $\delta$ compatible with the topology on $X$, by which, the Smirnov theorem, is constructed as the space of clusters of $(X, \delta)$, can alternatively be constructed as the space of ultrafilters of $B_\delta = \{ A \subseteq X : A \neq (X - A) \}$. More precisely, the homeomorphism between the space of clusters of $(X, \delta)$ and the Stone space of $B_\delta$ can explicitly be constructed as follows.

For an ultrafilter $\mathcal{V}$ on $X$ let $\sigma(\mathcal{V}) = \{ A \delta B \text{ for each } B \in \mathcal{V} \}$ denote the cluster associated with $\mathcal{V}$. Given a cluster $\sigma$ of $(X, \delta)$, by [13, Thm. 5.8], there exists an ultrafilter $\mathcal{U}$ such that $\sigma = \sigma(\mathcal{V})$. Clearly $\mathcal{U}_\sigma = \mathcal{V} \cap B_\delta$ is an ultrafilter of $B_\delta$, and we define a map $f$ from the space $Y$ of clusters of $(X, \delta)$ to the Stone space $S_\delta$ of $B_\delta$ by $f(\sigma(\mathcal{V})) = \mathcal{U}_\sigma$. To see that $f$ is well-defined, it is sufficient to show that for each ultrafilters $\mathcal{V}_1, \mathcal{V}_2$ with $\sigma(\mathcal{V}_1) = \sigma(\mathcal{V}_2)$ we have $\mathcal{V}_1 \cap \mathcal{V}_2 = \mathcal{V}_2 \cap \mathcal{V}_3$. If $\mathcal{V}_1 \cap \mathcal{V}_2 \neq \mathcal{V}_2 \cap \mathcal{V}_3$, then without loss of generality we may assume that there exists $A \in \mathcal{V}_1 \cap \mathcal{V}_2$ such that $A \notin \mathcal{V}_1 \cap \mathcal{V}_2$. Therefore, $A \notin \mathcal{V}_1$, $A \notin (X - A)$, and $X - A \in \mathcal{V}_2$. Thus, $(X - A) \cap B \neq \emptyset$ for each $B \in \mathcal{V}_2$. This, by (P5), implies that $(X - A) \delta B$ for each $B \in \mathcal{V}_2$. Therefore, $X - A \in \sigma(\mathcal{V}_2)$, thus $X - A \in \sigma(\mathcal{V}_1)$, which implies that $(X - A) \delta A$, a contradiction. Consequently, $\mathcal{V}_1 \cap \mathcal{V}_2 = \mathcal{V}_2 \cap \mathcal{V}_3$, and so $f$ is well defined.

To see that $f$ is 1-1, let $\sigma_1 \neq \sigma_2$. Without loss of generality we may assume that there exists $A \in \sigma_1$ such that $A \notin \sigma_2$. By [13, Thm. 5.8], there exist ultrafilters $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\sigma_1 = \sigma(\mathcal{V}_1)$ and $\sigma_2 = \sigma(\mathcal{V}_2)$. Then $A \delta \mathcal{V}_2$ for each $B \in \mathcal{V}_1$ and there exists $C \in \mathcal{V}_2$ such that $A \notin C$. This, by (SP4), implies that there exists $D \in \mathcal{V}_2$ such that $D \notin \mathcal{V}_1 \cap \mathcal{V}_2$ and $D \notin C$. Therefore, $D \notin \mathcal{V}_1 \cap \mathcal{V}_2$, and so $f(\sigma_1) \neq f(\sigma_2)$. To see that $f$ is onto, let $\mathcal{U}$ be an ultrafilter of $B_\delta$. Then there exists an ultrafilter $\mathcal{V}$ on $X$ such that $\mathcal{U} = \mathcal{V} \cap B_\delta$ (note that $\mathcal{V}$ is not unique). Clearly $f(\sigma(\mathcal{V})) = \mathcal{U}$, and so $f$ is onto.

Finally, to see that $f$ is continuous, let $U$ be a clopen subset of $S_\delta$. Then there exists $A \in B_\delta$ such that $U = \psi_\delta(A)$, where $\psi_\delta(A) = \{ u \in S_\delta : A \in u \}$. We show that if $f^{-1}(\psi_\delta(A)) = \psi(A)$. Let $\sigma$ be a cluster of $(X, \delta)$ and $V$ be an ultrafilter on $X$ such that $\sigma = \sigma(\mathcal{V})$. We have $\sigma \in f^{-1}(\psi_\delta(A))$ iff $f(\sigma) \in \psi_\delta(A)$ iff $A \in f(\sigma)$ iff $A \in \mathcal{V} \cap B_\delta$. On the other hand, $\sigma \in \mathcal{V} \cap B_\delta$ iff $A \in \sigma$. We show that $A \in \mathcal{V} \cap B_\delta$ if $A \in \sigma$. First suppose that $A \in \sigma$. Then $A \delta \mathcal{V}_2$ for each $B \in \mathcal{V}_1$. Therefore, $A \delta \mathcal{V}_1 \cap \mathcal{V}_2$, and so $\psi(\sigma) \neq \emptyset$. By Lemma 4.1, this implies that $A \delta B$ for each $B \in \mathcal{V}_1 \cap \mathcal{V}_2$. Thus, by (SP4), there exists $D \in \mathcal{V}_2$ such that $A \delta D$ and $(X - D) \notin \mathcal{V}_2$. Therefore, $A \in \mathcal{V} \cap B_\delta$, and so $f(\sigma) = \mathcal{U} = U$. Consequently, $f$ is a continuous bijection between compact Hausdorff spaces, so $f$ is a homeomorphism.

We recall that a subset $U$ of a topological space $X$ is regular open if $\text{int}(\text{cl}(U)) = U$. Therefore, $U$ is regular open iff it is the interior of a closed set. Regular closed sets are defined dually. Clearly each clopen is regular open (regular closed).

But the converse is not true in general. A topological space $X$ is called extremely disconnected if each regular open (regular closed) set is clopen. Equivalently, $X$ is extremely disconnected if the closure of each open set is clopen (the interior of each closed set is clopen). Extremally disconnected spaces were introduced by Stone [16]. He showed that extremely disconnected compact Hausdorff spaces are exactly the Stone duals of complete Boolean algebras. An important result about extremally disconnected spaces was obtained by Gleason [6] who showed that extremely disconnected compact Hausdorff spaces are exactly the projective objects in the category of compact Hausdorff spaces and continuous maps. It is well known (see, e.g., [5, Thm. 6.2.27]) that a completely regular space $X$ is extremally disconnected iff $\beta(X)$ is extremally disconnected. To this we add that for an extremally disconnected regular space $X$, the Stone–Čech compactification of $X$ is a unique up to equivalence extremally disconnected compactification of $X$.

Lemma 4.9. If $X$ is a regular extremally disconnected space, then $X$ is zero-dimensional.

Proof. Let $U$ be open and $x \in U$. Since $X$ is regular, there is an open subset $V$ of $X$ such that $x \in V \subseteq \text{cl}(V) \subseteq U$. As $X$ is extremely disconnected and $V$ is open, $\text{cl}(V)$ is clopen. Thus, there is a clopen subset $W = \text{cl}(V)$ of $X$ such that $x \in W \subseteq U$, and so $X$ is zero-dimensional.

Lemma 4.10. Let $X$ be zero-dimensional Hausdorff and let $\mathcal{B}$ be a Boolean basis of $X$. If $\mathcal{B}$ is a complete Boolean algebra, then for each family $\{ U_i : i \in I \} \subseteq \mathcal{B}$, we have $\bigvee \{ U_i : i \in I \} = \text{cl}(\bigcup \{ U_i : i \in I \})$.

Proof. Let $U = \bigvee \{ U_i : i \in I \}$. Then $U_i \subseteq U$ for each $i \in I$. Therefore, $\bigcup \{ U_i : i \in I \} \subseteq U$, and as $U$ is clopen, we obtain $\text{cl}(\bigcup \{ U_i : i \in I \}) \subseteq U$. Conversely, let $x \notin \text{cl}(\bigcup \{ U_i : i \in I \})$. Since $\mathcal{B}$ is a basis, there is a $W \in \mathcal{B}$ such that $x \in W$ and $W \cap \bigcup \{ U_i : i \in I \} = \emptyset$. Let $V = X - W$. Since $\mathcal{B}$ is closed under set-theoretic complement, $V \in \mathcal{B}$. Moreover, $x \notin V$ and $V$ is an upper bound of $\{ U_i : i \in I \}$. Therefore, $U \subseteq V$, so $x \notin U$, and so $U \subseteq \text{cl}(\bigcup \{ U_i : i \in I \})$. Thus, $U = \text{cl}(\bigcup \{ U_i : i \in I \})$. □
Lemma 4.11. Let $X$ be zero-dimensional Hausdorff and let $B$ be a Boolean basis of $X$. If $B$ is a complete Boolean algebra, then $B$ is the Boolean algebra of all clopen subsets of $X$.

Proof. It is sufficient to show that if $U$ is clopen, then $U \in B$. But since $B$ is a basis, there is a family $\{U_i: i \in I\} \subseteq B$ such that $U = \bigcup\{U_i: i \in I\}$. Therefore, $U = \text{cl}(\bigcup\{U_i: i \in I\})$, which, by Lemma 4.10, means that $U \in B$.

Theorem 4.12. If $X$ is a regular extremally disconnected space, then $\beta(X)$ is a unique up to equivalence extremally disconnected compactification of $X$.

Proof. Let $X$ be a regular extremally disconnected space. By Lemma 4.9, $X$ is zero-dimensional Hausdorff. Let $Y$ be an extremally disconnected compactification of $X$. By the Dwinger theorem, there is a Boolean basis $B$ of $X$ such that $Y$ is (equivalent to) the Stone space of $B$. Therefore, $B$ is isomorphic to the complete Boolean algebra of clopen subsets of $Y$. By Lemma 4.11, $B$ is the Boolean algebra of all clopen subsets of $X$. Therefore, $Y$ is the largest zero-dimensional compactification of $X$. Since $X$ is extremally disconnected, so is $\beta(X)$ (see, e.g., [5, Thm. 6.2.27]). Thus, $\beta(X)$ is zero-dimensional, hence equivalent to $Y$. Consequently, $\beta(X)$ is a unique up to equivalence extremally disconnected compactification of $X$.

Remark 4.13. Since the Stone–Čech compactification of $X$ is zero-dimensional iff $X$ is strongly zero-dimensional (see, e.g., [5, Thms. 6.2.12 and 6.2.10]), Theorem 4.12 implies that a regular extremally disconnected space is always strongly zero-dimensional.

References