Scattered, Hausdorff-reducible, and hereditarily irresolvable spaces

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Abstract

We show that a topological space is hereditarily irresolvable if and only if it is Hausdorff-reducible. We construct a compact irreducible $T_1$-space and a connected Hausdorff space, each of which is strongly irresolvable. Furthermore, we show that the three notions of scattered, Hausdorff-reducible, and hereditarily irresolvable coincide for a large class of spaces, including metric, locally compact Hausdorff, and spectral spaces.

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1. Introduction

Hewitt [9] investigated the question of when a topological space $X$ can be written as the union of two disjoint dense subspaces. He called a space that can be written in this way resolvable. A space is called irresolvable if it is not resolvable. It is obvious that a resolvable space is dense-in-itself. Hewitt also considered whether every dense-in-itself space is resolvable. He proved that this is true for large classes of topological spaces, including metric spaces and locally compact Hausdorff spaces. He also showed that there exist dense-in-itself spaces which are irresolvable. Padmavally [12] proved that there exists a connected dense-in-itself Hausdorff space which is irresolvable. Anderson
In [9] Hewitt also introduced the notions of maximal, MI, and strongly irresolvable spaces. We recall that a topological space \((X, \tau)\) is called maximal if every nonempty open set of \(X\) is infinite and each proper expansion \(\tau' \supseteq \tau\) contains a finite nonempty open set. A dense-in-itself space \(X\) is called MI if every dense subset of \(X\) is open, and is called strongly irresolvable, or SI, if every subspace of \(X\) is irresolvable. Hewitt proved that every maximal space is MI and that every MI space is SI. He also established the following decomposition result [9, Theorem 28]: Each dense-in-itself space \(X\) is the union of two disjoint subspaces \(C\) and \(U\) where \(C\) is closed and resolvable and \(U\) is open and SI. To work with a space that is not dense-in-itself, we recall that a space is called submaximal if every dense subset of \(X\) is open, and is hereditarily irresolvable, or HI, if every nonempty subspace of \(X\) is irresolvable. Hewitt’s decomposition theorem says that \(X\) is the union of two disjoint subspaces \(C\) and \(U\) where \(C\) is closed and resolvable and \(U\) is open and HI.

In his book [8], originally written in 1914, Hausdorff studied when, for a subspace \(Y\) of a topological space \(X\), there is a decreasing transfinite sequence \(\{A_\beta\}_{\beta < \kappa}\) of closed subsets of \(X\) such that \(Y = \bigcup (A_{\alpha+2n+1} - A_{\alpha+2n+2})\), where \(n\) ranges over natural numbers and \(\alpha < \kappa\) ranges over limit ordinals or 0. He called subspaces written in this way reducible in \(X\). There is a close connection between Hausdorff’s notion of reducible and Hewitt’s notion of irresolvable; in fact, we prove in Theorem 2.4 below that every subspace of \(X\) is reducible if and only if every subspace of \(X\) is irresolvable. In order to characterize reducible subspaces, Hausdorff defined the residue of a set \(A \subseteq X\) to be \(\rho(A) = A \cap \text{Cl}(\text{Cl}(A) - A)\), where we have denoted the closure of a subset \(Y\) of \(X\) by \(\text{Cl}(Y)\). He proved [8, §30.3] that a subspace \(Y\) is reducible in \(X\) if and only if, for each nonempty closed subset \(A\) of the subspace \(Y\), we have \(\rho(A) \subseteq A\). See also [11, §12, VII, Theorem 1] for a proof of this fact. Consequently, every subspace of \(X\) is reducible if and only if, for each nonempty subset \(A\) of \(X\), we have \(\rho(A) \subseteq A\). Esakia [4] defined a space to be Hausdorff-reducible, or HR, if \(\rho(A) \subseteq A\) for each nonempty subset \(A\) of \(X\). He showed that \(X\) is HR if and only if \(\text{Cl}(A) = \text{Cl}(A - \rho(A))\) for each \(A \subseteq X\). Recall that a space is scattered if every nonempty subspace has an isolated point. Esakia observed that every scattered space is HR, and Gabelaia [6] proved that the converse holds in some particular cases. It was left as an open question whether or not a topological space is scattered if and only if it is HR.

In this paper we show that a space is HR if and only if it is HI, and give several equivalent characterizations of HI and scattered spaces. This, together with El’kin’s example [2, P. 37], yields an HR space which is not scattered. We also show that for filtral topologies each of the three notions of submaximal, HI, and HR coincide with the fact that the nonempty open sets form an ultrafilter. In addition, we show that a space \((X, \tau)\) is maximal if and only if \(\tau - \{\emptyset\}\) is a free ultrafilter. We expand on El’kin’s example to exhibit a compact irreducible \(T_1\)-space which is SI, where a space is said to be irreducible if it cannot be written as the union of two proper closed subspaces. We also expand on Padmavally’s and Anderson’s examples to show that there exists a connected Hausdorff MI space. Finally, we extend Hewitt’s results to show that the three notions of scattered, HR, and HI coincide for the classes of metric spaces, Alexandroff spaces, first countable
spaces, locally compact Hausdorff spaces, and spectral spaces. It follows that, for these spaces, Hewitt’s decomposition theorem coincides with the Cantor–Bendixson Theorem [13, Theorem 8.5.2], which states that each space $X$ is the union of two disjoint subspaces $C$ and $U$ where $C$ is closed and dense-in-itself and $U$ is open and scattered.

2. The conditions HI and HR are equivalent

In this section we show that a topological space is HI if and only if it is HR, and we give filter-theoretic characterizations for a space to be HI or to be scattered. We denote by $P(X)$ the power set of $X$. Recall that a nonempty subset $F$ of $P(X)$ is called a filter on $X$ if $F$ is closed under finite intersections, and if $A \in F$ and $A \subseteq B$ imply $B \in F$. A filter maximal with respect to inclusion among all proper filters is said to be an ultrafilter. The principal filter generated by $A \subseteq X$ is $F_A = \{B \in P(X) : A \subseteq B\}$. If $x \in X$, then we denote by $F_x$ the principal ultrafilter generated by $\{x\}$. We write $F_{x,X}$ for this filter if we need to indicate the dependence on the set $X$. A non-principal ultrafilter is called free. If $S$ is a subset of $P(X)$, then the filter generated by $S$ is

$$\{B \subseteq X : \text{there are } A_1, \ldots, A_n \in S \text{ such that } A_1 \cap \cdots \cap A_n \subseteq B\}.$$ 

If $X$ is a topological space, let $D(X)$ denote the set of all dense subsets of $X$. We write $D(X, \tau)$ for $D(X)$ if we need to indicate the dependence on the topology $\tau$. Note that $D(X)$ is a filter if and only if the intersection of every pair of dense sets is again dense.

We give some notation to be used throughout the paper. For a subset $A$ of $X$, we denote the closure of $A$ by $\text{Cl}(A)$ or $\text{Cl}_X(A)$, the interior of $A$ by $\text{Int}(A)$ or $\text{Int}_X(A)$, the boundary (i.e., frontier) of $A$ by $\text{Fr}(A)$ or $\text{Fr}_X(A)$, the set of limit points of $A$ by $d(A)$, the isolated points of $X$ by $\text{Iso}(X)$, and the complement of $A$ by $A^c$.

Parts of the following proposition can be found in [9,2].

Proposition 2.1. Let $X$ be a topological space. Then the following conditions are equivalent.

1. $D(X)$ is a filter.
2. $D(X)$ is a filter generated by the dense open subsets of $X$.
3. If $A$ is dense in $X$, then so is $\text{Int}(A)$.
4. $\text{Int}(\text{Cl}(A)) \subseteq \text{Cl}(\text{Int}(A))$ for every subset $A$ of $X$.
5. $\text{Int}(\text{Fr}(A)) = \emptyset$ for every subset $A$ of $X$.
6. $U$ is irresolvable in its relative topology for every open subset $U$ of $X$.

Moreover, each of these implies the condition

(7) $X$ is irresolvable.

Proof. To begin, we show that (4) and (5) are equivalent. Note that

$$\text{Int}(\text{Fr}(A)) = \text{Int}(\text{Cl}(A) \cap \text{Cl}(A^c)) = \text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(A^c)).$$
Therefore, \( \text{Int}(\text{Fr}(A)) = \emptyset \) if and only if \( \text{Int}(\text{Cl}(A)) \cap \text{Int}(\text{Cl}(A^c)) = \emptyset \), which occurs if and only if \( \text{Int}(\text{Cl}(A)) \subseteq (\text{Int}(\text{Cl}(A^c)))^c \). Since, for every subset \( B \) of \( X \), we have \((\text{Int}(B))^c = \text{Cl}(B^c)\), it follows that \((\text{Int}(\text{Cl}(A^c)))^c = \text{Cl}(\text{Cl}(A^c) = \text{Cl}(\text{Int}(A))\). Thus, (4) and (5) are equivalent.

It is easy to see that (4) implies (3), since if \( A \) is dense in \( X \), then (4) yields \( X = \text{Int}(X) = \text{Int}(\text{Cl}(\text{Cl}(A))) \subseteq \text{Cl}(\text{Int}(A)) \), which shows that \( \text{Int}(A) \) is also dense in \( X \).

(3) implies (2): Let \( A \) and \( B \) be dense. Then \( \text{Int}(A) \) and \( \text{Int}(B) \) are dense by (3). Because the intersection of two dense open sets is always dense, \( \text{Int}(A) \cap \text{Int}(B) \) is dense. Therefore, \( A \cap B \) is also dense since it contains the dense set \( \text{Int}(A) \cap \text{Int}(B) \). Thus, \( \mathcal{D}(X) \) is a filter. Moreover, since (3) says that every dense set contains a dense open set, \( \mathcal{D}(X) \) is generated by the dense open subsets of \( X \).

(2) implies (1) is obvious.

(1) implies (6): Suppose \( U \) is a nonempty open subset of \( X \). If \( U \) is resolvable, then there exist two disjoint dense subsets \( A, B \) of \( U \). But then \( A \cup U^c, B \cup U^c \) are dense in \( X \), and (1) implies that \( (A \cup U^c) \cap (B \cup U^c) \) is dense in \( X \). However, \((A \cup U^c) \cap (B \cup U^c) = (A \cap B) \cup U^c = U^c \) is a proper closed subset of \( X \), and so cannot be dense in \( X \). This contradiction proves that \( U \) is irresolvable.

(6) implies (3): Suppose \( A \) is dense in \( X \) and \( U \) is an arbitrary nonempty open subset of \( X \). We want to show that \( \text{Int}(A) \cap U \neq \emptyset \). Observe that \( U = (U \cap A) \cup (U \cap A) \) and \( (U \cap A) \cap (U - A) = \emptyset \). Also \( U \subseteq \text{Cl}(U \cap A) \) since \( A \) is dense and \( U \) is open in \( X \). Now if \( \text{Int}(A) \cap U = \emptyset \), then \( U \subseteq \text{Cl}(U - A) \) and \( U \) is resolvable into the union \( U \cap A \) and \( U - A \), a contradiction. So \( \text{Int}(A) \cap U \neq \emptyset \) and \( \text{Int}(A) \) is dense in \( X \).

(3) implies (4): Let \( A \) be a subset of \( X \). We wish to prove that \( \text{Int}(\text{Cl}(A)) \subseteq \text{Cl}(\text{Int}(A)) \). If \( \text{Int}(\text{Cl}(A)) = \emptyset \), then there is nothing to prove, so suppose that \( \text{Int}(\text{Cl}(A)) \neq \emptyset \). Let \( x \in \text{Int}(\text{Cl}(A)) \), and let \( U \) be a neighborhood of \( x \). We need to show that \( U \cap \text{Int}(A) \neq \emptyset \).

Since \( A \cup \text{Cl}(A)^c \) is dense in \( X \), Condition (3) implies that \( \text{Int}(A \cup \text{Cl}(A)^c) \) is also dense in \( X \). Note that \( \text{Int}(A) = \text{Int}(\text{Cl}(A)) \cap \text{Int}(A \cup \text{Cl}(A)^c) \). Then
\[
U \cap \text{Int}(A) = U \cap \text{Int}(\text{Cl}(A)) \cap \text{Int}(A \cup \text{Cl}(A)^c)
= (U \cap \text{Int}(\text{Cl}(A))) \cap \text{Int}(A \cup \text{Cl}(A)^c) \neq \emptyset
\]
since \( U \cap \text{Int}(\text{Cl}(A)) \) is a nonempty open set. Therefore, \( x \in \text{Cl}(\text{Int}(A)) \), as desired. We have thus proven that \( \text{Int}(\text{Cl}(A)) \subseteq \text{Cl}(\text{Int}(A)) \) for every subset \( A \) of \( X \). This finishes the proof that the first six conditions are equivalent.

It is easy to see that the seven conditions each imply Condition (7); for example, (6) implies (7) trivially.

\[ \square \]

**Example 2.2.** We give an example to show that Condition (7) does not imply the six equivalent conditions of the proposition. Let \( X = [0, 1] \cup \{2\} \) with its usual topology. Then 2 is an isolated point of \( X \), so every dense set contains 2. Thus, \( X \) does not contain disjoint dense subsets, so Condition (7) holds. However, it is obvious that \( [0, 1] \), which is an open subspace of \( X \), is resolvable into the union of \([0, 1] \cap \mathbb{Q} \) and \([0, 1] - \mathbb{Q} \), so Condition (6) does not hold.

Note that the space in Example 2.2 is neither dense-in-itself nor connected. In Section 3 we will give an example of a dense-in-itself connected irresolvable space \( X \) for which
$\mathcal{D}(X)$ is not a filter. In spite of these counterexamples, it follows from [7, Theorem 3] that $X$ is irresolvable if and only if both $\mathcal{D}(X)$ and $\{\text{Int}(A) : A \in \mathcal{D}(X)\}$ form a filterbase on $X$. Another way of saying this is that $X$ is irresolvable if and only if $\text{Int}(A) \cap \text{Int}(B) \neq \emptyset$ for every $A, B \in \mathcal{D}(X)$.

While (7) does not imply the other conditions of Proposition 2.1, if we relativize (7) to subspaces of $X$, meaning that we require it to hold for every subspace of $X$, then it does imply them, as we observe in the following corollary.

**Corollary 2.3.** Let $X$ be a topological space. If $X$ is HI, then the six equivalent conditions of Proposition 2.1 hold.

**Proof.** If $X$ is HI, then Condition (6) of Proposition 2.1 holds trivially. □

We would like to stress that the relativization we performed in Corollary 2.3 is crucial for characterizing HR spaces. Indeed, we are in a position now to show that relativization of each of the six conditions of Proposition 2.1 to subspaces of $X$ is equivalent to $X$ being HR. We recall that the Hausdorff residue of a subset $A$ of $X$ is $\rho(A) = A \cap \text{Cl}(\text{Cl}(A) - A)$.

**Theorem 2.4.** Let $X$ be a topological space. Then the following conditions are equivalent.

1. The space $X$ is HR.
2. The space $X$ is HI.
3. For every subspace $Y$ of $X$, the set $\mathcal{D}(Y)$ of dense subsets of $Y$ is a filter on $Y$.
4. For every subspace $Y$ of $X$, the set $\mathcal{D}(Y)$ is a filter generated by the dense open subsets of $Y$.
5. For every subspace $Y$ of $X$, if $A$ is dense in $Y$, then $\text{Int}_Y(A)$ is dense in $Y$.
6. For every subspace $Y$ of $X$, we have $\text{Int}_Y(\text{Cl}_Y(A)) \subseteq \text{Cl}_Y(\text{Int}_Y(A))$ for every subset $A$ of $Y$.
7. For every subspace $Y$ of $X$, we have $\text{Int}_Y(\text{Fr}_Y(A)) = \emptyset$ for every subset $A$ of $Y$.
8. For every subspace $Y$ of $X$, the set $Y - \rho(Y)$ is dense in $Y$.

Furthermore, if we replace “subspace” in each of 3–7 by “closed subspace”, we obtain a statement equivalent to each of those above.

**Proof.** It follows from Proposition 2.1 and Corollary 2.3 that Conditions (2) through (7) are equivalent.

To show (1) implies (2) we prove the contrapositive of this implication. Suppose there is a subspace $Y$ containing disjoint dense subsets $A$ and $B$. Then $\text{Cl}(A) = \text{Cl}(Y) = \text{Cl}(B)$. Thus, $\text{Cl}(\text{Cl}(A) - A) = \text{Cl}(\text{Cl}(Y) - A) \supseteq \text{Cl}(B) = \text{Cl}(Y)$. This shows $A \subseteq \text{Cl}(\text{Cl}(A) - A)$, and so $\rho(A) = A$. The space $X$ therefore is not HR.

To prove (7) implies (8), observe that $Y - \rho(Y) = Y - \text{Cl}(\text{Cl}(Y) - Y)$ and that $\text{Fr}_{\text{Cl}(Y)}(Y) = \text{Cl}(\text{Cl}(Y) - Y)$. Since $\text{Cl}(Y)$ is a closed subspace of $X$, (7) implies that $\text{Int}_{\text{Cl}(Y)}(\text{Fr}_{\text{Cl}(Y)}(Y)) = \emptyset$. Since $\text{Int}(A) = \text{Cl}(A'')$ for every $A$, we see that $\text{Int}_{\text{Cl}(Y)}(\text{Fr}_{\text{Cl}(Y)}(Y)) = \text{Cl}(Y) - \text{Cl}[\text{Cl}(Y) - \text{Cl}(\text{Cl}(Y) - Y)]$. 

Thus, \( \text{Cl}(\text{Cl}(Y) - \text{Cl}(Y - Y)) = \text{Cl}(Y) \). Since \( Y - \text{Cl}(\text{Cl}(Y) - Y) = \text{Cl}(Y) - \text{Cl}(\text{Cl}(Y) - Y) \), we conclude that \( \text{Cl}(Y - \text{Cl}(\text{Cl}(Y) - Y)) = \text{Cl}(Y) \). Therefore, \( \text{Cl}(Y - \rho(Y)) = \text{Cl}(Y) \).

It is trivial to see that (8) implies (1) since if \( Y - \rho(Y) \) is dense in \( Y \), then \( Y - \rho(Y) \) is nonempty, and so \( \rho(Y) \) is a proper subset of \( Y \) for every nonempty subspace \( Y \) of \( X \).

Finally, it is easy to see that \( X \) is HI if and only if every nonempty closed subspace of \( X \) is irresolvable. From this and the first line of the argument above it is clear that Condition (2) is equivalent to each of the conditions obtained from (3) through (7) by replacing “subspace” by “closed subspace”. \( \square \)

It is clear that (8) is equivalent to the identity \( \text{Cl}(Y) = \text{Cl}(Y - \rho(Y)) \) for every subspace \( Y \) of \( X \). As we noted in the introduction, Esakia showed that this identity is equivalent to (1). Moreover, he [5] also observed that (6) and (7) are equivalent to (1).

**Example 2.5.** El’kin [2] gave the following example of a dense-in-itself irreducible \( T_1 \)-space that is maximal. Let \( X \) be an infinite set, \( \mathcal{F} \) a free ultrafilter on \( X \), and set \( \tau = \mathcal{F} \cup \{ \emptyset \} \). It follows from [9, Theorems 23, 24] that \( X \) is SI. This example and Theorem 2.4 imply that there exist HR spaces which are not scattered, providing a negative solution to the question stated in the introduction.

The space \( X \) is not compact; in fact, we show that the only compact subspaces of \( X \) are finite. Let \( Y \) be an infinite subspace of \( X \) and write \( Y = A_1 \cup A_2 \) for some disjoint infinite subsets \( A_1 \) and \( A_2 \) of \( Y \). Since \( \mathcal{F} \) is an ultrafilter, either \( A_1 \in \mathcal{F} \) or \( X - A_1 \in \mathcal{F} \). In either case, we have an open set \( U \) such that \( Y - U \) is infinite. Then \( \{ U \cup \{ y \}; \ y \in Y - U \} \) is an open cover of \( Y \) which does not have a finite subcover. In the next section we show how to use this example to produce a compact irreducible \( T_1 \)-space which is SI.

**Remark 2.6.** We will use Theorems 23 and 24 of [9] in the proof of the next proposition in a slightly more general context. Hewitt’s standing hypothesis was to assume a space is \( T_0 \). To avoid assuming either \( T_0 \) or \( T_1 \), we note two facts: First, if \( (X, \tau) \) is maximal, then it is a \( T_1 \)-space. Second, if \( (X, \tau) \) is submaximal, then it is \( T_0 \). To prove the first statement, let \( \tau' \) be the topology generated by the union of \( \tau \) with the collection of all cofinite subsets of \( X \). Since \( (X, \tau) \) is maximal, all nonempty open sets of \( (X, \tau) \) are infinite. Because every open set in \( \tau' \) is a union of sets of the form \( U \cap A \) with \( U \in \tau \) and \( A \) cofinite, we see that every nonempty open set in \( \tau' \) is infinite. Since \( (X, \tau) \) is maximal, \( \tau' = \tau \). Therefore, \( \tau \) contains every cofinite subset of \( X \), and so points are closed. Thus, \( (X, \tau) \) is \( T_1 \). For the second statement, suppose that \( X \) is not \( T_0 \). Then there are distinct points \( x, y \in X \) with \( [x] = [y] \). The set \( [x] - \{ x \} \) is dense in \( [x] \) since \( y \in [x] - \{ x \} \). Thus, \( X - \{ x \} = ([x] - \{ x \}) \cup ([x])^c \) is dense in \( X \). However, \( X - \{ x \} \) is not open since \( [x] \) is not closed. Therefore, \( X \) is not sub maximal. We also note that the proof of [9, Theorem 23] actually shows that a submaximal space is HI.

We call a topology \( \tau \) on a set \( X \) a *filtral topology* if \( \tau - \{ \emptyset \} \) is a filter on \( X \). The following result extends [2, Theorem 3].
**Proposition 2.7.** For a filtral topology $\tau$ on $X$ the following conditions are equivalent.

1. $X$ is submaximal.
2. $X$ is HI.
3. $X$ is HR.
4. $\tau - \{\emptyset\}$ is an ultrafilter.
5. $D(X) = \tau - \{\emptyset\}$.

Moreover, $\tau - \{\emptyset\}$ is a free ultrafilter if and only if $X$ is maximal.

**Proof.** (1) implies (2) follows from Remark 2.6.

(2) is equivalent to (3) by Theorem 2.4.

(2) implies (4): Suppose $\tau - \{\emptyset\}$ is not an ultrafilter. Then there exists $A \subseteq X$ such that $A \notin \tau - \{\emptyset\}$ and $A^c \notin \tau - \{\emptyset\}$. Since $\tau - \{\emptyset\}$ is a filter, no nonempty subset of $A$ or $A^c$ is open, so $\text{Int}(A) = \text{Int}(A^c) = \emptyset$. Hence, $\text{Cl}(A^c) = \text{Cl}(A) = X$. Therefore, $X$ is resolvable, so is not HI.

(4) implies (5): Suppose $\tau - \{\emptyset\}$ is an ultrafilter on $X$ and $A$ is a nonempty subset of $X$ such that $A \notin \tau - \{\emptyset\}$. Then $A^c \in \tau - \{\emptyset\}$, so $A^c$ is open. Then $A$ is a proper closed subset of $X$; hence, $A \notin D(X)$. Therefore, $D(X) \subseteq \tau - \{\emptyset\}$. For the reverse inclusion, if $U$ is a nonempty open set, then $U$ is dense since $\tau - \{\emptyset\}$ is a filter. Therefore, $D(X) = \tau - \{\emptyset\}$.

(5) implies (1) is obvious.

If $X$ is maximal, then it is $T_1$ by Remark 2.6. Thus, $X$ is MI by [9, Theorem 24]. Therefore, $\tau - \{\emptyset\}$ is an ultrafilter by what we have already proved. Moreover, since there are no finite nonempty open subsets of $X$, $\tau - \{\emptyset\}$ must be free. The converse was pointed out by El’kin [2, p. 37]. For the convenience of the reader, we give a proof. Let $F$ be a free ultrafilter and $\tau = F \cup \{\emptyset\}$. Then $\tau$ contains no finite nonempty set. Let $\tau'$ be a proper expansion of $\tau$ and choose $A \in \tau' - \tau$. Since $F$ is an ultrafilter, $A^c \in F$. Therefore, for any $x \in A$, we have $A^c \cup \{x\} \in F$, so $\{x\} = (A^c \cup \{x\}) \cap A \in \tau'$. Thus, $\tau$ is maximal.

We conclude this section by obtaining a characterization of scattered spaces similar to Theorem 2.4.

**Proposition 2.8.** Let $X$ be a topological space. Then the following two conditions are equivalent.

1. $\text{Iso}(X)$ is dense in $X$.
2. $D(X)$ is a filter generated by $\text{Iso}(X)$.

Moreover, these two conditions imply

3. $D(X)$ is a filter contained in a principal ultrafilter $F_x$ for some $x \in X$.

Finally, Condition (3) implies

4. $X$ has an isolated point.
Proof. To prove (1) implies (2), observe that if $A$ is dense in $X$, then $\text{Iso}(X) \subseteq A$. Thus, if $\text{Iso}(X)$ is dense in $X$, then $D(X) = F_{\text{Iso}(X)}$.

(2) implies (1) is trivial.

For (2) implies (3) observe that if $D(X) = F_{\text{Iso}(X)}$, then $\text{Iso}(X) \neq \emptyset$, and so $D(X) \subseteq F_x$ for every $x \in \text{Iso}(X)$.

Finally, to see (3) implies (4), observe that if $D(X) \subseteq F_x$, then $X - \{x\}$ is not dense. Therefore, $X - \{x\}$ must be closed, and so $\{x\}$ is open. $\blacksquare$

Example 2.9. Let $X$ be the space constructed in Example 2.2. We show that $X$ satisfies Condition (4) but not (3). The space $X$ contains a single isolated point, namely 2. However, $D(X)$ is not a filter since the two sets $(0, 1] \cap \mathbb{Q}) \cup \{2\}$ and $(0, 1) \setminus \mathbb{Q}) \cup \{2\}$ are dense, but their intersection $\{2\}$ is clearly not dense.

Example 2.10. Let $(X, \tau)$ be the space of Example 2.5. Let $Z$ be the disjoint union of $X$ and a singleton $\{p\}$. Then $\text{Iso}(Z) = \{p\}$. The dense sets of $Z$ are of the form $A \cup \{p\}$ with $A \in \mathcal{F}$. Therefore, $D(Z)$ is a filter, and $D(Z)$ is contained in the principal ultrafilter $F_p$. However, $\text{Iso}(Z)$ is not dense. Therefore, $Z$ satisfies (3) but not (2).

As in Corollary 2.3, relativization of Condition (4) to subspaces of $X$ will imply the other conditions of Proposition 2.8. However, observe that relativization of Condition (4) to all subspaces of $X$ is the definition of scattered spaces. Thus, we arrive at the following theorem.

**Theorem 2.11.** Let $X$ be a nonempty topological space. Then the following conditions are equivalent.

1. $X$ is scattered.
2. For every subspace $Y$ of $X$, the set $\text{Iso}(Y)$ is dense in $Y$.
3. For every subspace $Y$ of $X$, $D(Y)$ is a filter generated by $\text{Iso}(Y)$.
4. For every nonempty subspace $Y$ of $X$, $D(Y)$ is a filter contained in a principal ultrafilter $F_{y, Y}$ for some $y \in Y$.
5. For every subspace $Y$ of $X$, we have $d(Y) = d(Y - d(Y))$.

Moreover, if we replace “subspace” by “closed subspace” in (2)–(4), the resulting statements are equivalent to the five above.

**Proof.** It follows from Proposition 2.8 that (2) is equivalent to (3), that (3) implies (4), and that (4) implies (1).

To prove (1) implies (2), let $Y$ be a subspace of $X$. If $V$ is a nonempty open subset of $Y$, then, by hypothesis, $V$ contains an isolated point $y$. Since $V$ is open in $Y$, we conclude that $y \in \text{Iso}(Y)$. Consequently, $V \cap \text{Iso}(Y) \neq \emptyset$. Therefore, $\text{Iso}(Y)$ is dense in $Y$.

To prove (2) implies (5) observe that for each subspace $Y$ we have the inclusion $d(Y - d(Y)) \subseteq d(Y)$. To show that $d(Y) \subseteq d(Y - d(Y))$ we note that (2) implies $\text{Cl}(Y) = \text{Cl}(\text{Iso}(\text{Cl}(Y)))$. We also note that for every $A \subseteq X$ we have $\text{Cl}(A) = A \cup d(A)$ and $\text{Iso}(A) = A - d(A)$. Then
Since $d(Y)$ is disjoint from $\text{Cl}(Y) - d(\text{Cl}(Y))$, we obtain that
\[
d(Y) \subseteq d(\text{Cl}(Y) - d(\text{Cl}(Y)))
\]
in the third line we used the identity $d(A \cup B) = d(A) \cup d(B)$. Therefore, $d(Y) = d(Y - d(Y))$ for every subspace $Y$ of $X$.

It is trivial to see that (5) implies (1) since $d(Y) = d(Y - d(Y))$ implies $\text{Iso}(Y) = Y - d(Y) \neq \emptyset$ for every nonempty subspace $Y$ of $X$.

Finally, it is easy to see that $X$ is scattered if and only if every nonempty closed subspace has an isolated point. From this and the first line of the argument above, it is clear that Condition (1) is equivalent to each of the conditions obtained from (2) through (4) by replacing “subspace” by “closed subspace”. □

We note that Esakia [4] observed that (1) and (5) are equivalent, which gives an equational axiomatization of scattered spaces in terms of $d$.

It is clear that a scattered space is HI. Therefore, by Theorem 2.4, a scattered space is HR. This is a new proof that a scattered space is HR [4]. Moreover, Theorems 2.4 and 2.11 indicate exactly when an HR space $X$ is scattered: An HR space $X$ is scattered whenever $D(Y)$ is contained in a principal ultrafilter for every nonempty closed subspace $Y$ of $X$; and $X$ is not scattered if $D(X)$ is not contained in a principal ultrafilter.

3. Compact SI spaces and Hausdorff SI spaces

In this section we expand on El’kin’s example to construct an irreducible compact $T_1$-space that is SI. We also expand on the examples of Padmavally and Anderson to construct connected Hausdorff spaces that are MI.

To produce a compact space using Example 2.5 we use Alexandroff’s one-point compactification of a topological space. We note that many texts discuss the one-point compactification only for a locally compact Hausdorff space. Nevertheless, it can be constructed for every space: If $(X, \tau)$ is a topological space, and $X^* = X \cup \{\infty\}$, we define a topology $\tau^*$ on $X^*$ by
\[
\tau^* = \tau \cup \{(X - C) \cup \{\infty\}: C \subseteq X \text{ is compact and closed}\}.
\]
Then $(X^*, \tau^*)$ is a compact space containing $X$ as an open subspace. Moreover, if $X$ is not compact, then $X$ is dense in $X^*$. It is also easy to see that if $X$ is $T_1$, then $X^*$ is also $T_1$. 

\[
d(Y) \subseteq \text{Cl}(Y) = \text{Cl}((\text{Iso}(\text{Cl}(Y))) = \text{Cl}(\text{Cl}(Y) - d(\text{Cl}(Y)))
\]
Note that $X^*$ is Hausdorff if and only if $X$ is locally compact Hausdorff. Consequently, even if $X$ is completely regular, $X^*$ is not Hausdorff when $X$ is not locally compact. We now show that the one-point compactification of the space of Example 2.5 is an irreducible compact $T_1$-space which is SI.

**Proposition 3.1.** Let $F$ be a free ultrafilter on an infinite set $X$, and let $\tau = F \cup \{\emptyset\}$. Then the one-point compactification $(X^*, \tau^*)$ of $(X, \tau)$ is an irreducible compact $T_1$-space such that $D(X^*, \tau^*)$ is the filter on $X^*$ generated by $F$. Furthermore, $X^*$ is SI.

**Proof.** We have already noted that $X^*$ is compact and $T_1$. It is trivial to see that $X^*$ is irreducible since the dense open subspace $X$ is irreducible. We show that $D(X^*, \tau^*)$ is the filter $G$ on $X^*$ generated by $F$. To prove this, let $A \in F$. If $U$ is a nonempty open set of $X^*$, then $U \cap X$ is a nonempty open set of $X$, so $A \cap (U \cap X) \neq \emptyset$, which shows that $A \in D(X^*, \tau^*)$. Therefore, $F \subseteq D(X^*, \tau^*)$, and so $G \subseteq D(X^*, \tau^*)$. For the converse, let $A \in D(X^*, \tau^*)$. Then $A$ intersects every nonempty open set of $X^*$. Since open sets of $X$ are open in $X^*$, $A - \{\infty\}$ intersects every nonempty open set of $X$, so $A - \{\infty\} \in F$. Hence, $A \in G$. We have thus proven that $D(X^*, \tau^*) = G$. Note that $\infty$ is not an isolated point of $X^*$ since $X$ is not compact. Therefore, $X^*$ is dense-in-itself because $X$ is a dense-in-itself open subspace of $X^*$.

Finally, to show $X^*$ is SI, let $Y$ be a nonempty closed subset of $X^*$. Then $Z = Y \cap X = Y - \{\infty\}$ is a closed subset of $X$. If $Z = X$, then $Y = X$ or $Y = X^*$. In either case $D(Y)$ is a filter. Suppose instead $Z$ is a proper subset of $X$. We show that every point of $Z$ is an isolated point of $Y$. To see this, let $y \in Z$. Since $\tau - \{\emptyset\}$ is a filter on $X$, the set $(X - Z) \cup \{y\}$ is open in $X$, and so is also open in $X^*$. Thus, $((X - Z) \cup \{y\}) \cap Y = \{y\}$ is open in $Y$. Because every dense set of $Y$ contains every isolated point of $Y$, we either have $D(Y) = \{Y\}$ or $D(Y) = \{Z, Y\}$, depending on whether $\infty \in Y$ and if it is an isolated point of $Y$. In each case, we see that $D(Y)$ is a filter on $Y$. Therefore, by Theorem 2.4, $X^*$ is SI. \qed

The space $(X, \tau)$ of Example 2.5 satisfies $\tau - \{\emptyset\} = D(X, \tau)$. We now show that the space $(X^*, \tau^*)$ above satisfies $\tau^* - \{\emptyset\} \subseteq D(X^*, \tau^*)$. It is easy to see that the filter $G$ on $X^*$ generated by $F$ is equal to $F \cup \{A \cup \{\infty\}: A \in F\}$ since $X^* = X \cup \{\infty\}$. Suppose $U$ is a nonempty open subset of $X^*$. Then either $U \in F$ or $U = (X - C) \cup \{\infty\}$, where $C$ is compact and closed in $X$. In each case it is obvious that $U \in G$ since $F \subseteq G$ and $X - C \in F$. Therefore, every nonempty open set of $X^*$ is dense, and so $\tau^* - \{\emptyset\} \subseteq D(X^*, \tau^*)$. However, the reverse inclusion does not hold, since there are dense sets of $X^*$ that are not open; to see this, let $C$ be a proper infinite subset of $X$ that is not in $F$. Then $C$ is closed in $X$ and not compact, since, as we proved in Example 2.5, the only compact subsets of $X$ are finite. Thus, $U = X - C \in F$, and so $U \cup \{\infty\} \in G$. On the other hand, $U \cup \{\infty\}$ is not open in $X^*$ by the definition of $\tau^*$.

**Example 3.2.** We now give an example, as promised in Section 2, of a dense-in-itself connected irresolvable space $X$ such that $D(X)$ is not a filter. Let $Y$ be a dense-in-itself connected resolvable $T_1$-space with $|Y| > 1$ and let $Z$ be a dense-in-itself connected irresolvable $T_1$-space. Choose $y \in Y$ and $z \in Z$ such that $Z - \{z\}$ is also irresolvable. For
example, we may choose \( Y \) to be \( \mathbb{R} \) and \( Z \) to be the space of Proposition 3.1 and \( z \) the point at infinity. We let \( X \) denote the wedge \( Y \cup Z \) of the pointed sets \( (Y, y) \) and \( (Z, z) \). This is the gluing of \( Y \) and \( Z \) by identifying \( y \in Y \) with \( z \in Z \). Let \( x \in X \) denote the identified point. It is elementary to see that \( Y \) and \( Z \) are homeomorphic to closed subspaces of \( X \) and that, under the identification of \( Y \) and \( Z \) with subspaces of \( X \), that \( Y \setminus \{ y \} \) and \( Z \setminus \{ z \} \) are open in \( X \). Then \( X \) is irresolvable since \( Z \setminus \{ z \} \) is an irresolvable open subspace of \( X \).

Furthermore, \( X \) is connected since \( Y \) and \( Z \) are intersecting connected subspaces whose union is \( X \). Finally, \( X \) is dense-in-itself since each of \( Y \) and \( Z \) is dense-in-itself. To prove that \( D(X) \) is not a filter, let \( A \) and \( B \) be disjoint dense sets in \( Y \). Then \( A \cup Z \) and \( B \cup Z \) are dense sets in \( X \). Their intersection \( Z \) is not dense in \( X \) since it is a proper closed subset of \( X \), since \( |Y| > 1 \). Thus, \( D(X) \) is not a filter.

The first example of a connected dense-in-itself Hausdorff space that is irresolvable was given by Padmavally [12]. Later, Anderson [1] showed that for each infinite cardinal \( \kappa \) there exists a connected dense-in-itself irresolvable Hausdorff space of dispersion character \( \geq \kappa \). He made use of the notion of an admissible expansion of a topology: If \( (X, \tau) \) is a topological space, then an expansion \( \tau' \supseteq \tau \) of \( \tau \) is said to be admissible if \( \tau' \) has a subbasis of the form \( \tau \cup D \) with \( D \subseteq D(X, \tau) \) such that each \( D \in D \) is dense in \( (X, \tau') \). He proves [1, Lemma 1] that if \( \tau' \) is an admissible expansion of \( \tau \), and if \( (X, \tau) \) is connected, then so is \( (X, \tau') \).

We will expand on these examples to show that there exists a connected Hausdorff SI space. To build the example we will start with a dense-in-itself space \( (X, \tau) \) such that for every nonempty open set \( U \) and every dense set \( A \), the intersection \( U \cap A \) is infinite. An easy argument shows that every dense-in-itself \( T_1 \)-space satisfies this condition; thus, we will assume \( (X, \tau) \) is \( T_1 \). We will construct a new topology \( \hat{\tau} \) containing \( \tau \) such that \( (X, \hat{\tau}) \) is SI. Moreover, if \( (X, \tau) \) is Hausdorff, then clearly \( (X, \hat{\tau}) \) is Hausdorff. We will build \( \hat{\tau} \) from \( \tau \) by adding a certain filter of dense sets of \( (X, \tau) \) to \( \tau \). We show that the appropriate filter exists in the following lemma.

**Lemma 3.3.** Let \( X \) be a topological space. Then there is a filter \( F \) on \( X \) maximal among filters consisting of dense sets.

**Proof.** Let \( S \) be the set of all filters on \( X \) consisting of dense sets. Then \( S \) is nonempty, since if \( A \) is any dense set, then \( F_A = \{ B \in \mathcal{P}(X) : A \subseteq B \} \) is a filter on \( X \) consisting of dense sets. Note that \( S \) is partially ordered by inclusion. We wish to apply Zorn’s lemma to \( S \). To do this, let \( \{ F_\alpha \} \) be a chain in \( S \), where each \( F_\alpha \) is a filter consisting of dense sets. It is elementary to see that \( \bigcup_\alpha F_\alpha \) is a filter and consists of dense sets. This union is then an element of \( S \). By Zorn’s lemma, there is a maximal element \( F \) of \( S \).

**Proposition 3.4.** Let \( (X, \tau) \) be a connected dense-in-itself \( T_1 \)-space. Let \( F \) be a filter consisting of dense sets, maximal among filters consisting of dense sets. Define \( \hat{\tau} \) to be the topology generated by \( \tau \cup F \). Then \( D(X, \hat{\tau}) = F \). Furthermore, \( (X, \hat{\tau}) \) is connected and MI. Finally, if \( (X, \tau) \) is Hausdorff, then \( (X, \hat{\tau}) \) is Hausdorff.
Proof. We observe that a basis of \( \hat{\tau} \) consists of all finite intersections of elements of \( \tau \cup \mathcal{F} \). Since both \( \tau \) and \( \mathcal{F} \) are closed under finite intersections, this basis is \( \{ U \cap A: U \in \tau, A \in \mathcal{F} \} \). To prove the inclusion \( \mathcal{F} \subseteq \mathcal{D}(X, \hat{\tau}) \), let \( B \in \mathcal{F} \). If \( U \in \tau \) and \( A \in \mathcal{F} \), then \( B \cap (U \cap A) = U \cap (A \cap B) \) is nonempty since \( A \cap B \in \mathcal{F} \) and \( \mathcal{F} \subseteq \mathcal{D}(X, \tau) \). For the reverse inclusion, let \( B \in \mathcal{D}(X, \hat{\tau}) \). Then \( B \in \mathcal{D}(X, \tau) \) because \( \tau \) is a smaller topology. Let \( A \in \mathcal{F} \) and \( U \in \tau \). Then \( A \cap U \in \hat{\tau} \). Since \( B \) is dense in \((X, \hat{\tau})\), we have \( B \cap (A \cap U) \neq \emptyset \). Thus, \( A \cap B \) intersects each \( U \in \tau \) nontrivially. Therefore, \( A \cap B \in \mathcal{D}(X, \tau) \). This shows that the filter \( \mathcal{F}' \) generated by \( \mathcal{F} \cup \{ B \} \) is a filter of dense sets. The maximality of \( \mathcal{F} \) implies that \( \mathcal{F}' = \mathcal{F} \), so \( B \in \mathcal{F} \). Therefore, we proved that \( \mathcal{D}(X, \hat{\tau}) = \mathcal{F} \). Since \( \mathcal{F} \subseteq \hat{\tau} \), we see that \((X, \hat{\tau})\) is submaximal. To show that \((X, \hat{\tau})\) is dense-in-itself, recall that \( \{ U \cap A: U \in \tau, A \in \mathcal{F} \} \) is a basis of \( \hat{\tau} \), and that, since \( X \) is \( T_1 \), for each \( U \in \tau - \{ \emptyset \} \) and each \( A \in \mathcal{F} \), the intersection \( U \cap A \) is infinite. It follows that \((X, \hat{\tau})\) has no isolated points, and so \((X, \hat{\tau})\) is dense-in-itself. Thus, \((X, \hat{\tau})\) is MI. Since \( \tau \subseteq \hat{\tau} \), it is obvious that if \((X, \tau)\) is Hausdorff, then so is \((X, \hat{\tau})\). Moreover, by definition of \( \hat{\tau} \) and the equality \( \mathcal{D}(X, \hat{\tau}) = \mathcal{F} \), this topology is an admissible expansion of the connected topology \( \tau \), so \( \hat{\tau} \) is connected, by [1, Lemma 1]. \( \square \)

Note that \( \mathbb{R} \) with its standard topology satisfies the hypotheses of Proposition 3.4. Therefore, there is a connected Hausdorff topology \( \tau \) on \( \mathbb{R} \) that is stronger than the standard topology such that \((\mathbb{R}, \tau)\) is MI.

We have three examples of SI spaces. In the space of Example 2.5, the dense sets are the nonempty open sets. For the space of Proposition 3.1, the nonempty open sets are dense but not vice-versa. Finally, for the space of Proposition 3.4, the dense sets are open but the converse is not true. The first statement follows from Proposition 2.7, we verified the second after Proposition 3.1, and the third follows from Proposition 3.4.

There are examples of non-scattered SI spaces \((Z, \tau)\) in which there is no containment relation between \( \tau - \{ \emptyset \} \) and \( \mathcal{D}(Z, \tau) \), which we now verify. Let \((X, \tau_X)\) be SI, and \((Y, \tau_Y)\) be scattered. Denote by \((Z, \tau_Z)\) the disjoint union of \( X \) and \( Y \). Since the disjoint union of SI spaces is SI, \( Z \) is SI. Furthermore, \( Z \) is not scattered since \( X \) is dense-in-itself. Moreover, the dense sets of \( Z \) are precisely the unions of a dense set in \( X \) and a dense set in \( Y \). We choose \((Y, \tau_Y)\) so that there is no inclusion relation between \( \mathcal{D}(Y) \) and \( \tau_Y \). Then there will be no inclusion relation between \( \mathcal{D}(Z) \) and \( \tau_Z \). An example of a scattered space with this property is the ordinal space \( Y = \omega^2 + 1 \). The isolated points of \( Y \) are the non-limit ordinals. Not every open set in \( Y \) is dense; for example, if \( x \) is an isolated point, then \( \{ x \} \) is not dense. In addition, there are dense sets that are not open; for example, if \( B = Y - \{ \text{on}: n \in \omega \} \), then \( B \) is not open since the sequence \( \text{on}: n \in \omega \) is disjoint from \( B \) and converges to \( \omega^2 \in B \). However, since \( B \) contains all of the isolated points of \( Y \), the set \( B \) is dense, by Theorem 2.11.

4. When do HI, HR, and scattered coincide?

In this last section we show that the three notions of scattered, HR, and HI coincide for rather large classes of topological spaces. We recall that the weight of a topological space \( X \) is the smallest cardinality of a basis of \( X \) and we denote the weight of \( X \) by \( \omega(X) \).
Definition 4.1. A topological space $X$ is said to be light if $\omega(X) \leq |X|$.

We give several classes of light spaces. We will prove below in Corollary 4.7 that each space in one of these classes is scattered if and only if it is HI.

Example 4.2. If $X$ is a finite metric space, then $X$ is discrete, so it is light since $\{\{x\} : x \in X\}$ is a basis for $X$. On the other hand, if $X$ is an infinite metric space, then $\{B(x, 1/n) : x \in X, n \geq 1\}$ is a basis for $X$ of cardinality $|X|$, so $X$ is light. Thus, every metric space is light.

For another example, recall that $X$ is said to be an Alexandroff space if the intersection of every collection of open sets is open. If $X$ is an Alexandroff space, then every point $x$ has a unique smallest open neighborhood $U_x$. It is easy to see that $\{U_x : x \in X\}$ is a basis for $X$, so $X$ is light. It follows that every finite space is light since a finite space is clearly an Alexandroff space.

It is clear that an infinite first countable space is light. Hence, every first countable space is light because we noted above that finite spaces are light. For a final example, if $X$ is locally compact and Hausdorff, then $X$ is light by [3, Corollary 3.3.6].

We recall that a space $X$ is called spectral if it is a compact $T_0$-space such that the set of compact open subsets of $X$ is closed under finite intersections and forms a basis for the topology, and that every nonempty closed irreducible subspace of $X$ is the closure of a singleton. For a spectral space $(X, \tau)$, the patch topology $\tau^#$ on $X$ is defined by setting the compact open sets of $\tau$ and their complements to be a subbasis for $\tau^#$. The space $(X, \tau^#)$ is compact, Hausdorff [10, Theorem 1], and has a basis of clopen sets. Therefore, the compact open sets in $\tau$ are clopen in $\tau^#$.

Example 4.3. We show that a spectral space $X$ is light. If $X$ is finite, then this follows from Example 4.2; so, we may assume that $X$ is infinite. Since $(X, \tau^#)$ is compact Hausdorff, by [3, Corollary 3.3.6] it has a basis $B$ with $|B| \leq |X|$. Note that, by compactness, each clopen subset of $(X, \tau^#)$ is a finite union of elements of $B$. Therefore, the set $C$ of clopen subsets of $(X, \tau^#)$ satisfies $|C| = \aleph_0 \cdot |B|$, which is less than or equal to $|X|$ since $X$ is infinite. Since the set $E$ of all compact open subsets of $(X, \tau)$ is contained in $C$, we see that $|E| \leq |C|$, and hence $|E| \leq |X|$. So, every spectral space is light.

Hewitt proved [9, Theorem 42] that if an infinite $T_1$-space $X$ satisfies $\omega(X) \leq \Delta(X)$, then $X$ is resolvable if and only if $X$ is dense-in-itself. In fact, the proof of [9, Theorem 42] uses neither the $T_1$ nor the $T_0$ assumption. The condition $\omega(X) \leq \Delta(X)$ implies that $X$ is light. However, we give an example below to see that Hewitt’s result is false if the condition $\omega(X) \leq \Delta(X)$ is replaced by the assumption that $X$ is light. Nevertheless, we do prove below that if $X$ has a basis of light spaces, then $X$ is resolvable if and only if it is dense-in-itself.

Example 4.4. We give an example of a dense-in-itself light space which is irresolvable. Let $(U, \tau_U)$ be an SI space. Choose a set $X$ containing $U$ with $|X| \geq \max(\omega(U), |U|)$. We
define a topology $\tau$ on $X$ by
\[
\tau = \tau_U \cup \{ A \subseteq X : U \subseteq A \} = \tau_U \cup [U, X].
\]
The space $(X, \tau)$ is light since if $B$ is a basis of $(U, \tau_U)$ with $|B| = \omega(U)$, then
\[
B \cup \{ U \cup \{ x \} : x \in X - U \}
\]
is a basis for $X$, and its cardinality is less than or equal to $|X|$ by the choice of $X$. Furthermore, since $U$ is dense-in-itself and open in $X$, and since every point in $X - U$ is not isolated, $X$ is dense-in-itself. Finally, $X$ is irresolvable because it contains a nonempty open subspace that is irresolvable.

As the proof of Theorem 42 of [9], the proof of Theorem 20 of [9] uses neither the $T_0$ nor the $T_1$ assumption. We will then quote these results in the proof of the next proposition for spaces which need not be $T_0$ nor $T_1$.

**Proposition 4.5.** Suppose that $X$ has a basis of light spaces. Then $X$ is dense-in-itself if and only if $X$ is resolvable.

**Proof.** If $X$ is resolvable, then it is clearly dense-in-itself. Conversely, suppose that $X$ is dense-in-itself. We first assume that $X$ is finite. Then every $x \in X$ has a unique smallest open neighborhood $U_x$. Since $X$ is dense-in-itself, $|U_x| > 1$. Therefore, we can construct disjoint sets $A$ and $B$ such that $A \cap U_x \neq \emptyset$ and $B \cap U_x \neq \emptyset$ for every $x \in X$. The collection $\{ U_x : x \in X \}$ is a basis for $X$; this shows that $A$ and $B$ are dense in $X$. Thus, $X$ is resolvable.

Next, we assume that $X$ is infinite. To prove that $X$ is resolvable, we will show that every nonempty open set of $X$ contains a nonempty resolvable subspace and apply [9, Theorem 20]. Let $U$ be a nonempty open set in $X$, and let $V$ be an open subset of $U$ with $|V| = \Delta(U)$. By hypothesis on $X$, there is an open set $W \subseteq V$ with $\omega(W) \leq |W|$. By the choice of $V$, we see that $|W| = |V|$, so $\Delta(W) = |W|$, and so $\omega(W) \leq \Delta(W)$. If $W$ is infinite, then $W$ is resolvable by [9, Theorem 42], and if $W$ is finite, then it is resolvable by the finite case proved above. Therefore, by [9, Theorem 20], the space $X$ is resolvable.

**Theorem 4.6.** Let $X$ be a topological space for which every closed subspace has a basis of light subspaces. Then the following conditions are equivalent.

1. $X$ is scattered.
2. $X$ is HI.
3. $X$ is HR.

**Proof.** We already know that (1) implies (2) and (2) is equivalent to (3). It is left to show that (2) implies (1). Suppose $X$ is not scattered. By the Cantor–Bendixson Theorem [13, Theorem 8.5.2], there is a nonempty closed dense-in-itself subspace $Y$ of $X$. By Proposition 4.5, $Y$ is resolvable. Therefore, $X$ is not HI.

The hypothesis of Theorem 4.6 may seem somewhat artificial; however, it is just what we need to apply the result to several classes of topological spaces. Hewitt [9,
Theorems 41, 47, Corollary to Theorem 48] proved that a space $X$ is dense-in-itself if and only if it is resolvable, provided that $X$ is metric, first countable, or locally compact Hausdorff. The following corollary contains a relativized version of these results of Hewitt.

**Corollary 4.7.** The three notions of scattered, HR, and HI coincide provided $X$ is either (1) metric, (2) Alexandroff, (3) first countable, (4) locally compact Hausdorff, or (5) spectral.

**Proof.** We saw in Examples 4.2 that metric spaces, Alexandroff spaces, and first countable spaces are light. Moreover, every subspace of a metric space (respectively Alexandroff, first countable) is again metric (respectively Alexandroff, first countable). Thus, the corollary holds for these classes. Next, to see (4), we note that open subsets and closed subsets of a locally compact Hausdorff space are locally compact Hausdorff. We saw in Example 4.2 that a locally compact Hausdorff space is light. Thus, (4) follows from Theorem 4.6. Finally, suppose that $X$ is spectral. If $Y$ is a closed subspace of $X$, then $Y$ is a spectral space. Furthermore, every spectral space has a basis of open sets each of which is itself a spectral space. Thus, (5) follows from the theorem. 

We note that Gabelaia [6, Theorem 5] proved that an Alexandroff space is scattered if and only if it is HR. Since every finite space is an Alexandroff space, it follows from the previous corollary that the three notions of scattered, HI, and HR coincide for finite topological spaces. In the next corollary we show that they are in fact equivalent to $X$ being a $T_0$-space.

**Corollary 4.8.** For a finite space $X$, the conditions of Theorem 4.6 are each equivalent to $X$ being a $T_0$-space.

**Proof.** Suppose $X$ is scattered, and let $x$ and $y$ be distinct points in $X$. Then $\{x, y\}$ contains an isolated point, say $x$. Therefore, there exists an open set $U$ of $X$ such that $U \cap \{x, y\} = \{x\}$. Thus, $x \in U$ and $y \notin U$, so $X$ is a $T_0$-space.

Conversely, suppose $X$ is a $T_0$-space. Since every subspace of $X$ is also a finite $T_0$-space, it is enough to show that $X$ has an isolated point. However, this is proved in [9, Theorem 8]. Thus, $X$ is scattered.

We conclude the paper by mentioning that for all the spaces covered in Theorem 4.6, and in particular for all spaces in Corollary 4.7, the Cantor–Bendixson Theorem agrees with Hewitt’s decomposition theorem. That is to say, if $X$ is the disjoint union $X = C \cup U$ with $C$ closed and dense-in-itself and $U$ open and scattered, then $C$ is resolvable and $U$ is HI. Therefore, this is precisely Hewitt’s decomposition.

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References