LITTLEWOOD-PALEY AND MULTIPLIER THEOREMS
ON WEIGHTED $L^p$ SPACES\textsuperscript{1}

BY

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ABSTRACT. The Littlewood-Paley operator $\gamma(f)$, for functions $f$ defined on $\mathbb{R}^n$, is shown to be a bounded operator on certain weighted $L^p$ spaces. The weights satisfy an $A_p$ condition over the class of all $n$-dimensional rectangles with sides parallel to the coordinate axes. The necessity of this class of weights demonstrates the 1-dimensional nature of the operator. Results for multipliers are derived, including weighted versions of the Marcinkiewicz Multiplier Theorem and Hörmander's Multiplier Theorem.

1. Introduction. Let $m(x)$ be a bounded function on $\mathbb{R}^n$. The operator $Tf$ defined by the Fourier transform equation $(Tf)(x) = m(x)\hat{f}(x)$ is called a multiplier operator with multiplier $m(x)$. Let $\rho$ be an $(n$-dimensional) interval and $\chi_\rho(x)$ the characteristic function of $\rho$. The operator $S_\rho f$, having multiplier $m(x) = \chi_\rho(x)$ and defined by the equation

$$(S_\rho f)(x) = \chi_\rho(x)\hat{f}(x),$$

is called a partial sum operator.

We define the operator $\gamma(f)$ by

DEFINITION 1.1. Let a collection of disjoint intervals $\Delta = \{\rho\}$ be a decomposition of $\mathbb{R}^n$ (i.e., $\bigcup_\Delta \rho = \mathbb{R}^n$). Given a function, $f$, in the Schwartz class $S(\mathbb{R}^n)$, define

$$\gamma(f)(x) = \gamma(f, \Delta)(x) = \left(\sum_\Delta |S_\rho f(x)|^2\right)^{1/2}.$$  \hspace{1cm} (1.1)

By taking Fourier transforms, for any decomposition $\Delta$, we have the obvious $L^2$ equality

$$\|\gamma(f)\|_2 = \|f\|_2.$$  \hspace{1cm} (1.2)

Given an appropriate $\Delta$, we will show that (1.2) can be extended to certain weighted $L^p$ spaces as an equivalence between norms.

A sequence $\{n_k\}_{k=-\infty}^\infty$, $n_k > 0$, is called a lacunary sequence if there is an $\alpha > 1$ such that $n_{k+1}/n_k \geq \alpha$ for all $k$. The dyadic sequence, $n_k = 2^k$, is an example of such a sequence.

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DEFINITION 1.2. Let \( \{n_k\}_{k=-\infty}^{+\infty} \) be a lacunary sequence. Let \( \Delta \) be the collection of all intervals of the form \( [n_k, n_{k+1}] \) and \( [-n_{k+1}, -n_k], -\infty < k < \infty \). Then, \( \Delta \) is called a lacunary decomposition of \( \mathbb{R}^1 \).

It follows from the definition of a lacunary sequence that \( \bigcup \Delta \rho = \mathbb{R}^1 \) (or, to be exact, \( \mathbb{R}^1 - \{0\} \)). When \( \{n_k\} \) is the dyadic sequence, the resulting \( \Delta \) is called the dyadic decomposition of \( \mathbb{R}^1 \).

DEFINITION 1.3. Let \( \Delta_i, i = 1, 2, \ldots, n \), be \( n \) lacunary decompositions of \( \mathbb{R}^1 \). Let \( \Delta \) be the collection of the intervals, \( \rho \), of the form \( \rho = \rho_1 \times \rho_2 \times \cdots \times \rho_n \) where \( \rho_i \in \Delta_i \). Then, \( \Delta \) is called a lacunary decomposition of \( \mathbb{R}^n \).

It is well known (see [21] and [24]) that if \( \Delta \) is a lacunary decomposition of \( \mathbb{R}^n \), then \( \|\gamma(f)\|_p \) is equivalent to \( \|f\|_p \) for \( 1 < p < \infty \); i.e., there are constants \( A(p, \Delta) \) and \( B(p, \Delta) \) such that

\[
A(p, \Delta)\|f\|_p \leq \|\gamma(f)\|_p \leq B(p, \Delta)\|f\|_p.
\]

The weight functions we will consider satisfy the following definition.

DEFINITION 1.4. Let \( \mathcal{R} \) be a collection of bounded sets in \( \mathbb{R}^n \) and \( w \) a nonnegative, locally integrable function. If \( 1 < p < \infty \), then \( w \) is in \( A_p(\mathbb{R}^n, \mathcal{R}) \) if there is a constant, \( c \), such that

\[
\left( \frac{1}{|R|} \int_R w(x) \, dx \right) \left( \frac{1}{|R|} \int_R w(x)^{-1/(p-1)} \, dx \right)^{\frac{p-1}{2}} \leq c
\]

for any \( R \in \mathcal{R} \). We say \( w \) is in \( A_1(\mathbb{R}^n, \mathcal{R}) \) if there is an constant, \( c \), such that \( w^*(x) \leq cw(x) \) for almost every \( x \), where

\[
w^*(x) = \sup_{R \in \mathcal{R}} \frac{1}{|R|} \int_R w(x) \, dx
\]

is the Hardy-Littlewood maximal function of \( w \) with respect to the collection \( \mathcal{R} \).

This class of functions was first introduced by Rosenblum [17] and Muckenhoupt [11]. The basic properties of \( A_p \) functions can be found in Muckenhoupt [11] and Coifman and C. Fefferman [1].

Let \( \mathcal{Q}_n \) and \( \mathcal{Q}'_n \) denote the collections of all \( n \)-dimensional cubes and all \( n \)-dimensional intervals with sides parallel to the coordinate axes, respectively. \( A_p(\mathbb{R}^n, \mathcal{Q}_n) \) is the \( A_p \) class of Muckenhoupt. We note that when \( n = 1, A_p(\mathbb{R}^1, \mathcal{Q}_1) = A_p(\mathbb{R}^1, \mathcal{Q}_1) \). However, for \( n > 1 \), we have \( A_p(\mathbb{R}^n, \mathcal{Q}_n) \subsetneq A_p(\mathbb{R}^n, \mathcal{Q}_n) \). That the containment is strict is demonstrated by the fact that \( |x|^\alpha \in A_p(\mathbb{R}^n, \mathcal{Q}_n) \) for \( -n < \alpha < n(p - 1) \) while \( |x|^\alpha \in A_p(\mathbb{R}^n, \mathcal{Q}_n) \) for \( -1 < \alpha < p - 1 \). In other words, the values of \( \alpha \) for which \( |x|^\alpha \) is in \( A_p(\mathbb{R}^n, \mathcal{Q}_n) \) lie in the 1-dimensional range, \( -1 < \alpha < p - 1 \). As we will see in the next section, this is a consequence of the fact that \( A_p(\mathbb{R}^n, \mathcal{Q}_n) \) can be described as the class of functions which are in \( A_p(\mathbb{R}^1, \mathcal{Q}_1) = A_p(\mathbb{R}^1, \mathcal{Q}_1) \) in each variable uniformly with respect to the other variables.

Let \( w(x) \) be a nonnegative function. We define \( L^p_\mathcal{Q}(\mathbb{R}^n), 1 < p < \infty, \) to be the collection of all functions \( f \) such that \( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx < + \infty \). For \( f \in L^p_\mathcal{Q}(\mathbb{R}^n) \), we define

\[
\|f\|_{p,w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p}.
\]
\( \| \cdot \|_{p,w} \) is a norm which makes \( L^p_w(\mathbb{R}^n) \) a Banach space. Closely allied to the \( L^p_w \) spaces are the weighted analogs of the Hardy-Stein-Weiss spaces \( H^p \). We assume the reader is familiar with the theory of \( H^p \) spaces and their relationships to \( L^p \) spaces. For \( p > 1 \) and \( w \in A_p(\mathbb{R}^n, \mathcal{B}_n) \), \( H^p_w \) is naturally isomorphic to \( L^p_w \). When \( p = 1 \) and \( w \in A_1(\mathbb{R}^n, \mathcal{B}_n) \), \( H^1_w \) is isomorphic to a subspace of \( L^1_w \). For definitions and details, see [3], [12], [21], [22], and [24]. The Schwartz class, \( \mathcal{S}(\mathbb{R}^n) \), of infinitely differentiable functions of rapid decrease at infinity is dense in all of the previously mentioned spaces, in the appropriate norms.

We now state our main result.

**Theorem 1.** Let \( \Delta = \{ \rho \} \) be a lacunary decomposition of \( \mathbb{R}^n \), \( 1 < p < \infty \), and \( w \in A_p(\mathbb{R}^n, \mathcal{B}_n) \). Then, there are constants \( A \) and \( B \), depending on \( p \), \( w \), and \( \Delta \), such that

\[
A \| f \|_{p,w} \leq \| \gamma(f)(x) \|_{p,w} \leq B \| f \|_{p,w}
\]  

(1.4)

When \( \Delta \) is the dyadic decomposition of \( \mathbb{R}^n \), (1.4) implies \( w \in A_p(\mathbb{R}^n, \mathcal{B}_n) \).

Hirschman [4] proved this theorem in the periodic case, where \( \Delta \) is the dyadic decomposition of the integers and \( w(\theta) = |\theta|^a, -1 < a < p - 1 \). When \( n = 1 \) and \( \Delta \) is the dyadic decomposition of \( \mathbb{R}^1 \), the theorem extends to a weak type result for \( p = 1 \). That is, there exists a constant, \( c \), depending on \( w \) and \( \Delta \), such that

\[
w(\{ x \in \mathbb{R}^n : |\gamma(f)(x)| > \lambda \}) \leq (c/\lambda) \| f \|_{H^1_w},
\]

where, given a measurable set \( E \), \( w(E) = \int_E w(x) \, dx \).

The proof of Theorem 1 is divided into several parts. We first show that in \( \mathbb{R}^1 \), \( w \in A_p \) and \( f \in L^p_w \) imply (1.4). From this we derive the \( n \)-dimensional version. Next we show that (1.4) implies \( w \in A_p \) when \( \Delta \) is the dyadic decomposition of \( \mathbb{R}^n \).

The proof is completed by showing that if \( w \in A_p \) and \( f \notin L^p_w \) then \( \gamma(f) \notin L^p_w \).

In order to go from the dyadic decomposition of \( \mathbb{R}^1 \) to a general lacunary one, we need a generalization of the Marcinkiewicz Multiplier Theorem.

**Theorem 2.** Let \( m \) be bounded on \( \mathbb{R}^1 \) and of bounded variation on every finite interval not containing the origin. Let \( \| m \|_\infty \leq B \) and \( \int_I |dm(x)| \leq B \) for every dyadic interval \( I \). If \( 1 < p < \infty \) and \( w \in A_p(\mathbb{R}^1, \mathcal{B}_1) \), then \( m \) is a bounded multiplier from \( L^p_w(\mathbb{R}^1) \) to \( L^p_w(\mathbb{R}^1) \), with norm depending only on \( B, p, \) and \( w \).

As in the unweighted case, Theorem 1 is equivalent to Theorem 2 when \( n = 1 \). Using the result for the \( \gamma \)-function in \( \mathbb{R}^n \), we can get a generalization of Theorem 2. We think of \( \mathbb{R}^n \) as divided into \( 2^n \) "quadrants" by the coordinate axes. For example, the first "quadrant" is the set \( \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \ldots, n \} \).

**Theorem 3.** Let \( m \in C^n \) in each "quadrant" of \( \mathbb{R}^n \) and such that \( \| m \|_\infty \leq B \),

\[
\sup_{x_{k+1}, \ldots, x_n} \int_{\mathbb{R}^n} \left| \frac{\partial^k m(x)}{\partial x_1 \cdots \partial x_k} \right| dx_1 \cdots dx_k \leq B
\]

for \( 0 < k < n \), \( m \) any dyadic interval in \( \mathbb{R}^n \), and any permutation of \( (x_1, \ldots, x_n) \). If \( 1 < p < \infty \) and \( w \in A_p(\mathbb{R}^n, \mathcal{B}_n) \) then \( m \) is a bounded multiplier from \( L^p_w(\mathbb{R}^n) \) to \( L^p_w(\mathbb{R}^n) \).
The proof of Theorem 1 relies on a weighted version of Hörmander’s Multiplier Theorem.

**Theorem 4.** Let \( k > \lceil n/2 \rceil \) and \( m \in C^k(\mathbb{R}^n - \{0\}) \). Suppose that

\[
\sup_{r > 0} \frac{r^{2|\alpha| - n}}{r < |x| < 2r} \left| \left( \frac{\partial}{\partial x} \right)^{\alpha} m(x) \right|^2 dx \leq B, \text{ for } |\alpha| < k.
\]

When \( k < n \) and \( n/k < p < \infty \), \( m \) is a bounded multiplier from \( L^p_w(\mathbb{R}^n) \) to \( L^p_w(\mathbb{R}^n) \) and maps \( L^{p/k}_w(\mathbb{R}^n) \) to weak \( L^{n/k}_w(\mathbb{R}^n) \) if \( w \in A_{p/k}(\mathbb{R}^n, \mathcal{D}_n) \). For \( k = n \), the strong type result is the same, but now \( H^1_w(\mathbb{R}^n) \) gets mapped into weak \( L^1_w(\mathbb{R}^n) \). Finally, if \( k > n \), \( m \) is a bounded multiplier from \( L^p_w(\mathbb{R}^n) \) to \( L^p_w(\mathbb{R}^n) \), \( 1 < p < \infty \), and from \( H^1_w(\mathbb{R}^n) \) to \( H^1_w(\mathbb{R}^n) \). The norm of the operator depends only on \( B, p, w, k, \) and \( n \).

We will prove Theorem 4 using Littlewood-Paley theory. A more general version of the theorem appears in [10]. That proof involves the sharp function of Fefferman and Stein [2].

In Chapter 2, we prove several preliminary results, including Theorem 4. Chapter 3 consists of the proof of Theorem 1.

Chapter 4 is devoted to obtaining multiplier theorems. As a consequence, we obtain the following weighted, \( n \)-dimensional variant of the Hausdorff-Young Theorem.

\[
\left( \sum_{\Delta} \| (S_{\rho/f})^*(x) |x|^{-\alpha} \|_p^2 \right)^{1/2} \leq C \| f(x) |x|^{-\alpha} \|_p,
\]

for \( 1 < p < 2 \) and \( 0 < \alpha < 1/p' \) (see (4.5)). Typical of the results (though easier to state) is the following one.

**Theorem 5.** Let \( 1 < p < 2 < q < \infty \), \( 1/r = 1/p - 1/q \), \( 0 < \alpha < 1/q \), and \( 0 < \beta < 1/p' \). Given a bounded function \( m(x) \), let \( Tf \) be the multiplier operator defined by \( (Tf)^*(x) = m(x)\hat{f}(x) \). If \( m(x) |x|^{-\alpha+\beta} \in L^r(\mathbb{R}^n) \), then \( T \) is a bounded operator from \( L^{q/r}_{\alpha+\beta}(\mathbb{R}^n) \) to \( L^{q/r}_{\alpha+\beta}(\mathbb{R}^n) \).

We note that the main theorem and the applications are all true when carried out in the periodic case.

Throughout this paper, \( C \) will denote a positive constant, not necessarily the same for each occurrence, depending only on the parameters mentioned or implied but not on \( f \) (except in the proof of Theorem 4.2). All sets and functions mentioned are assumed to be measurable and we take \( 0 \cdot \infty \) to be 0.

I would like to express my warmest thanks to Dr. Richard L. Wheeden. This work is truly a product of his patience and guidance.

2. Preliminary results. Let \( f = \{ f_k \} \) be a vector-valued function on \( \mathbb{R}^n \). We say \( f \in L^p_w(\mathbb{R}^n, I^2) \) if \( \| f \| = (\Sigma_k |f_k|^2)^{1/2} \in L^p_w(\mathbb{R}^n) \), and then \( \| f \|_{L^p_w(\mathbb{R}^n, I^2)} = \| (\Sigma |f_k|^2)^{1/2} \|_{p,w} \). In addition to Theorem 4, we will obtain the following
Theorem 2.1. Let \( 1 < p < \infty \) and \( w \in A_p(\mathbb{R}^n, \mathbb{R}_n) \). Let \( \Delta = \{ \rho_k \} \) be any collection of intervals in \( \mathbb{R}^n \). For \( f = \{ f_k \} \in L^p_\Delta (\mathbb{R}^n, l^2) \), set \( S(f) = \{ S_{\rho_k} f_k \} \). Then

(i) \( \| S(f) \|_{L^p_\Delta (\mathbb{R}^n, l^2)} \leq C \| f \|_{L^p_\Delta (\mathbb{R}^n, l^2)} \), \( 1 < p < \infty \),

(ii) \( w(\{ x \in \mathbb{R}^n : |S(f)(x)| > \lambda \}) \leq (C/\lambda) \| f \|_{L^p_\Delta (\mathbb{R}^n, l^2)} \).

The \( C \) depends on \( w, p, \) and \( n \).

Lemma 2.2. Let \( w \in A_p(\mathbb{R}^n, \mathbb{R}_n) \), \( 1 < p < \infty \). Then, there exists a constant, \( C \), depending only on \( w \), such that for almost every fixed \( (n - 1) \)-tuple, \((x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\), and any interval \( I \subset \mathbb{R}^1 \),

\[
\left( \frac{1}{|I|} \int_I w(x_1, \ldots, x_j, \ldots, x_n) \, dx_j \right) \cdot \left( \frac{1}{|I|} \int_I \left( w(x_1, \ldots, x_j, \ldots, x_n) \right)^{-1/(p-1)} \, dx_j \right)^{p-1} \leq C.
\]

Lemma 2.2 says that \( A_p \) over arbitrary rectangles implies \( A_p \) in each variable uniformly with respect to the other variables. The two conditions are actually equivalent.

Let \( f(x) \in \mathbb{R}(\mathbb{R}^n) \) and let \( f(x, y) \) be its Poisson integral. Let \( \nabla f(x, y) \) be the full gradient of \( f(x, y) \) and define the \( k \)th gradient of \( f \) by

\[
\nabla^k f(x, y) = \left( \nabla^{k-1} \frac{\partial}{\partial x_1} f(x, y), \ldots, \nabla^{k-1} \frac{\partial}{\partial x_n} f(x, y), \nabla^{k-1} \frac{\partial}{\partial y} f(x, y) \right).
\]

In the proof of Theorem 4, we will need the following variants of the Littlewood-Paley \( g \)-function of \( f \):

\[
S_k(f)(x) = \left( \int \int_{|x-t| < y} |\nabla^k f(t, y)|^2 y^{2k-n} \, dt \, dy \right)^{1/2},
\]

\[
g_k^*(f)(x) = \left( \int_0^\infty \int_{R_n} \left( \frac{y}{|t| + y} \right)^\lambda |\nabla f(x - t, y)|^2 y^{1-n} \, dt \, dy \right)^{1/2}.
\]

\( S_k(f) \) satisfies the inequality

\[
S_1(f)(x) < C_k S_k(f)(x)
\]

(see [21, p. 216]).

We now begin the proof of Theorem 4. Let \( f \in \mathbb{R}(\mathbb{R}^n) \) and define \( g(x) \) by \( \hat{g}(x) = m(x) \hat{f}(x) \). By standard arguments (see e.g., [21, pp. 96–99 and pp. 232–235]), we deduce that \( S_{k+1}(g)(x) \leq C g_k^*(f)(x), \lambda = 2k/n \). Therefore, by (2.1), \( S_1(g)(x) \leq C g_k^*(f)(x) \). Applying Corollary 1 of [3] and the corollary to Theorem 2 of [12], we get \( \| g \|_{H^2} \leq C \| f \|_{H^2} \) if \( p > n/k \), or \( p \geq 1 \) if \( k > n \). The result extends to \( k > n \) by continuity; for \( p = 1 \), see [13].

To prove the weak type inequalities, we need the nontangential maximal function of \( f \), defined by

\[
N(f)(x) = \sup_{\{(t, y) : |x-t| < y/2\}} |f(t, y)|.
\]

Since \( f \in \mathbb{R}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \), \( f(x) \leq N(f)(x) \) for almost every \( x \). It follows from Gundy and Wheeden [3] that, given \( 0 < \varepsilon < 1 \) and \( \beta > 1 \), there exists a \( \delta > 0 \)
such that, for all $\alpha > 0$,
\[
\omega(\{x \in \mathbb{R}^n : N(f)(x) > \beta \alpha\}) \leq \epsilon \omega(\{x \in \mathbb{R}^n : N(f)(x) > \alpha\}) + \omega(\{x \in \mathbb{R}^n : S_1(f)(x) > \delta \alpha\}).
\] (2.2)

Multiply both sides of (2.2) by $(\beta \alpha)^{n/k}$ and take the sup over $\alpha > 0$. We then get
\[
\sup_{\alpha > 0} (\beta \alpha)^{n/k} \omega(\{x \in \mathbb{R}^n : N(f)(x) > \beta \alpha\})
\]
\[
\leq \epsilon \beta^{n/k} \sup_{\alpha > 0} \alpha^{n/k} \omega(\{x \in \mathbb{R}^n : N(f)(x) > \alpha\})
\]
\[
+ \sup_{\alpha > 0} (\beta \alpha)^{n/k} \omega(\{x \in \mathbb{R}^n : S_1(f)(x) > \delta \alpha\}).
\]

Since the first sup on the right is the same as the one on the left, if we choose $\epsilon$ and $\beta$ so that $\epsilon \beta^{n/k} = \frac{1}{2}$ and change $\beta \alpha$ to $\alpha$, we get
\[
\sup_{\alpha > 0} \alpha^{n/k} \omega(\{x \in \mathbb{R}^n : N(f)(x) > \alpha\})
\]
\[
\leq C \sup_{\alpha > 0} \alpha^{n/k} \omega\left(\left\{x \in \mathbb{R}^n : S_1(f)(x) > \frac{\delta \alpha}{\beta}\right\}\right).
\]

From the previous remarks and the corollary of [12], since $n/k = 2/\lambda$, and $g \in L^2(\mathbb{R}^n)$,
\[
\sup_{\alpha > 0} \alpha^{n/k} \omega(\{x \in \mathbb{R}^n : |g(x)| > \alpha\}) \leq \sup_{\alpha > 0} \alpha^{n/k} \omega(\{x \in \mathbb{R}^n : N(g)(x) > \alpha\})
\]
\[
\leq C \sup_{\alpha > 0} \alpha^{n/k} \omega\left(\left\{x \in \mathbb{R}^n : S_1(g)(x) > \frac{\delta}{\beta} \alpha\right\}\right)
\]
\[
\leq C \sup_{\alpha > 0} \alpha^{n/k} \omega\left(\left\{x \in \mathbb{R}^n : g^*_h(f)(x) > C' \frac{\delta}{\beta} \alpha\right\}\right)
\]
\[
\leq C \|f\|^{n/k}_{H^1}\|g\|^{n/k}_{H^1}.
\]

It is of interest to compare Theorem 4 to the original version of Hörmander's theorem. In the unweighted case, $k > [n/2]$ is sufficient to get a multiplier on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. This uses the fact that multipliers on $L^p$ and $L^\infty$ are the same. Although there is a similar duality for weighted $L^p$ spaces, we cannot use it because we also need $w \in A_{pk/n}(\mathbb{R}^n, \mathcal{D}_n)$. When $n = 1$ and $n = 2$, the two theorems agree since $k > [n/2]$ implies $k > n$ in these cases. Also, to get the $H^1$ result, both need the same $k$.

Let $\rho = (a, b) \subset \mathbb{R}^1$ and let $H(f)$ denote the Hilbert transform of $f$. By comparing the Fourier transforms, it is not hard to see that
\[
S_{\rho}f(x) = S_{(a, b)}f(x)
\]
\[
= \frac{i}{2} \left[ e^{2\pi i x \cdot b} H(e^{-2\pi i x \cdot a} f(t))(x) - e^{2\pi i x \cdot a} H(e^{-2\pi i x \cdot b} f)(x) \right].
\] (2.3)

Consider now Theorem 2.1. By (2.3), for the proof when $n = 1$, it is enough to know the result when $S(f)$ is replaced by the vector-valued Hilbert transform. This result was proved by John [9].
Next, let \( \rho = (a_1, b_1) \times \cdots \times (a_n, b_n) \subseteq \mathbb{R}^n \) and \( H_j(f) \) be the 1-dimensional Hilbert transform in the \( x_j \)-variable. If \( S_{\rho}^j \) is the operator acting only on the \( x_j \) variable, by considering functions of the form \( f(x) = f_1(x_1)f_2(x_2) \cdots f_n(x_n) \), it follows that \( S_{\rho}f = S_{\rho}^1S_{\rho}^{n-1} \cdots S_{\rho}^1(f) \). But, as in (2.3),

\[
S_{\rho}^j(f)(x) = \frac{i}{2} \left[ e^{2\pi i x \cdot b_j}H_j(e^{-2\pi i \cdot f(t)})(x) - e^{2\pi i x \cdot a_j}H_j(e^{-2\pi i \cdot f(t)})(x) \right].
\]

Thus the theorem is proved by an \( n \)-fold application of the 1-dimensional result once we establish Lemma 2.2; i.e., once we know that \( w \in A_p(\mathbb{R}^n, \mathcal{R}_n) \) implies \( w \) is in \( A_p \) uniformly in each variable.

For simplicity in the proof of the lemma, we will assume \( n = 2 \). Let \( w \in A_p(\mathbb{R}^2, \mathcal{R}_2) \) and \( I \subset \mathbb{R}^1 \) be any interval. We want to show

\[
\left( \frac{1}{|I|} \int_I w(x, y) \, dy \right) \left( \frac{1}{|I|} \int_I w(x, y)^{-1/(p-1)} \, dy \right)^{p-1} < C
\]

(2.4)
for almost every \( x \). If \( J \subset \mathbb{R}^1 \) is any interval, then by assumption

\[
\left[ \frac{1}{|J|} \int_J \left( \frac{1}{|I|} \int_I w(t, y) \, dy \right) dt \left[ \frac{1}{|J|} \int_J \left( \frac{1}{|I|} \int_I w(t, y)^{-1/(p-1)} \, dy \right) dt \right]^{p-1}
\]

\[
= \left( \frac{1}{|J \times I|} \int_{J \times I} w(t, y) \, dy \, dt \right) \left( \frac{1}{|J \times I|} \int_{J \times I} w(t, y)^{-1/(p-1)} \, dy \, dt \right)^{p-1} < C.
\]

Letting \( J \) shrink to \( x \) and using Lebesgue’s Differentiation Theorem, we get (2.4) for almost every \( x \), depending on \( I \). Considering only intervals, \( I \), with rational endpoints and taking limits, the result follows.

3. Proof of Theorem 1. The proof of our main theorem will use Khinchine’s inequality for Rademacher series. Let \( r_m(t) = \text{sgn}(\sin 2^m \pi t) \), \( m = 0, 1, 2, \ldots \), be the Rademacher functions, and set \( f(t) = \sum_{m=0}^{\infty} a_m r_m(t) \). Then there are constants \( B_p \) and \( C_p \) such that for \( 0 < p < \infty \)

\[
B_p \left( \int_0^1 |f(t)|^p \, dt \right)^{1/p} < \left( \sum |a_m|^2 \right)^{1/2} < C_p \left( \int_0^1 |f(t)|^p \, dt \right)^{1/p}
\]

(3.1)
(see [24, Vol. I, p. 213]). If \( m = (m_1, \ldots, m_n) \) is a multi-index of nonnegative integers, we define the \( n \)-dimensional Rademacher functions by

\[
r_m(t) = r_{m_1}(t_1) \cdots r_{m_n}(t_n) = r_{m_1}(t_1) r_{m_2}(t_2) \cdots r_{m_n}(t_n),
\]

where \( r_{m_i}(t_i) \) is a 1-dimensional Rademacher function. This collection satisfies an inequality similar to (3.1).

We begin the proof of Theorem 1 with

**Theorem 3.1.** Let \( \Delta \) be the dyadic decomposition of \( \mathbb{R}^1 \).
(i) If \( 1 < p < \infty, w \in A_p(\mathbb{R}^1, \mathcal{R}_1) \), and \( f \in L^p_\mathcal{R}(\mathbb{R}^1) \), then

\[
A\|f\|_{p,w} \leq \|\gamma(f)\|_{p,w} \leq B\|f\|_{p,w}.
\]

(ii) If \( w \in A_1(\mathbb{R}^1, \mathcal{R}_1) \) and \( f \in H^1_w(\mathbb{R}^1) \), then

\[
w\left( \{ x \in \mathbb{R}^1 : |\gamma(f)(x) > \lambda \} \right) < (C/\lambda)\|f\|_{H^1_w}.
\]

The constants \( A \), \( B \), and \( C \) depend only on \( p \) and \( w \).
Let $\phi \in C^2(\mathbb{R}^1)$ be equal to 1 on $(1, 2)$ and 0 on the complement of $(\frac{1}{2}, 4)$. For $\rho \in \Delta$, $\rho = [2^k, 2^{k+1}]$ say, set $\phi_\rho(x) = \phi(2^{-k}x)$ and define $\Phi_\rho f$ to be the operator with multiplier $\phi_\rho(x)$; i.e., $(\Phi_\rho f)(x) = \phi_\rho(x)f(x)$. Since $\phi_\rho(x) = 1$ for $x \in \rho$, it follows that

$$S_\rho \Phi_\rho f = S_\rho f.$$ (3.2)

Let $\{r_\rho(t)\}$ be the Rademacher functions indexed by $\rho \in \Delta$ and define

$$\psi f = \sum_\Delta r_\rho(t) \Phi_\rho f.$$

The multiplier associated with $\psi$ is $m_\rho(x) = \sum_\Delta r_\rho(t) \phi_\rho(x)$. Since at most three $\phi_\rho$’s are nonzero for any given $x$, there is a constant $B$, depending only on $\phi$, such that

$$|m_\rho(x)| < B, \quad \left| \frac{d}{dx} m_\rho(x) \right| < \frac{B}{|x|}, \quad \text{and} \quad \left| \frac{d^2}{dx^2} m_\rho(x) \right| < \frac{B}{|x|^2}.$$ (3.3)

Thus, $m_\rho(x)$ satisfies the conditions of Theorem 4, so that

$$||\psi f||_{p,w}^p \leq C^p ||f||_{p,w}^p$$

(if $p = 1$, these are $H^p_\rho$ norms). Since $B$ can be chosen independent of $t$, $C$ does not depend on $t$. Now, integrate (3.3) in $t$ from 0 to 1 and change the order of integration on the left. By (3.1),

$$\left\| \left( \sum_\Delta |\Phi_\rho f|^2 \right)^{1/2} \right\|_{p,w} \leq C \|f\|_{p,w}.$$ (3.4)

From Theorem 2.1 and (3.2), we get

$$\|\gamma(f)\|_{p,w} = \left\| \left( \sum_\Delta |S_\rho f|^2 \right)^{1/2} \right\|_{p,w} = \left\| \left( \sum_\Delta |S_\rho \Phi_\rho f|^2 \right)^{1/2} \right\|_{p,w} \leq C \left\| \left( \sum_\Delta |\Phi_\rho f|^2 \right)^{1/2} \right\|_{p,w} \leq C \|f\|_{p,w}$$

for $p > 1$, and

$$\left\{ \{w \in \mathbb{R}^n : |\gamma(f)(x)| > \lambda \} \right\} = w \left( \left\{ x \in \mathbb{R}^n : \left( \sum_\Delta \left| S_\rho \Phi_\rho f(x) \right|^2 \right)^{1/2} > \lambda \right\} \right) \leq (C/\lambda) \left\| \left( \sum_\Delta |\Phi_\rho f|^2 \right)^{1/2} \right\|_{1,w} \leq (C/\lambda) \|f\|_{H^w_\rho}.$$ (3.5)

The proof of part (i) is completed by a duality argument (see [21, p. 105]).

Notice that Theorem 3.1 remains true if $\Delta$ is defined by the sequence $n_k = \alpha^k$, $\alpha > 1$, instead of $n_k = 2^k$. The proof is the same with trivial modifications.

In order to extend the result to a general lacunary decomposition of $\mathbb{R}^1$, we need Theorem 2. Assuming its validity for the moment, we will prove

**Theorem 3.2.** Let $\Delta$ be a lacunary decomposition of $\mathbb{R}^1$. If $1 < p < \infty$, $w \in A_p(\mathbb{R}^1, R_1)$, and $f \in L^p_w(\mathbb{R}^1)$, then there are constants $A$ and $B$, depending only on $p$, $w$, and $\Delta$, so that

$$A \|f\|_{p,w} \leq \|\gamma(f)\|_{p,w} \leq B \|f\|_{p,w}.$$
Let $\Delta$ be defined by the sequence $\{ \pm n_k \}_{k=-\infty}^{+\infty}$ with $n_{k+1}/n_k > \alpha > 1$. Let $M$ be the least integer such that $\alpha^M > 2$. If $m(x)$ is any function such that $m|_\rho$ is identically $+1$ or $-1$ for each $\rho \in \Delta$, then $\|m\|_{\infty} = 1$ and $\int_I |dm(x)| \leq 2M$ for any dyadic interval $I$. By Theorem 2, $m$ is a multiplier on $L^p_\rho(\mathbb{R}^1)$.

Define $m_t(x)$ by $m_t(x) = r_\rho(t)$ if $x \in \rho$. Then $m_t(x) = \sum_\Delta r_\rho(t)X_\rho(x)$. Also, let $G_N = \{ x: 1/N < |x| < N \}$. $\Delta_N = \{ \rho \in \Delta : \rho \subset G_N \}$ and define $m^N_t(x) = \sum_\Delta r_\rho(t)X_\rho(x)$. Clearly, $m^N_t$ satisfies the same bounds as $m_t$. If we define $g$ by $\hat{g}(x) = m^N_t(x)\hat{f}(x)$, then

$$\hat{g}(x) = \left( \sum_\Delta r_\rho(t)X_\rho(x) \right)\hat{f}(x) = \sum_\Delta r_\rho(x)X_\rho(x)\hat{f}(x)$$

$$= \sum_\Delta r_\rho(t)(S_\rho f)^*(x) = \left( \sum_\Delta r_\rho(t)S_\rho f \right)^*(x).$$

By Theorem 2, $\|\sum_\Delta r_\rho(t)S_\rho f\|_{p,w} \leq C\|f\|_{p,w}$. The $C$ here depends on $\Delta, p$ and $w$, but not $t$ or $N$. Applying (3.1), as in the method following (3.3), $\|\left( \sum_\Delta |S_\rho f|^2 \right)^{1/2}\|_{p,w} \leq \|f\|_{p,w}$. Letting $N \to \infty$ and using the Monotone Convergence Theorem yields

$$\|\gamma(f)\|_{p,w} = \left\| \left( \sum_\Delta |S_\rho f|^2 \right)^{1/2} \right\|_{p,w} \leq C\|f\|_{p,w}.$$ 

The proof of the opposite inequality is proved in the same manner as in Theorem 3.1. This completes the proof of Theorem 3.2.

Theorem 3.2 is the 1-dimensional version of inequality (1.4). We now proceed with the proof for general $n$.

From Theorem 3.2 we deduce the 1-dimensional inequality

$$\left\| \sum_\Delta r_\rho(t)S_\rho f \right\|_{p,w} \leq C\|f\|_{p,w}. \quad (3.5)$$

In fact, if we order $\Delta = \{ \rho_j \}_{j=-\infty}^{+\infty}$, then for $f \in L^p_\rho(\mathbb{R}^1)$, $(\sum_1^N |S_\rho f|^2)^{1/2}$ is Cauchy in $L^p_\rho(\mathbb{R}^1)$. If $N > M$, using Theorem 3.2 again,

$$\left\| \sum_1^N r_\rho(t)S_\rho f - \sum_1^M r_\rho(t)S_\rho f \right\|_{p,w} = \left\| \sum_1^N r_\rho(t)S_\rho f \right\|_{p,w} < C\left\| \left( \sum_1^N |S_\rho f|^2 \right)^{1/2} \right\|_{p,w},$$

which implies $(\sum_1^N r_\rho(t)S_\rho f)^{1/2}_{N=1}$ is Cauchy in $L^p_\rho(\mathbb{R}^1)$. (3.5) follows from this.

Define $T_x f(x) = \sum_\Delta r_\rho(t)S_\rho f(x)$. Let $T_t$ be the operator above acting only on the $t_i$-variable, with the other variables fixed. By considering functions of the form $f(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n)$, we see that

$$T_f = T_{x_1}T_{x_{n-1}} \cdots T_{x_1}f. \quad (3.6)$$

If $f \in (L^2 \cap L^p)(\mathbb{R}^n)$, then for almost every fixed $(n - 1)$-tuple $(x_2, \ldots, x_n)$, $f(x_1, x_2, \ldots, x_n) \in (L^2 \cap L^p)(\mathbb{R}^1)$ and $w(x_1, x_2, \ldots, x_n) \in A_p(\mathbb{R}^1, \mathcal{F}_1)$, with an $A_p$ constant independent of $x_2, \ldots, x_n$. Applying the 1-dimensional result (3.5) gives

$$\int_{\mathbb{R}^1} |T_{x_1}f(x_1, x_2, \ldots, x_n)|^p w(x_1, x_2, \ldots, x_n) \, dx_1 \leq C^p \int_{\mathbb{R}^1} |f(x_1, x_2, \ldots, x_n)|^p w(x_1, x_2, \ldots, x_n) \, dx_1,$$
Integrating in the other \( n - 1 \) variables, we have
\[
\| T_{i, f} \|_{p, w} \leq C \| f \|_{p, w}.
\]  
(3.7)
The \( C \) here depends on \( p, w \) and the lacunary constant \( \alpha > 1 \) for the sequence defining the decomposition for the real line related to the \( x_i \)-variable. Similarly, (3.7) holds with \( T_i \) replaced by \( T_{i, i} \), \( i = 2, \ldots, n \), with a constant depending on the decomposition related to the \( x_i \)-variable.

Using (3.6) and applying (3.7) successively in each variable, we obtain \( \| T_{f} \|_{p, w} \leq C \| f \|_{p, w} \), with a constant independent of \( t \). Integrating in \( t \) and using the \( n \)-dimensional analog of (3.1), we get
\[
\| \gamma(f) \|_{p, w} = \left\| \left( \sum_{\Delta} |S_{\rho f}|^2 \right)^{1/2} \right\|_{p, w} \leq C \| f \|_{p, w}.
\]
The proof is now completed as in the case of Theorem 3.1.

Let \( \hat{f} \) denote the inverse Fourier transform of \( f \). We now prove Theorem 2.

Let \( f \in \mathcal{S}(\mathbb{R}^1) \) and define \( g \) by \( \hat{g}(x) = m(x)\hat{f}(x) \). Let \( \Delta \) be the dyadic decomposition of \( \mathbb{R}^1 \). For \( \rho \in \Delta \) and \( \xi \in \rho \), set \( \chi_{\rho, \xi}(x) = \chi(\{ x : x \in \rho \) and \( x < \xi \}) \) and define \( S_{\rho, f} \) by \( (S_{\rho, f})(x) = \chi_{\rho, \xi}(x)\hat{f}(x) \). Then
\[
(S_{\rho g})(x) = (S_{\rho g})^\ast(x) = \int_{\mathbb{R}^1} e^{2\pi i x \xi}(S_{\rho g})^\ast(x)\, d\xi
\]
\[
= \int_{\mathbb{R}^1} e^{2\pi i x \xi} \chi_{\rho}(\xi)\hat{g}(\xi)\, d\xi
\]
\[
= \int_{\rho} m(\xi)\hat{f}(\xi) e^{2\pi i x \xi} d\xi.
\]  
(3.8)

Suppose now that \( \rho = [2^k, 2^{k+1}] \) and set \( F(\xi) = \int_\xi^\infty \hat{f}(t)e^{2\pi i t \xi} \, dt \). Then, \( F'(\xi) = \hat{f}(\xi)e^{2\pi i x \xi} \) almost everywhere. Integrating by parts the right-hand side of (3.8) gives
\[
(S_{\rho g})(x) = m(\xi)F(\xi)\frac{2^{k+1}}{2} - \int_\rho F(\xi)\, dm(\xi)
\]
\[
= m(2^{k+1})(S_{\rho f})^\ast(x) - \int_\rho \chi_{\rho, \xi}(t)\hat{f}(t)e^{2\pi i t \xi} \, dt\, dm(\xi)
\]
\[
= m(2^{k+1})S_{\rho f}(x) - \int_\rho ((S_{\rho f})^\ast(x) \, dm(\xi)
\]
\[
= m(2^{k+1})S_{\rho f}(x) - \int_\rho (S_{\rho, \xi}f)(x) \, dm(\xi).
\]  
(3.9)
Therefore, by the condition on \( m \),
\[
\| (S_{\rho g})(x) \|^2 \leq \left( \int_\rho |(S_{\rho, \xi}f)(x)|^2 \, dm(\xi) + |(S_{\rho f})(x)|^2m(2^{k+1}) \right)
\]
\[\cdot \left( \int_\rho |dm(\xi)| + |m(2^{k+1})| \right)
\]
\[
\leq 2B\left( \int_\rho |(S_{\rho, \xi}f)(x)|^2 \, dm(\xi) + B|(S_{\rho f})(x)|^2 \right).
\]
From here, it follows that
\[
\int \gamma(g)^p(x)w(x) \, dx = \int \left( \sum_{\Delta} |(S_{\rho g}(x)|^2 \right)^{p/2} w(x) \, dx \\
\leq (2B)^{p/2} \int \left\{ \sum_{\Delta} \left( \int \rho \left|(S_{\rho \xi})(x)|^2 \right| dm(\xi) \right| + B \left|(S_{\rho f}(x)|^2 \right) \right\}^{p/2} w(x) \, dx \\
\leq (2B)^{p/2} C^p \int \left\{ \sum_{\Delta} |(S_{\rho f}(x)|^2 \left( \int \rho \left|dm(\xi)\right| + B \right) \right\}^{p/2} w(x) \, dx \\
\leq (2BC)^p \int \gamma(f)^p(x)w(x) \, dx,
\]

once we show that
\[
\int \left( \sum_{i=1}^N \int \rho_i (S_{\rho \xi})(x)|^2 \left|dm(\xi)\right| \right)^{p/2} w(x) \, dx \\
\leq C \int \left\{ \sum_{i=1}^N |(S_{\rho f})(x)|^2 \left( \int \rho_i \left|dm(\xi)\right| \right) \right\}^{p/2} w(x) \, dx. \tag{3.10}
\]

To conclude \( \|g\|_{p,w} \leq C \|\gamma(g)\|_{p,w} \) we need Lemma 3.7 and the version of Lemma 3.6 with \( \Delta \) the dyadic decomposition of \( \mathbb{R}^1 \). The proof for this special case depends only on Theorems 4 and 3.1. Thus, proving (3.10) will complete the proof of Theorem 2 since we know by Theorem 3.1 that \( \|\gamma(f)\|_{p,w} \) is equivalent to \( \|f\|_{p,w} \) when \( \Delta \) is the dyadic decomposition of \( \mathbb{R}^1 \).

Notice that for \( \xi \in \rho \),
\[
(S_{\rho \xi})(x) = \chi_{\rho \xi}(x)f(x) = \chi_{\rho \xi}(x)\chi_{\rho}(x)f(x) = (S_{\rho \xi}(S_{\rho f}))^*(x),
\]
so that \( S_{\rho \xi} \) is a partial sum of \( S_{\rho f} \). Now, divide each \( \rho_i \) of (3.10) into \( m \) equal parts by partitions \( \xi_j, j = 0, 1, \ldots, m, i = 1, \ldots, N \). By Theorem 2.1,
\[
\int \left\{ \sum_{i=1}^N \left( \sum_{j=1}^m \left|(S_{\rho \xi})(x)|^2 \int \rho_{\xi_j} \left|dm(t)\right| \right) \right\}^{p/2} w(x) \, dx \\
\leq C^p \int \left\{ \sum_{i=1}^N \left( \sum_{j=1}^m \left|(S_{\rho f})(x)|^2 \int \rho_{\xi_j} \left|dm(t)\right| \right) \right\}^{p/2} w(x) \, dx \\
\leq C^p \int \left\{ \sum_{i=1}^N \left|(S_{\rho f})(x)|^2 \left( \int \rho_i \left|dm(t)\right| \right) \right\}^{p/2} w(x) \, dx.
\]

Letting \( m \to \infty \) proves (3.10) and thus Theorem 2.

We note that the proof of Theorem 3 is the same as Theorem 2. We decompose \( S_{\rho g} \) into a sum of \( 2^n \) pieces each of which is handled as in Theorem 2.

The proof that inequality (1.4) implies \( w \in A_p \) when \( \Delta \) is the dyadic decomposition of \( \mathbb{R}^n \) is contained in the following theorem. The result is true in the case
where $\Delta$ is defined by the lacunary sequence $n_k = \alpha^k$ (or $n$ lacunary decompositions $n_k = (\alpha_j)^k$); we consider only the dyadic case for simplicity. We follow the proof of Theorem 8 in [8].

**Theorem 3.3.** Let $\Delta$ be the dyadic decomposition of $\mathbb{R}^n$ and $1 \leq p < \infty$. If there exists a $c$ such that

$$w\{x \in \mathbb{R}^n : |S_\rho f(x)| > \lambda\} \leq (c/\lambda^p)\|f\|_{p,w}^p$$

for all $\rho \in \Delta$, then $w \in A_p(\mathbb{R}^n, \mathcal{R}_n)$.

Fix a rectangle $R = I_1 \times \cdots \times I_n$ and let $f$ be positive on $R$ and 0 elsewhere. Let $l_j$ be the greatest integer such that $2^{l_j} < 1/4n|I_j|$ and let $\rho$ be the dyadic rectangle $[2^{l_1}, 2^{l_1+1}] \times \cdots \times [2^{l_n}, 2^{l_n+1}]$. Note

$$\hat{\chi}_\rho (x) = \prod_{j=1}^n \hat{\chi}_{[2^{l_j}, 2^{l_j+1}]}(x_j) = \prod_{j=1}^n \left\{ \frac{1}{2^{l_j}} \left( \frac{\sin 2^{l_j+1} x_j}{x_j} e^{-i 32^{l_j+1} x_j} \right) \right\}$$

$$= \left\{ \prod_{j=1}^n 2^{l_j} \left( \frac{\sin 2^{l_j+1} x_j}{2^{l_j} x_j} \right) \right\} \exp \left( -i \sum_{j=1}^n 32^{l_j+1} x_j \right).$$

Since $S_\rho f(x) = (\hat{\chi}_\rho * f)(x)$, for $x \in R$ we have

$$|S_\rho f(x)| = \left| \int_R \frac{\rho}{2^n} \prod_{j=1}^n \left( \frac{\sin 2^{l_j+1} (x_j - y_j)}{2^{l_j+1} (x_j - y_j)} \right) \exp \left( -i \sum_{j=1}^n 32^{l_j+1} (x_j - y_j) \right) f(y) \, dy \right|$$

$$\geq \left| \text{Re} \left[ \int_R \frac{\rho}{2^n} \prod_{j=1}^n \left( \frac{\sin 2^{l_j+1} (x_j - y_j)}{2^{l_j+1} (x_j - y_j)} \right) \exp \left( -i \sum_{j=1}^n 32^{l_j+1} (x_j - y_j) \right) f(y) \, dy \right] \right|$$

$$\geq \left| \int_R \frac{\rho}{2^n} \prod_{j=1}^n \left( \frac{\sin 2^{l_j+1} (x_j - y_j)}{2^{l_j+1} (x_j - y_j)} \right) \cos \left( \sum_{j=1}^n 32^{l_j+1} (x_j - y_j) \right) f(y) \, dy \right|. \quad (3.11)$$

By the definition of $\rho$ and the fact that $x, y \in R$,

$$\left| \sum_{j=1}^n 32^{l_j+1} (x_j - y_j) \right| \leq \sum_{j=1}^n 2^{2l_j} |I_j| \leq \sum_{j=1}^n \frac{1}{2n} = \frac{1}{2}.$$
since, by the definition of $k_j$,
\[
\frac{1}{|R|} = \prod_{j=1}^{n} \frac{1}{|I_j|} \leq (4n)^n \prod_{j=1}^{n} 2^{k_j + 1} = (8n)^n |\rho|.
\]

Thus, by the assumed inequality,
\[
\int_{R} w(x) \, dx \leq w\left(\left\{ x \in \mathbb{R}^n : |S\rho f(x)| > \frac{C}{|R|} \int_{R} f(y) \, dy \right\}\right)
\]
\[
\leq \frac{C}{\left(\frac{1}{|R|} \int_{R} f(y) \, dy\right)^p} \|f\|_{p,w}^p,
\]
which can be rewritten
\[
\left(\int_{R} w(x) \, dx\right)\left(\int_{R} f(y) \, dy\right)^p \leq C |R|^p \int_{R} |f(x)|^p w(x) \, dx. \tag{3.12}
\]

To see this implies $w \in A_p$, we will show
\[
\left(\frac{1}{|R|} \int_{R} w(x) \, dx\right)\left(\frac{1}{|R|} \int_{R} w(x)^{1/(p-1)} \, dx\right)^{p-1} \leq C, \tag{3.13}
\]
with the $C$ in (3.12). Set $A = \int_{R} w(x)^{-1/(p-1)} \, dx$. If $A = 0$, the left-hand side of (3.13) is 0. If $0 < A < \infty$, set $f(x) = w(x)^{-1/(p-1)}$. The right-hand side of (3.12) equals $C |R|^p A$. Dividing both sides of (3.12) by $|R|^p A$ yields (3.13). If $A = \infty$, $w^{-1/p} \notin L^p(R)$. Thus, there is a function $g \in L^p(R)$ such that $gw^{-1/p} \notin L^1(R)$. Let $f(x) = g(x)w(x)^{-1/p}$. From (3.12), we get $\int_{R} w(x) \, dx = 0$, so the left-hand side of (3.13) is 0.

To complete the proof of Theorem 1, we need to know that if $f \notin L^p_w(\mathbb{R}^n)$ then $\gamma(f) \notin L^p_w(\mathbb{R}^n)$, or equivalently, $\gamma(f) \in L^p_w(\mathbb{R}^n)$ implies $f \in L^p_w(\mathbb{R}^n)$. Since we need to be able to define the Fourier transform of $f$ in order to form $\gamma(f)$, we may assume that $f$ is at least in $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions. With this in mind, we will show that for $1 < p < \infty$ and $w \in A_p(\mathbb{R}^n, \mathcal{O}_n)$, $\gamma(f) \in L^p_w(\mathbb{R}^n)$ implies $f \in L^p_w(\mathbb{R}^n)$ and $\|f\|_{p,w} \leq C \|\gamma(f)\|_{p,w}$. The proof will be a consequence of a few lemmas. Let $C_c^\infty(\mathbb{R}^n)$ be the space of $C^\infty$ functions with compact support.

**Lemma 3.4.** Let $\mathcal{Z} = \{\phi \in C_c^\infty(\mathbb{R}^n) : 0 \notin \text{supp} \phi\}$. Then $\mathcal{Z}$ is dense in $L^2(\mathbb{R}^n)$.

Let $\beta \in C^\infty(\mathbb{R}^n)$ be such that $\beta(x) \equiv 1$ for $|x| > 1$ and $\beta(x) \equiv 0$ for $|x| < \frac{1}{2}$. Let $f \in L^2(\mathbb{R}^n)$ and $r_j \in C_c^\infty(\mathbb{R}^n)$ such that $r_j$ converges to $f$ in $L^2$. Set $\phi_j(x) = \beta(jx)r_j(x)$, so $\phi_j \in C_c^\infty(\mathbb{R}^n)$ and $\phi_j \equiv 0$ for $|x| < 1/(2j)$, hence $\phi_j \in \mathcal{Z}$. If $C = \sup_{x \in \mathbb{R}^n} |\beta(x)|$, then
\[ \| f - \phi_j \|_2^2 = \int_{\mathbb{R}^n} |f(x) - \phi_j(x)|^2 \, dx \]

\[ \leq \int_{|x| > 1/j} |f(x) - r_j(x)|^2 \, dx + 2 \int_{|x| < 1/j} |f(x)|^2 \, dx \]

\[ + 2 \int_{|x| < 1/j} |\beta(jx) r_j(x)|^2 \, dx \]

\[ \leq \| f - r_j \|_2^2 + 2 \int_{|x| < 1/j} |f(x)|^2 \, dx \]

\[ + 2 C^2 \left( \int_{|x| < 1/j} |r_j(x) - f(x)|^2 \, dx + \int_{|x| < 1/j} |f(x)|^2 \, dx \right) \]

\[ \leq (2 C^2 + 2) \left( \| f - r_j \|_2^2 + \int_{|x| < 1/j} |f(x)|^2 \, dx \right) . \]

Since \[ r_j \to f \text{ in } L^2 \text{ and } \int_{|x| < 1/j} |f(x)|^2 \, dx \to 0 \text{ as } j \to \infty , \| f - \phi_j \|_2 \to 0. \]

From Lemma 3.4 we get

**Corollary 3.5.** The set of \( \phi \in \mathcal{S}(\mathbb{R}^n) \) such that \( \hat{\phi} \in \mathcal{S} \) is dense in \( L^2(\mathbb{R}^n) \).

Now, let \( C = \{ x \in \mathbb{R}^n : 1 \leq x_i \leq 2 , i = 1, 2, \ldots, n \} \) and \( \phi \in C_c^\infty(\mathbb{R}^n) \) be identically 1 on \( C \) and supported in \( \{ x \in \mathbb{R}^n : \frac{1}{4} \leq x_i \leq 4 , i = 1, 2, \ldots, n \} \). For a dyadic interval, \( R_k \), let \( \phi_k \) be the function \( \phi \) adjusted to \( R_k \), as in proof of Theorem 3.1. The \( \phi_k \)'s have bounded overlaps; in fact, \( 1 < \sum \phi_k(x) < 2^n + 1 \). Therefore, if we set \( \psi_k = \phi_k / \sum \phi_k \), then \( \psi_k \in C_c^\infty(\mathbb{R}^n) \) and \( \sum \psi_k \equiv 1 \). Let \( T_k \) be the operator with multiplier \( \psi_k \).

**Lemma 3.6.** Let \( 1 < p < \infty , w \in A_p(\mathbb{R}^n, \mathcal{O}_w) \), and \( \gamma(f) \in L_p^w(\mathbb{R}^n) \). Then

\[ \left\| \sum T_k f \right\|_{p,w} \leq C \| \gamma(f) \|_{p,w} . \]

Since \( \gamma(f) \in L_p^w(\mathbb{R}^n) \), \( \sum_{\Delta} S_{\rho} f \in L_p^w(\mathbb{R}^n) \) and \( \| \sum_{\Delta} S_{\rho} f \|_{p,w} \leq C \| \gamma(f) \|_{p,w} \). Let \( T = \sum T_k \), so \( T \) is the operator with multiplier \( \sum \psi_k = 1 \). By Theorem 4, \( T \) is a bounded multiplier operator on \( L_p^w(\mathbb{R}^n) \), so that

\[ \left\| \sum T_k \left( \sum_{\Delta} S_{\rho} f \right) \right\|_{p,w} \leq C \left\| \sum_{\Delta} S_{\rho} f \right\|_{p,w} \leq C \| \gamma(f) \|_{p,w} . \]

The proof is complete once we know \( T_k f = T_k(\sum_{\Delta} S_{\rho} f) \). Note that for \( M \) large enough, \( \text{supp } \psi_k \subset \bigcup_{\rho \in \Delta_M} \rho \) (\( \Delta_M \) as in the proof of Theorem 3.2). Then, for all \( N > M \),

\[ (T_k f)^* = \psi_k \hat{f} = \psi_k \sum_{\rho \in \Delta_N} \chi_{\rho_N} \hat{f} = \psi_k \left( \sum_{\rho \in \Delta_N} S_{\rho} f \right)^* = \left( T_k \left( \sum_{\rho \in \Delta_N} S_{\rho} f \right) \right)^* , \]

so that \( T_k f = T_k(\sum_{\rho \in \Delta_N} S_{\rho} f) \). But \( T_k \) is a bounded multiplier operator on \( L_p^w(\mathbb{R}^n) \) and \( \sum_{\rho \in \Delta_N} S_{\rho} f \) converges to \( \sum_{\rho \in \Delta} S_{\rho} f \) in \( L_p^w \). Therefore, since \( T_k(\sum_{\rho \in \Delta_N} S_{\rho} f) \) is constant for \( N > M \),

\[ T_k \left( \sum_{\rho \in \Delta} S_{\rho} f \right) = T_k \left( \sum_{\rho \in \Delta_N} S_{\rho} f \right) = T_k f . \]
Lemma 3.7. Under the hypothesis of Lemma 3.6, \( \|f\|_{p,w} \leq \|\Sigma T_k f\|_{p,w} \).

Since \( f \in S'(\mathbb{R}^n) \), there are \( g_j \in C_c^\infty(\mathbb{R}^n) \) such that \( g_j \) converges to \( f \) in \( S' \). Let \( \phi \in S(\mathbb{R}^n) \) be such that \( \hat{\phi} \in \mathcal{F} \). Then

\[
(g_j, \phi) = (\hat{g}_j, \hat{\phi}) = \int \hat{g}_j \hat{\phi} = \int \left( \sum_k \psi_k \right) \hat{g}_j \hat{\phi} = \sum_k \int \hat{g}_j (\psi_k \hat{\phi}) = \sum_k (\hat{g}_j, \psi_k \hat{\phi}).
\]

Because \( \psi_k \hat{\phi} \in S(\mathbb{R}^n) \) and the Fourier transform is continuous on \( S' \), letting \( j \to \infty \), we get

\[
(f, \phi) = \sum_k (\hat{f}, \psi_k \hat{\phi}) = \sum_k (\psi_k \hat{f}, \hat{\phi}) = \sum_k ((T_k f)^*, \hat{\phi}) = \sum_k (T_k f, \phi).
\]

This is actually a finite sum since \( \hat{\phi} \in \mathcal{F} \) implies \( \psi_k \hat{\phi} \equiv 0 \) and consequently \( (T_k f, \phi) = 0 \) for almost every \( k \). Therefore, \( \Sigma(T_k f, \phi) = (\Sigma T_k f, \phi) \) and

\[
|(f, \phi)| = \left| \left( \sum T_k f, \phi \right) \right| < \| \sum T_k f \|_{p,w} \| \phi \|_{p',w'^{-1}(p^{-1})}.
\]

Since \( \{ \phi \in S(\mathbb{R}^n) : \phi \in \mathcal{F} \} \) is dense in \( L^2 \), taking the sup over such \( \phi \) with \( \| \phi \|_{p',w'^{-1}(p^{-1})} = 1 \), we get \( \|f\|_{p,w} < \|\Sigma T_k f\|_{p,w} \).

4. Applications. We now consider applications of Theorem 1. In particular, we will generalize Theorem 6 of Stein [19] and Theorems 3 and 4 of Riviere and Sager [16].

Let \( \{f_k\} \) be a sequence of functions defined on \( \mathbb{R}^n \). By \( \Sigma f_k \in L_p^c(\mathbb{R}^n) \) we mean the partial sums \( \Sigma f_k \) converge in \( L_p^c(\mathbb{R}^n) \).

Theorem 4.1. Let \( 1 < p < \infty \), \( w \in A_p(\mathbb{R}^n, \mathcal{R}_n) \), and \( \{S_k\} \) be any collection of lacunary partial sums. Then, \( (\Sigma_k |S_k f|^2)^{1/2} \in L_p^c(\mathbb{R}^n) \) implies \( \Sigma_k \epsilon_k S_k f \in L_p^c(\mathbb{R}^n) \) for all \( \{\epsilon_k\} \in l^\infty \). Moreover, there is a constant, \( c \), independent of \( f \) and \( \{\epsilon_k\} \), such that

\[
\left\| \sum_k \epsilon_k S_k f \right\|_{p,w} < c \|\{\epsilon_k\}\|_{l^\infty} \left( \sum_k |S_k f(x)|^2 \right)^{1/2} \|_{p,w}.
\]

Notice that since \( (\Sigma_k |S_k f|^2)^{1/2} \in L_p^c(\mathbb{R}^n) \) and

\[
\left( \sum_k |\epsilon_k S_k f(x)|^2 \right)^{1/2} < \|\epsilon_k\|_{l^\infty} \left( \sum_k |S_k f(x)|^2 \right)^{1/2},
\]

\( (\Sigma_k |\epsilon_k S_k f|^2)^{1/2} \in L_w^c(\mathbb{R}^n) \). Thus, we may assume \( \epsilon_k = 1 \) for all \( k \). Using Theorem 1, the proof is the same as for inequality (3.5).

The converse of Theorem 4.1 is more general and easier to prove. We have

Theorem 4.2. Let \( p > 0 \) and \( w > 0 \). Let \( \{f_k\} \) be any collection of functions and assume that \( \Sigma \epsilon_k f_k \in L_p^c(\mathbb{R}^n) \) for all \( \{\epsilon_k\} \in l^\infty \). Then \( (\Sigma_k |f_k|^2)^{1/2} \in L_p^c(\mathbb{R}^n) \) and there exists a constant, \( c \), independent of \( \{f_k\} \) such that

\[
\left\| \left( \sum_k |f_k|^2 \right)^{1/2} \right\|_{p,w} < c \sup_{\|\{\epsilon_k\}\|_{l^\infty} = 1} \left\| \sum_k \epsilon_k f_k \right\|_{p,w}.
\]
It is enough to show that there exists a constant $c$, depending on $\{f_k\}$, such that
\[ \left\| \sum_k \epsilon_k f_k \right\|_{p,w} < c \left\| \{ \epsilon_k \} \right\|_\infty \tag{4.1} \]
for all $\{\epsilon_k\} \in l^\infty$. For, if (4.1) is valid, it follows that $M = \sup_{\|\{\epsilon_k\}\|_\infty = 1} \|\sum_k \epsilon_k f_k\|_{p,w}$ is finite. Let $\epsilon_k = r_k(t)$ for $0 < t < 1$. Then $\|\{\epsilon_k\}\|_\infty = 1$ and
\[ M^p > \int_{\mathbb{R}^n} \left| \sum_k r_k(t)f_k(x) \right|^p w(x) \, dx. \]
Integrating in $t$, from 0 to 1, and using (3.1), we have
\[ M^p > \int_0^1 \int_{\mathbb{R}^n} \left| \sum_k r_k(t)f_k(x) \right|^p w(x) \, dx \, dt \]
\[ = \int_{\mathbb{R}^n} \left( \int_0^1 \left| \sum_k r_k(t)f_k(x) \right|^p dt \right) w(x) \, dx \]
\[ > c \int_{\mathbb{R}^n} \left( \sum_k |f_k(x)|^p \right)^{p/2} w(x) \, dx. \]

In order to prove (4.1), let $p > 1$ and consider the collection of maps $\{H_N: l^\infty \to L^p_w(\mathbb{R}^n)\}$ defined by $H_N((\epsilon_k)) = \sum_{k=1}^N \epsilon_k f_k$. Let $H = H_\infty$. Since $H_N((\epsilon_k))$ is a finite sum, each $H_N$ is continuous and by assumption $H_N((\epsilon_k))$ converges to $H((\epsilon_k))$ in $L^p_w(\mathbb{R}^n)$, for each $\{\epsilon_k\} \in l^\infty$. Therefore, $\|H_N((\epsilon_k))\|_{p,w}^\infty$ is bounded for each $\{\epsilon_k\} \in l^\infty$. By the Principle of Uniform Boundedness, there exists a constant, $c > 0$, such that $\|H_N\| < c$ for all $N$. It follows that $\|H\| < c$.

For $0 < p < 1$, the proof is the same, using the extension of the Principle of Uniform Boundedness to quasinormed spaces (see [23]).

Let $\{S_k\}$ be any collection of lacunary partial sums and set $f_k = S_k f$. Then, combining Theorems 1, 4.1, and 4.2, we obtain

**Theorem 4.3.** Let $1 < p < \infty$, $w \in A_p(\mathbb{R}^n, \mathcal{R}_n)$, and $\{S_k\}$ be any collection of lacunary partial sum operators. Then $f \in L^p_w(\mathbb{R}^n)$ if and only if $\sum_k \epsilon_k S_k f$ converges in $L^p_w(\mathbb{R}^n)$ for any sequence $\{\epsilon_k\} \in l^\infty$. Moreover, $\|f\|_{p,w}$ is equivalent to $\sup_{\|\{\epsilon_k\}\|_\infty = 1} \|\sum_k \epsilon_k S_k f\|_{p,w}$.

Let $f$ be a measurable function and $\lambda(s) = m(\{x \in \mathbb{R}^n: |f(x)| > s\})$ be the distribution function of $f$ (with respect to Lebesgue measure). We define the nonincreasing rearrangement $f_\rho$ of $f$ by $f_\rho(t) = \inf_{\lambda(s) < t} s$ for $t > 0$. Next, set
\[ \|f\|_{p,q}^* = \left( \int_0^\infty \left[ t^{1/p} f_\rho(t) \right]^q \frac{dt}{t} \right)^{1/q} \]
if $1 < p < \infty$, and
\[ \|f\|_{p,q}^* = \sup_{t > 0} t^{1/p} f_\rho(t) \]
if $1 < p < \infty$ and $q = \infty$. We then define the space $L^{p,q}$ as $\{f: \|f\|_{p,q}^* < \infty\}$. We note in passing that $L^{p,p} = L^p$ and for $q_2 < q_1$,
\[ \|f\|_{p,q_1}^* < \|f\|_{p,q_2}^*. \tag{4.2} \]
For details, see [7].
Let $\Delta = \{\rho\}$ be a lacunary decomposition of $\mathbb{R}^n$. Define
\[ \|f\|_{l^\infty(L^r,\infty)} = \sup_{\rho \in \Delta} \|f x_\rho\|_{r,\infty}^* \]
and $l^\infty(L^r,\infty) = l^\infty(L^r,\infty, \Delta) = \{f: \|f\|_{l^\infty(L^r,\infty)} < \infty\}$. The generalizations of the results of Riviere and Sagher [16] are the following two theorems.

**Theorem 4.4.** Let $1 < p < \infty$ and $\Delta = \{\rho\}$ be a lacunary decomposition of $\mathbb{R}^n$.
(i) If $1 < p < 2$, $0 < \alpha < 1/p'$, and $f \in L_{p,\infty}^p(\mathbb{R}^n)$, then
\[ \left( \sum_\Delta \|(S_{\rho} f)^\vee(x)|x|^{-\alpha} \|_{p,p'}^{2} \right)^{1/2} \leq c \|f\|_{p,\infty} \].
(ii) If $2 < p < \infty$, $0 < \alpha < 1/p$, and $\sum_\Delta \|(S_{\rho} f)^\vee(x)|x|^\alpha \|_{p,p'}^{2} < \infty$, then
\[ \|f\|_{p,\infty} \leq c \left( \sum_\Delta \|(S_{\rho} f)^\vee(x)|x|^\alpha \|_{p,p'}^{2} \right)^{1/2} \].

**Theorem 4.5.** Let $1 < p < 2 < q < \infty$, $1/r = 1/p - 1/q$, $0 < \alpha < 1/q$, and $0 < \beta < 1/p'$. Given a bounded function $m(x)$, let $Tf$ be the multiplier operator defined by $(Tf)^\vee(x) = m(x)\hat{f}(x)$. If $m(x)|x|^\alpha + \beta \in l^\infty(L^r,\infty)$, then $T$ is a bounded operator from $L_p^p(\mathbb{R}^n)$ to $L_q^q(\mathbb{R}^n)$.

Two results are needed to prove Theorems 4.4 and 4.5. We will use Theorem 1 and the following versions of Pitt’s Theorem (see [5], [14], and [20]):
(i) if $1 < p < 2$, and $0 < \alpha < 1/p'$, then
\[ \left( \int_{\mathbb{R}^n} |\hat{f}(x)|^p |x|^{-\alpha p/n} dx \right)^{1/p} \leq c(n, p, \alpha) \left( \int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha dx \right)^{1/p} \] \hspace{1cm} (4.3)
(ii) if $1 < p < \infty$, $0 < \alpha < 1/p'$, and $\lambda = 2/p + \alpha - 1 > 0$, then
\[ \left( \int_{\mathbb{R}^n} |\hat{f}(x)|^p |x|^{-\lambda p/n} dx \right)^{1/p} \leq c(n, p, \alpha) \left( \int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha dx \right)^{1/p} \] \hspace{1cm} (4.4)

These inequalities are also true with the roles of $f$ and $\hat{f}$ reversed.

Riviere and Sagher were interested in the unweighted version of Theorem 4.4 in order to find a unified proof of Paley’s Theorem and a generalization of the Hausdorff-Young Theorem, known as Kellog’s Theorem:
\[ \left( \sum_\kappa \left( \sum_{n \in B_\kappa} |\hat{f}(n)|^p \right)^{2/p'} \right)^{1/2} \leq C_p \|f\|_p \]
for $1 < p < 2$, where $\{B_\kappa\}$ is the dyadic decomposition of the integers. When $p > 2$, the inequality sign is reversed.

Notice, first, that (4.4) is already a weighted version of Paley’s Theorem—if $1 < p < 2$, say, we have
\[ \int_{\mathbb{R}^n} |\hat{f}(x)|^p |x|^{2-p} |x|^{-\alpha p/n} dx \leq c \int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha dx \].
As for Kellog's Theorem, if we use (4.2) with (i) and (ii) of Theorem 4.4, we obtain
\[
\left( \sum_{\Delta} \| (S_{\rho f})^\gamma(x)|x|^{-\alpha}\|_p^2 \right)^{1/2} \leq c \| f(x)|x|^\alpha \|_p, \quad 1 < p < 2, \tag{4.5}
\]
and
\[
\| f(x)|x|^{-\alpha}\|_p \leq c \left( \sum_{\Delta} \| (S_{\rho f})^\gamma(x)|x|^\alpha\|_p^2 \right)^{1/2}, \quad 2 < p < \infty. \tag{4.6}
\]
Writing these at length, we see that (4.5) and (4.6) are weighted, \(n\)-dimensional analogs of Kellog's Theorem; e.g., (4.5) becomes
\[
\left( \sum_{\rho \in \Delta} \left( \int_{\rho} |\hat{f}(x)|^p |x|^{-\alpha p'} \, dx \right)^{2/p'} \right)^{1/2} \leq c \| f(x)|x|^\alpha \|_{\rho \times \rho'.}
\]
In order to prove Theorem 4.4, fix a \(p\) and \(\alpha\) satisfying the conditions of (i) and choose an \(r\) for which \(1 < r < p\) and \(\alpha < 1/r'\). By (4.3),
\[
\| \hat{f}(x)|x|^{-\alpha}\|_{p', r} \leq c \| f(x)|x|^\alpha\|_{p, r}
\]
and
\[
\| \hat{f}(x)|x|^{-\alpha}\|_{p', 2} \leq c \| f(x)|x|^\alpha\|_{p, 2}.
\]
By the interpolation theorem for \(L^{p,q}\) spaces (see [7]), these imply
\[
\| \hat{f}(x)|x|^{-\alpha}\|_{p', q} \leq c \| f(x)|x|^\alpha\|_{p, q}, \tag{4.7}
\]
for \(1 < q < \infty\).
Since \(0 < \alpha < 1/p'\), \(0 < \alpha p < p - 1\) so that \(|x|^{\alpha p} \in A_p(\mathbb{R}^n, \mathbb{R}_n)\). Therefore, by (4.7), Minkowski's inequality and Theorem 1,
\[
\left( \sum_{\Delta} \| (S_{\rho f})^\gamma(x)|x|^{-\alpha}\|_{p', p}^2 \right)^{1/2} \leq c \left( \sum_{\Delta} \| S_{\rho f}(x)|x|^{\alpha}\|_{p', p}^2 \right)^{1/2} = c \left( \sum_{\Delta} \| S_{\rho f}(x)|x|^{\alpha}\|_{p}^2 \right)^{1/2} \leq c \| \gamma(f)|x|^\alpha \|_p \leq c \| f(x)|x|^\alpha \|_{p \times \rho' \times \rho}. \]
To prove (ii), we proceed as before, only now we use the version of (4.3) with the roles of \(f\) and \(\hat{f}\) interchanged, obtaining (note \(p > 2\))
\[
\| f(x)|x|^{-\alpha}\|_{p, q} \leq c \| \hat{f}(x)|x|^\alpha\|_{p', q}, \quad 1 < q < \infty. \tag{4.8}
\]
Now \(0 < \alpha < 1/p\), so that \(-1 < -\alpha p < 0\) and \(|x|^{-\alpha p} \in A_p(\mathbb{R}^n, \mathbb{R}_n)\). Using Theorem 1, Minkowski's inequality and (4.8),
\[
\| f(x)|x|^{-\alpha}\|_p \leq c \left( \sum_{\Delta} \| S_{\rho f}(x)|x|^\alpha\|_p^2 \right)^{1/2} \leq c \left( \sum_{\Delta} \| S_{\rho f}(x)|x|^{-\alpha}\|_p^2 \right)^{1/2} \leq c \left( \sum_{\Delta} \| (S_{\rho f})^\gamma(x)|x|^\alpha\|_{p', p}^2 \right)^{1/2}.
\]
This completes the proof of Theorem 4.4.

Theorem 4.5 follows from Theorem 4.4 and Hölder's inequality for \(L^{p,q}\) spaces.
\[ ||Tf||_{q,|x|^{-\alpha}} = ||Tf(x)|x|^{-\alpha}||_q = ||Tf(x)|x|^{-\alpha}||^{*}_{q,q} \]
\[ \leq c\left( \sum_{\Delta} \|(S_{\rho}(Tf))^\ast(x)|x|^\alpha||^{*}_{q',q} \right)^{1/2} \]
\[ = c\left( \sum_{\Delta} \|X_{\rho}(x)m(x)f(x)|x|^\alpha||^{*}_{q',q} \right)^{1/2} \]
\[ \leq c\left( \sum_{\Delta} \left\{ \|m(x)|x|^\alpha + \beta X_{\rho}(x) \right\} \left\{ X_{\rho}(x)f(x)|x|^{-\beta} \right\}||^{*}_{q',q} \right)^{1/2} \]
\[ \leq c\left( \sum_{\Delta} \left\{ \|m(x)|x|^\alpha + \beta X_{\rho}(x) \right\} \left\{ X_{\rho}(x)f(x)|x|^{-\beta} \right\} \right)^{1/2} \]
\[ \leq c\left( \sum_{\Delta} \left\{ \|m(x)|x|^\alpha + \beta \right\} \left\{ X_{\rho}(x)f(x)|x|^{-\beta} \right\} \right)^{1/2} \]
\[ \leq c\left( \sum_{\Delta} \left\{ \|m(x)|x|^\alpha + \beta \right\} \left\{ X_{\rho}(x)f(x)|x|^{-\beta} \right\} \right)^{1/2} \]

The main steps in the previous proof are to apply a variant of Pitt's Theorem, Hölder's inequality for $L^{p,q}$ spaces, and another variant of Pitt's Theorem. If we use this procedure in the context of $L^p$ spaces, we can prove

**Theorem 4.6.** Let $1 < p, q < \infty$. Given a bounded $m(x)$, define $Tf(x)$ by $(Tf)^\ast(x) = m(x)f(x)$. If

(i) $1 < s < q, p < t < \infty$, and $1/r = 1/s - 1/t > 0$,
(ii) max(0, $1/s - 1/q'$) $\leq \alpha < \min(1/q, 1/q + 1/s - 1/q')$,
(iii) max(0, $1/p' - 1/t$) $\leq \beta < \min(1/p', 1/t),$
(iv) $m(x)|x|^{|a+b|+1/p+1/t-1/s-1/q|n} \in L^\infty(\mathbb{R}^n)$,

then $T$ is a bounded operator from $L^p_{|x|^{\alpha+n}}(\mathbb{R}^n)$ to $L^q_{|x|^{-\alpha}}(\mathbb{R}^n)$. Moreover, if $s = q > 2$, we may take $\alpha < 1/q$; if $t = p > 2$, we may take $\beta < 1/p'$.

Taking $s = q'$ and $t = p'$, we get Theorem 5. In 1-dimension, Theorem 4.5 is clearly better than Theorem 5 because $L^\infty(L^r) \subset L'$. However, since Theorem 5 allows for a greater range of powers of $|x|$ for $n > 1$, in higher dimensions the two overlap. Finally, setting $p = q = s = t$, and noting the remark at the end of Theorem 4.6, we get

**Theorem 4.7.** Let $1 < p < \infty$, max(0, $(2 - p)/p) < \alpha < 1/p$, and
max(0, $(p - 2)/p) < \beta < 1/p'$.

Let $m(x)$ be bounded and $T$ the multiplier operator defined by $m$. If $m(x)|x|^{|a+b|n} \in L^\infty(\mathbb{R}^n)$, then $T$ is a bounded operator from $L^p_{|x|^{\alpha+n}}(\mathbb{R}^n)$ to $L^p_{|x|^{-\alpha}}(\mathbb{R}^n)$.

**Bibliography**


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