RESULTS ON WEIGHTED NORM INEQUALITIES FOR MULTIPLIERS

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Abstract. Weighted $L^p$-norm inequalities are derived for multiplier operators on Euclidean space. The multipliers are assumed to satisfy conditions of the Hörmander-Mikhlin type, and the weight functions are generally required to satisfy conditions more restrictive than $A_p$ which depend on the degree of differentiability of the multiplier. For weights which are powers of $|x|$, sharp results are obtained which indicate such restrictions are necessary. The method of proof is based on the function $f^s$ of C. Fefferman and E. Stein rather than on Littlewood-Paley theory. The method also yields results for singular integral operators.

1. Let $m(x)$ be a bounded function on $\mathbb{R}^n$ and consider the multiplier operator $Tf$ defined initially for functions $f$ in the Schwartz space $\mathcal{S}$ by $(Tf^*)(x) = m(x)\hat{f}(x)$, where $\hat{g}$ is the Fourier transform of $g$. Denote by $s$ a real number greater than or equal to 1, $l$ a positive integer, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index of nonnegative integers $\alpha_j$ with length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We say $m \in M(s,l)$ if

$$\sup_{R > 0} \left( R^{|\alpha|-n} \int_{R < |x| < 2R} |D^\alpha m(x)|^s \, dx \right)^{1/s} < +\infty \quad \text{for all } |\alpha| < l. \quad (1.1)$$

The condition (1.1) has been known to be related to multiplier theorems for some time. The classic works in this direction are the theorems of Marcinkiewicz (see [18]) and Hörmander-Mikhlin (see [7]):

Theorem A. Let $n = 1, 1 < p < \infty$, and $m \in M(1,l)$. Then there exists a constant $C$, independent of $f$, such that $\|Tf\|_p \leq C\|f\|_p$.

Theorem B. Let $l > n/2, 1 < p < \infty$, and $m \in M(2,l)$. Then there exists a constant $C$, independent of $f$, such that $\|Tf\|_p \leq C\|f\|_p$.

Much work has been done to extend these results. Using interpolation methods, Calderón and Torchinsky [2] have considered the condition $m \in M(s,l)$ for $s > 2$ and $l > n/s$. Hirschman [6], Krée [11], and Triebel [20] have extended these results in various directions to weighted $L^p$ spaces for weights which are powers of $|x|$. More recently, Kurtz [12] extended Theo-

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rems A and B to $L^p$ spaces with more general weights by using the weighted norm inequalities derived in [15] for the function $g^*_\Lambda$.

The purpose of this paper is two-fold. We consider $s < 2$ and present a method of proof based on the function $f^*$ of Fefferman and Stein [5] rather than on Littlewood-Paley theory.

We say $f \in L^p_w(\mathbb{R}^n), 1 < p < \infty$ and $w(x) > 0$, if

$$\|f\|_{p,w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < +\infty.$$  

The weights $w$ we will consider satisfy an $A_r$ condition; i.e., $w \in A_r$ if there is a constant $C$ such that

$$\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(r-1)} \, dx \right)^{r-1} \leq C, \quad 1 < r < \infty,$$

$$\frac{1}{|Q|} \int_Q w(x) \, dx \leq C \, \text{ess inf}_Q w, \quad r = 1,$$

for all cubes $Q \subset \mathbb{R}^n$. When $r = 1$, the condition that $w \in A_1$ means $w^*(x) \leq Cw(x)$ for almost every $x$, where $g^*$ is the Hardy-Littlewood maximal function of $g$. Finally, $w \in A_\infty$ if there exist positive constants $C$ and $\delta$ such that for any cube $Q \subset \mathbb{R}^n$ and for any measurable set $E \subset Q$,

$$\frac{m_w(E)}{m_w(Q)} \leq C \left( \frac{|E|}{|Q|} \right)^\delta,$$

where $m_w(E) = \int_E w(x) \, dx$. Results concerning $A_p$ functions can be found in Muckenhoupt [13] and Coifman and Fefferman [3]. Note, in particular, that $w \in A_p$ implies $w \in A_\infty$.

We use $p'$ to denote the index conjugate to $p$: $1/p + 1/p' = 1, p > 1$.

The main result of this paper is:

**Theorem 1.** Let $1 < s < 2, n/s < l \leq n$, and $m \in M(s, l)$. If

1. $n/l < p < \infty$ and $w \in A_{pl/n}$, or
2. $1 < p < (n/l)'$ and $w^{-1/(p-1)} \in A_{p'/l/n}$,

then there is a constant $C$, independent of $f$, such that

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w}.$$  

When $l < n$, we may take $p = n/l$ in (1) and $p = (n/l)'$ in (2). If

3. $w^{n/l} \in A_1$,

there is a constant $C$, independent of $f$ and $\lambda$, such that

$$m_w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{1,w}, \quad \lambda > 0.$$

Using interpolation, other conditions on the weight can be found which guarantee that $T$ is a bounded operator. One result which we will prove is:
THEOREM 2. If \( 1 < p < \infty, \ 1 < s < 2, \ n/s < l < n, \ m \in M(s, l), \) and \( w^{n/l} \in A_p \) then
\[
\| Tf \|_{p,w} \leq C \| f \|_{p,w}
\]
for a constant independent of \( f \).

This result does not give the best possible condition on the weight. When \( w(x) = |x|^\beta \), we have \( w \in A_p \) if \(-n < \beta < n(p - 1)\). Interpreting Theorem 1 for such \( w \) and using interpolation with change of measures, we will show:

THEOREM 3. Let \( 1 < s < 2, \ n/s < l < n, \) and \( m \in M(s, l) \). If \( 1 < p < \infty \) and \( \max\{-n, -lp\} < \beta < \min\{n(p-1), lp\} \), then there is a constant \( C \), independent of \( f \), such that
\[
\| Tf \|_{p,|x|^\beta} \leq C \| f \|_{p,|x|^\beta}.
\]
In particular, if \( n/l < p < (n/l)' \), we get \(-n < \beta < n(p-1)\); we may also take \( p = n/l \) and \( p = (n/l)' \) if \( l < n \).

We will show that this result is sharp with the possible exception of the endpoint values of \( \beta \).

Let \( \hat{g} \) denote the inverse Fourier transform of \( g \). If we set \( K = \hat{m} \), then for \( f \in \mathcal{S}, \ Tf(x) = (K * f)(x) \). Our proof of Theorem 1 is based on using information about \( m \) to get estimates on approximations to \( K \), so it is not surprising that the technique carries over to convolution operators.

Denote by \( \Sigma = \Sigma_{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \}, \ x' = x/|x| \in \Sigma (x \neq 0) \), and \( \rho \) any rotation of \( \Sigma \) with magnitude \( |\rho| = \sup_{x \in \Sigma} |\rho x - x| \). Let \( 1 < r < \infty \) and \( \Omega \in L^r(\Sigma) \) be positively homogeneous of degree zero. We say that \( \Omega \) satisfies the \( L^r \)-Dini condition if
\[
\int_0^1 \omega_r(\delta) \frac{d\delta}{\delta} < +\infty,
\]
where
\[
\omega_r(\delta) = \sup_{|\rho| < \delta} \left( \int_\Sigma |\Omega(\rho x) - \Omega(x)|^r \, d\sigma_x \right)^{1/r}.
\]
Set \( K(x) = \Omega(x')/|x|^n \), with \( \int_\Sigma \Omega(x) \, d\sigma_x = 0 \), and \( Tf(x) = (K * f)(x) \) in the usual principal-value sense. If \( \Omega \) satisfies the \( L^r \)-Dini condition then it also satisfies the \( L^1 \)-Dini condition, which by [1] implies \( T \) is a bounded operator on \( L^p \), \( 1 < p < \infty \). Recently, Kaneko and Yano [10] have shown that if \( \Omega \) satisfies the \( L^\infty \)-Dini condition then \( T \) maps \( L^p \) into itself for \( 1 < p < \infty \) and \( w \in A_p \). We have extended this to:

THEOREM 4. Let \( 1 < r < \infty, \ \Omega \in L^r(\Sigma), \) and \( \int_\Sigma \Omega(x) \, d\sigma_x = 0 \). Suppose \( \Omega \) satisfies the \( L^r \)-Dini condition. If
\( r' < p < \infty \) and \( w \in A_{p/r}' \), or
(2) \(1 < p < r\) and \(w^{-1/(p-1)} \in A_{p'/r'}\), then there is a constant \(C\), independent of \(f\), such that

\[ \|Tf\|_{p,w} \leq C\|f\|_{p,w}. \]

When \(r < \infty\), we may take \(p = r'\) in (1) and \(p = r\) in (2). If \(w' \in A_1\), then

\[ m_w(\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{1,w}, \quad \lambda > 0, \]

where \(C\) is independent of \(f\) and \(\lambda\).

Theorem 4 is a direct analogue of Theorem 1. (We could also have stated a version of Theorem 3. See also [14]). In fact, when \(r > 2\), \(r'\) plays the same role as \(n/l\). For example, notice the similarity between \(m \in M(s, n), 1 < s < 2\), and \(\Omega\) satisfying the \(L^\infty\)-Dini condition. Our technique, however, does not allow for either \(r\) or \(s\) to be equal to 1.

§2 contains the basic lemma and a collection of results used in the proof of Theorem 1. This theorem and Theorems 2 and 3 are proved in §3. The proof of Theorem 4 is found in §4. The paper concludes with a counterexample showing Theorem 3 is best possible except for the question of endpoint equalities for \(\beta\). The basic lemma and the counterexample are generalizations to \(n > 1\) of results in [16], and we gratefully acknowledge many helpful discussions with W.-S. Young and B. Muckenhoupt.

2. Following [7], we select an approximation to the identity

\[ \sum_{j=-\infty}^{+\infty} \phi(2^{-j}x) = 1, \quad x \neq 0, \]

where \(\phi\) is an infinitely differentiable, nonnegative function supported in \(\frac{1}{2} < |x| < 2\). Let \(m_j(x) = m(x)\phi(2^{-j}x)\), so that

\[ m(x) = \sum_{j=-\infty}^{+\infty} m_j(x), \quad x \neq 0. \]

Notice that \(m_j(x)\) is supported in \(2^{j-1} < |x| < 2^{j+1}\) and that for such \(x\), \(m_k(x) = 0\) unless \(k = j - 1, j,\) or \(j + 1\). It follows easily that if \(m \in M(s, l)\) and \(|\alpha| < l\), then

\[ \left(\int_{\mathbb{R}^n} |D^\alpha m_j(x)|^s \, dx\right)^{1/s} \leq C(2^j)^{a/s-|\alpha|}, \]

with \(C\) independent of \(j\).

We also have that \(m_j \in L^1 \cap L^\infty\). Define \(k_j(x)\) by \(k_j(x) = \check{m}_j(x)\), and let

\[ m_N(x) = \sum_{j=-N}^{N} m_j(x), \quad K_N(x) = (m_N)^*(x) = \sum_{j=-N}^{N} k_j(x). \]

It follows that \(\|m_N\|_{\infty} \leq C\), uniformly in \(N\), and that \(m_N(x) \to m(x), x \neq 0,\)
as \( N \to \infty \). Now define \( T_N f \) by \( T_N f = (m_N^N)' \), so that \( T_N f = f \ast K_N \) for \( f \in L^2 \), say. The following lemma shows how conditions on \( m \) can be interpreted as conditions on \( K_N \).

**Lemma 1.** Let \( 1 < s \leq 2 \), \( m \in M(s, l) \) for a positive integer \( l \), and let \( K_N \) be defined as above. If \( d \) is an integer such that \( 0 < d < l \), \( 1 < t < s \), \( n/t < d < n/t + 1 \), and \( 1 < p < t' \), then

\[
\left( \int_{R < |x| < 2R} |K_N(x - y) - K_N(x)|^p \, dx \right)^{1/p} \leq CR^{-d} + n/p - n/t|y|^{d - n/t}
\]

for all \( |y| < \frac{R}{2} \),

with \( C \) independent of \( N, R, \) and \( y \).

**Proof.** Since \( K_N(x) = \sum_{j=-N}^N k_j(x) \),

\[
\left( \int_{R < |x| < 2R} |K_N(x - y) - K_N(x)|^p \, dx \right)^{1/p} \leq \sum_j \left( \int_{R < |x| < 2R} |k_j(x - y) - k_j(x)|^p \, dx \right)^{1/p}.
\]

Also, \( |y| < R/2 \) and \( R < |x| < 2R \) imply \( R/2 < |x - y| < 5R/2 \), so that

\[
\left( \int_{R < |x| < 2R} |k_j(x - y)|^p \, dx \right)^{1/p} \leq \left( \int_{R < |x| < 2R} |k_j(x - y)|^p \, dx \right)^{1/p} + \left( \int_{R < |x| < 2R} |k_j(x)|^p \, dx \right)^{1/p}
\]

\[
\leq 2 \left( \int_{R/2 < |x| < 5R/2} |k_j(x)|^p \, dx \right)^{1/p}.
\]

Therefore, we need to estimate

\[
\left( \int_{R/2 < |x| < 5R/2} |k_j(x)|^p \, dx \right)^{1/p} \quad \text{and} \quad \left( \int_{R < |x| < 2R} |k_j(x - y) - k_j(x)|^p \, dx \right)^{1/p}.
\]

Let \( d \) be an integer such that \( 0 < d < l \) and \( 1 < t < s \) such that \( p < t' \). It is easy to see that \( m \in M(i, d) \). Let \( x^* = x_1^* \cdots x_n^* \). Then

\[
\left( \int_{R/2 < |x| < 5R/2} |k_j(x)|^p \, dx \right)^{1/p} \leq CR^{-d} \left( \int_{R/2 < |x| < 5R/2} |x|^d k_j(x) \, dx \right)^{1/p}
\]

\[
\leq CR^{-d} \sum_{|\alpha| = d} \left( \int_{R/2 < |x| < 5R/2} |x|^\alpha k_j(x) \, dx \right)^{1/p}.
\]

Using the fact that \( \tilde{m}_j = k_j \), Hölder's inequality, and the Hausdorff-Young
(\int_{R/2 < |x| < 5R/2} |x^\alpha k_j(x)|^p \, dx)^{1/p} = \left( \int_{R/2 < |x| < 5R/2} |(D^\alpha m_j)(x)|^p \, dx \right)^{1/p}
= R^{n/p} \left( R^{-n} \int_{R/2 < |x| < 5R/2} |(D^\alpha m_j)(x)|^p \, dx \right)^{1/p}
< CR^{n/p} \left( R^{-n} \int_{R/2 < |x| < 5R/2} |(D^\alpha m_j)(x)|^{p/r} \, dx \right)^{1/r}
< CR^{n/p - n/r} \left( \int_{R^n} |D^\alpha m_j(x)|^r \, dx \right)^{1/r}
< CR^{n/p - n/r} (2^j)^{n/r - d}.

Combining these estimates gives

\left( \int_{R/2 < |x| < 5R/2} |k_j(x)|^p \, dx \right)^{1/p} < CR^{-d + n/p - n/r} (2^j)^{n/r - d}. \quad (2.2)

For the integral of the difference of the \(k_j\)'s we have

\left( \int_{R < |x| < 2R} |k_j(x - y) - k_j(x)|^p \, dx \right)^{1/p}
= \left( \int_{R < |x| < 2R} |\{ m_j(x)(e^{iy} - 1) \}^p \, dx \right)^{1/p}
< CR^{-d} \left( \int_{R < |x| < 2R} |x|^{d} \{ m_j(x)(e^{iy} - 1) \}^p \, dx \right)^{1/p}
< CR^{n/p - d} \sum_{|\alpha| = d} \left( R^{-n} \int_{R < |x| < 2R} |x^\alpha \{ m_j(x)(e^{iy} - 1) \}|^p \, dx \right)^{1/p}
< CR^{n/p - d} \sum_{|\alpha| = d} \left( R^{-n} \int_{R < |x| < 2R} |D^\alpha \{ m_j(x)(e^{iy} - 1) \}|^p \, dx \right)^{1/p}
< CR^{n/p - d - n/r} \sum_{|\alpha| = d} \left( \int_{R^n} |D^\alpha \{ m_j(x)(e^{iy} - 1) \}|^r \, dx \right)^{1/r}
< CR^{n/p - d - n/r} \sum_{|\beta| + |\gamma| = d} \left( \int_{R^n} |D^\beta m_j(x) \cdot D^\gamma (e^{iy} - 1)|^r \, dx \right)^{1/r}.\n
Consider first \(|\gamma| = 0, |\beta| = d\). Since \(|e^{ixy} - 1| < |x||y|\),

\[
\left( \int_{\mathbb{R}^n} (D^\beta m_j(x))(e^{ixy} - 1)|^t \, dx \right)^{1/t} < \left( \int_{\mathbb{R}^n} |x||y|D^\beta m_j(x)|^t \, dx \right)^{1/t} \leq C \gamma^2|y|(2^j)^{n/t-d} = C\gamma|y|(2^j)^{n/t-d+1}.
\]

If \(|\gamma| > 0, |D^\gamma(e^{ixy} - 1)| < |\gamma|^{|\gamma|}\) and \(|\beta| = d - |\gamma|\), so that

\[
\left( \int_{\mathbb{R}^n} |D^\beta m_j(x)\cdot D^\gamma(e^{ixy} - 1)|^t \, dx \right)^{1/t} < \left( \int_{\mathbb{R}^n} |\gamma|^{|\gamma|}D^\beta m_j(x)|^t \, dx \right)^{1/t} \leq C|\gamma|^{|\gamma|}\gamma(2^j)^{n/t-d+1} = C|\gamma|^{|\gamma|}(2^j)^{n/t-d+1}.
\]

Adding these estimates, we obtain

\[
\left( \int_{R < |x| < 2R} |k_j(x-y) - k_j(x)|^p \, dx \right)^{1/p} \leq CR^{n/p-d-n/r} \sum_{m=1}^{d} |y|^m(2^j)^{n/t-d+m}.
\]

But, if \(2^j < |\gamma|^{-1} \) (|y| \(\leq 2^{-j}\)),

\[
|y|^m(2^j)^{n/t-d+m} \leq |\gamma|(2^j)^{n/t-d+1},
\]

so for these values of \(j\), the estimate (2.3) becomes

\[
CR^{n/p-d-n/r}|\gamma|(2^j)^{n/t-d+1}.
\]

Using (2.2) and (2.3) in (2.1), we get

\[
\left( \int_{R < |x| < 2R} |K_N(x-y) - K_N(x)|^p \, dx \right)^{1/p} \leq C \sum_{2^j < |\gamma|^{-1}} R^{n/p-d-n/r}|\gamma|(2^j)^{n/t-d+1} + C \sum_{2^j > |\gamma|^{-1}} R^{n/p-d-n/r}(2^j)^{n/t-d+1} \leq CR^{n/p-d-n/r}|\gamma|^{d-n/t}
\]
as long as \(n/t < d < n/t + 1\). This completes the proof of Lemma 1.

Although we will not use it, we would like to point out that if \(l > \max\{n/p', n/s\}\), then

\[
\left( \int_{R < |x| < 2R} |K_N(x)|^p \, dx \right)^{1/p} \leq CR^{n/p-n}.
\]

This follows from (2.2) with \(d = l\) and the estimate

\[
\left( \int_{R < |x| < 2R} |k_j(x)|^p \, dx \right)^{1/p} < C2^{jn}R^{n/p},
\]

which is a consequence of \(|k_j(x)| = |\hat{m}_j(x)| \leq \|m_j\|_1 \leq C2^{jn}R^{n/p}.

**Remark 1.** We may replace the domain of integration in Lemma 1 by \(\{x \in \mathbb{R}^n: R < |x|\}\); that is, under the conditions of Lemma 1,
\[
\left( \int_{R < |x|} |K_N(x - y) - K_N(x)|^p \, dx \right)^{1/p} \lesssim CR^{-d+n/p-n/t'} |y|^{d-n/t}.
\]
For, if \( t, d, \) and \( y \) satisfy the conditions of Lemma 1,

\[
\left( \int_{R < |x|} |K_N(x - y) - K_N(x)|^p \, dx \right)^{1/p} \leq \sum_{j=0}^{\infty} \left( \int_{2^{j} < |x| < 2^{j+1}R} |K_N(x - y) - K_N(x)|^p \, dx \right)^{1/p} \\
\leq \sum_{j=0}^{\infty} C(2/R)^{-d+n/p-n/t'} |y|^{d-n/t} \\
= CR^{-d+n/p-n/t'} |y|^{d-n/t} \sum_{j=0}^{\infty} (2^j)^{-d+n/p-n/t'} \\
= CR^{-d+n/p-n/t'} |y|^{d-n/t},
\]
since \( -d + n/p - n/t' < 0 \) for \( n/t < d \).

Remark 2. The Hörmander-Mikhlin theorem follows easily from Lemma 1. To see this, let \( m \in M(s, l), 1 < s < 2 \) and \( l > n/s \). Choose \( t < s \) so that \( n/t < l < n/t + 1 \). By Remark 1 with \( p = 1 \) and \( R = 2|y| \), we have

\[
\int_{|x| > 2|y|} |K_N(x - y) - K_N(x)| \, dx \leq C(2|y|)^{-l-n/n-t} |y|^{l-n/t} = C.
\]
Thus, the kernels \( K_N \) satisfy, uniformly in \( N \), the Hörmander condition

\[
\int_{|x| > 2|y|} |K(x - y) - K(x)| \, dx \leq C \quad \text{for all } y \neq 0,
\]
so that \( T_{N}f = K_{N} * f \) is bounded on \( L^p \), uniformly in \( N \), for \( 1 < p < \infty \).

For \( f \in \mathcal{S} \), we have \( Tf = (m\hat{f})^{*} \). It follows that

\[
\| Tf - T_{N}f \|_{\infty} \lesssim \| (m - m^{N})\hat{f} \|_{1} \rightarrow 0
\]
since \( m^{N} \) converges pointwise and boundedly to \( m \). Then, applying Fatou's lemma, we get

\[
\| Tf \|_{p} \leq C \| f \|_{p},
\]
for \( f \in \mathcal{S} \), where \( C \) is the uniform bound for the \( T_{N} \) on \( L^p \). The result extends to all of \( L^p \) by continuity.

Part (1) of Theorem 1 is proved using Lemma 1 and the following three known results.

Lemma 2. Set \( f^{*}_r(x) = ((f^{*})^{*})^{1/r}(x) \). If \( 0 < r < p < \infty \) and \( w \in A_{p/r} \), then

\[
\| f^{*}_r \|_{p, w} \lesssim C \| f \|_{p, w}
\]
with \( C \) independent of \( f \).
This is an immediate corollary of results in [13].

**Lemma 3.** Let

\[ f^*(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y) - \text{avg}_Q f| \, dy, \]

where \( \text{avg}_Q f = |Q|^{-1} \int_Q f(z) \, dz \). Let \( 0 < p < \infty \) and \( w \in A_\infty \). Then

\[ \|f^*\|_{p, w} \leq C \|f^*\|_{p, w} \]

with \( C \) independent of \( f \).

This is proved in [4]. The following result is a special case of interpolation with change of measures. It is proved in [17] and [19].

**Lemma 4.** Let \( 1 < r < q < \infty \) and let \( w_0 \) and \( w_1 \) be two positive weights. If \( T \) is a bounded linear operator from \( L^r_{w_0} \) into itself and \( L^q_{w_1} \) into itself, then \( T \) is bounded from \( L^q \) into itself for \( r < p < q \) and \( w = w_0^{1-t} w_1^{-t} \), provided \( t = (q-p)/(q-r) \) for \( r \neq q \) and \( 0 < t < 1 \) for \( r = q \).

We would like to point out that \( w^{n/l} \in A_p, n/l > 1 \), if and only if \( w \in A_p \) and satisfies the reverse Hölder's inequalities

\[ \frac{1}{|Q|} \int_Q w^{n/l}(x) \, dx \leq C \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{n/l} \]

and

\[ \frac{1}{|Q|} \int_Q (w(x))^{1/(p-1)} \, dx \leq C \left( \frac{1}{|Q|} \int_Q w(x)^{1/(p-1)} \, dx \right)^{n/l}; \]

when \( p = 1 \), we only need the first inequality. For \( p > 1 \), if \( w \in A_p \) and satisfies the above inequalities, then

\[ \left( \frac{1}{|Q|} \int_Q w^{n/l}(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q (w^{n/l}(x))^{1/(p-1)} \, dx \right)^{p-1} \]

\[ \leq C \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{n/l} \left( \frac{1}{|Q|} \int_Q w(x)^{1/(p-1)} \, dx \right)^{(p-1)n/l} \]

so that \( w^{n/l} \in A_p \). For \( p = 1 \), if \( w \in A_1 \) and satisfies the first inequality above, then

\[ \frac{1}{|Q|} \int_Q w^{n/l}(x) \, dx \leq C \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{n/l} \]

\[ \leq C \left( \text{ess inf}_Q w \right)^{n/l} = C \text{ess inf}_Q w^{n/l}, \]

so that \( w^{n/l} \in A_1 \). For the other implication, note first that \( w^{n/l} \in A_p \) implies \( w \in A_p \) since \( n/l > 1 \). If \( p > 1 \), by the \( A_p \) condition,
\[
\left( \frac{1}{|Q|} \int_Q w^{n/l}(x) \, dx \right)^{l/n} \leq C \left( \frac{1}{|Q|} \int_Q w(x)^{-(n/l)(1/(p-1))} \, dx \right)^{-(l/n)(p-1)}.
\]

Thus, the first reverse Hölder’s inequality will follow if we show
\[
\left( \frac{1}{|Q|} \int_Q w(x)^{-(n/l)(1/(p-1))} \, dx \right)^{-(l/n)(p-1)} \leq \frac{1}{|Q|} \int_Q w(x) \, dx,
\]
or equivalently
\[
1 \leq \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-n/l(p-1)} \, dx \right)^{(p-1)/n}.
\]

But, if \( s > 1 \), using Hölder’s inequality, we have
\[
1 = \frac{1}{|Q|} \int_Q dx = \frac{1}{|Q|} \int_Q w^{1/s}(x) w^{-1/s}(x) \, dx
\]
\[
\leq \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{1/s} \left( \frac{1}{|Q|} \int_Q w(x)^{-1/(s-1)} \, dx \right)^{(s-1)/s}.
\]

Setting \( s - 1 = l(p - 1)/n \), or \( s = 1 + l(p - 1)/n > 1 \), we get the desired inequality. Since \( w^{n/l} \in A_p \) implies \( (w^{-1/(p-1)} w^{n/l}) \in A_p \), we also obtain the other reverse Hölder’s inequality from the argument above. Finally, when \( p = 1 \), by the \( A_1 \) condition
\[
\frac{1}{|Q|} \int_Q w^{n/l}(x) dx \leq c \text{ess inf } w^{n/l} = c (\text{ess inf } w)^{n/l} \leq C \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{n/l}.
\]

Notice that the above is true if we replace \( n/l \) by any \( t \geq 1 \).

3. We begin the proof of Theorem 1 by noting that (2) is a consequence of (1) by duality. To see this, suppose \( 1 < p < (n/l)' \) and \( w^{-1/(p-1)} \in A_{p'/n} \). Then, for \( f \in \mathcal{S} \),
\[
\| T f \|_{p,w} = \left( \int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \right)^{1/p} = \sup \int_{\mathbb{R}^n} Tf(x) g(x) \, dx,
\]
where the supremum is taken over all functions \( g \in \mathcal{S} \) such that \( \| g \|_{p', w^{-1/(p-1)}} = 1 \).

Let \( \overline{T} \) be the operator with multiplier \( \overline{m} \), the complex conjugate of \( m \). Then \( \overline{m} \) satisfies the same estimates as \( m \) and we have
\[
\| T f \|_{p,w} = \sup \int_{\mathbb{R}^n} f(x) \overline{T} g(x) \, dx \leq \sup \| f \|_{p,w} \| \overline{T} g \|_{p', w^{-1/(p-1)}}
\]
\[
\leq C \| f \|_{p,w} \sup \| g \|_{p', w^{-1/(p-1)}} = C \| f \|_{p,w}
\]
by (1), since \( p' > n/l \) and \( w^{-1/(p-1)} \in A_{p'/n} \).

Turning to the proof of (1), fix \( p > n/l \) and \( w \in A_{p'/n} \). Choose an \( r < s \) such that \( n/r \) is not an integer, \( n/l < r < p \) and \( w \in A_{p/r} \). There is an
integer \( d < l \) for which \( n/r < d < n/r + 1 \). We will show
\[
(T_N f)^*(x) \leq C f^*(x)
\] (3.1)
with a \( C \) independent of \( f \) and \( N \).

Fix \( x \in \mathbb{R}^n \) and let \( Q \) be a cube centered at \( x \) with diameter \( \delta \). Write
\[
f(y) = f_0(y) + \sum_{j=1}^{\infty} f_j(y),
\]
where
\[
f_0(y) = f(y) \chi(\{ y \in \mathbb{R}^n : |x - y| \leq 2\delta \})
\]
and
\[
f_j(y) = f(y) \chi(\{ y \in \mathbb{R}^n : 2^j \delta < |x - y| \leq 2^{j+1} \delta \}), \quad j = 1, 2, \ldots
\]
For \( y \in Q \),
\[
(K_N * f)(y) = (K_N * f_0)(y) + \sum_{j=1}^{\infty} (K_N * f_j)(y).
\]
By Hölder’s inequality and Remark 2, for any \( q > 1 \) we have
\[
\frac{1}{|Q|} \int_Q |(K_N * f_0)(y)| dy \leq \left( \frac{1}{|Q|} \int_Q |(K_N * f_0)(y)|^q dy \right)^{1/q} \leq C \frac{\|f_0\|_q}{|Q|^{1/q}} \leq C f^*_q(x),
\]
with \( C \) independent of \( N \). For any \( j \),
\[
(K_N * f_j)(y) = (K_N * f_j)(x) + \int \{ K_N(y - z) - K_N(x - z) \} f_j(z) dz
\]
\[
\equiv c_j + \varepsilon_j,
\]
say. Note that \( c_j \) is independent of \( y \) and
\[
|\varepsilon_j| \leq \int_{2^j \delta < |x - z| < 2^{j+1} \delta} |K_N(y - z) - K_N(x - z)| |f(z)| dz
\]
\[
\leq \left( \int_{2^j \delta < |x - z| < 2^{j+1} \delta} |K_N(y - z) - K_N(x - z)|^r dz \right)^{1/r} \cdot \left( \int_{|x - z| < 2^{j+1} \delta} |f(z)|^r dz \right)^{1/r}.
\]
Applying Lemma 1 with \( p = r' \) and \( t = r \) and noting that \( |x - y| < \delta \), we obtain
\[
|\varepsilon_j| \leq C |x - y|^{d - n/r} (2^j \delta)^{-d} (2^{j+1} \delta)^{n/r} \left\{ (2^j + 1) \delta \right\}^{-n} \int_{|x - z| < 2^{j+1} \delta} |f(z)|^r dz \right)^{1/r} \leq C (2^j)^{n/r - d} f^*_r(x).
\]
Therefore,
\[
\frac{1}{|Q|} \int_Q \left( (K_N \ast f)(y) - \sum_{j=1}^{\infty} c_j \right) dy \leq \frac{1}{|Q|} \int_Q \left[ \left( (K_N \ast f_0)(y) \right) dy + \sum_{j=1}^{\infty} \frac{1}{|Q|} \int_Q \left( (K_N \ast f_j)(y) - c_j \right) dy \right.
\]
\[
\leq C f_r^p(x) + C \sum_{j=1}^{\infty} \left( 2^j \right)^{n/r-d} f_r^p(x) = C f_r^p(x),
\]
since \( n/r - d < 0 \). The fact that this estimate is true for any cube centered at \( x \) implies (3.1). Now, using Lemmas 2 and 3, since \( w \in A_{p/r} \), we obtain
\[
\| (K_N \ast f) \|_{p,w} \leq \| (K_N \ast f_0) \|_{p,w} \leq C \| (K_N \ast f) \|_{p,w} \leq C \| f \|_{p,w} \leq C \| f \|_{p,w},
\]
uniformly in \( N \). Arguing as in Remark 2, we have
\[
\| T f \|_{p,w} = \| (K \ast f) \|_{p,w} \leq C \| f \|_{p,w}.
\]

When \( l < n \) and \( p = n/l \), the above proof fails. However, using Lemma 4 and the fact that \( w \in A_1 \) implies there is a \( b > 1 \) such that \( w^b \in A_1 \), we will prove the result. So, fix such a \( b \). Then \( w^b \in A_{q/l} \) for any \( q > n/l \). Setting \( w_0(x) = 1 \) and \( w_1(x) = w^b(x) \), we need to find \( q \) and \( r \) so that \( r < n/l < q \) and \( w(x) = (w^b(x))^{(n/l-r)/(q-r)} \). Thus we need \( b((n/l-r)/(q-r)) = 1 \) or \( b((n/l-1)/(q-r)) = q - r \). Then, choosing \( r, 1 < r < n/l \), and solving for \( q \), which is necessarily greater than \( n/l \) since \( b > 1 \), completes the proof.

The proof of Theorem 1 will be finished once we show the weak-type \((1,1)\) result. This will be done using standard techniques which are included for completeness. Fix a nonnegative \( f \) in \( L^1 \cap L^1_w \) and \( \lambda > 0 \). Applying the Calderón-Zygmund decomposition to \( f \), we get a sequence of disjoint cubes \( \{Q_k\} \) and functions \( g \) and \( b(x) = g(x) + b(x) \), satisfying
\[
\begin{align*}
(i) & \quad |Q_k| \leq (C/\lambda) \int_{Q_k} f(y) dy, \\
(ii) & \quad \| g \|_{L^2, w}^2 \leq \lambda \| f \|_{L^1, w}, \\
(iii) & \quad b(y) = f(y) - |Q_k|^{-1} \int_{Q_k} f(z) dz \quad \text{for } y \in Q_k, \quad \text{supp } b \subset \bigcup Q_k \quad \text{and} \\
& \quad \int_{Q_k} b(y) dy = 0.
\end{align*}
\]

Since \( T_N f = T_N g + T_N b \),
\[
m_w(\{ x \in \mathbb{R}^n : |T_N f(x)| > 2\lambda \}) < m_w(\{ x \in \mathbb{R}^n : |T_N g(x)| > \lambda \}) + m_w(\{ x \in \mathbb{R}^n : |T_N b(x)| > \lambda \}).
\]

We can apply (1) of Theorem 1 to the first term on the right because \( w \in A_1 \). Then, using (ii), we get
\[
m_w(\{ x \in \mathbb{R}^n : |T_N g(x)| > \lambda \}) \leq \frac{C}{\lambda^2} \| g \|_{L^2, w}^2 \leq \frac{C}{\lambda} \| f \|_{L^1, w}.
\]

Let \( Q'_k \) be \( Q_k \) expanded concentrically twice. Then using (i) and the fact that
\( w \in A_1 \), we have

\[
 m_w\left( \bigcup Q_k \right) \leq \sum m_w(Q_k^*) \leq C \sum m_w(Q_k) \leq C \sum \frac{1}{\lambda} \int_{Q_k} f(y) \frac{m_w(Q_k)}{|Q_k|} \, dy
\]

\[
 \leq \frac{C}{\lambda} \sum \int_{Q_k} f(y) w(y) \, dy \leq \frac{C}{\lambda} \|f\|_{1,w}.
\]

Thus, we have only to show

\[
m_w\left( \{ x \in \bigcup Q_k^* : |T_N b(x)| > \lambda \} \right) \leq \frac{C}{\lambda} \|f\|_{1,w}. \tag{3.2}
\]

Let \( y_k \) and \( \delta_k \) be the center and diameter of \( Q_k \). Then

\[
\int_{x \in \bigcup Q_k^*} |T_N b(x)| w(x) \, dx = \int_{x \in \bigcup Q_k^*} \left| \int_{\mathbb{R}^n} K_N(x - y) b(y) \, dy \right| w(x) \, dx
\]

\[
= \int_{x \in \bigcup Q_k^*} \left| \sum_k \int_{Q_k} K_N(x - y) b(y) \, dy \right| w(x) \, dx
\]

\[
= \int_{x \in \bigcup Q_k^*} \left| \sum_k \int_{Q_k} \{ K_N(x - y) - K_N(x - y_k) \} b(y) \, dy \right| w(x) \, dx
\]

\[
\leq \sum_k \int_{Q_k} \left( \int_{x \in Q_k^*} |K_N(x - y) - K_N(x - y_k)| w(x) \, dx \right) |b(y)| \, dy.
\]

If we can show, for any \( y \in Q_k \), that the inner integral is bounded by a constant independent of \( k \) and \( N \) times \( \text{ess inf}_{Q_k} w \), then our result will follow, as we now show. For, by (iii),

\[
m_w\left( \{ x \in \bigcup Q_k^* : |T_N b(x)| > \lambda \} \right) \leq \frac{1}{\lambda} \int_{x \in \bigcup Q_k^*} |T_N b(x)| w(x) \, dx
\]

\[
\leq \frac{C}{\lambda} \sum_k \int_{Q_k} |b(x)| \text{ess inf}_{Q_k} w \, dx \leq \frac{C}{\lambda} \sum_k \int_{Q_k} |b(x)| w(x) \, dx
\]

\[
\leq \frac{C}{\lambda} \sum_k \int_{Q_k} f(x) w(x) \, dx + \frac{C}{\lambda} \sum_k \left( \frac{1}{|Q_k|} \int_{Q_k} f(z) \, dz \right) w(x) \, dx
\]

\[
\leq \frac{C}{\lambda} \|f\|_{1,w} + \frac{C}{\lambda} \sum_k \int_{Q_k} f(z) w(z) \, dz
\]

\[
\leq \frac{C}{\lambda} \|f\|_{1,w} + \frac{C}{\lambda} \sum_k \int_{Q_k} f(z) w(z) \, dz \leq \frac{2C}{\lambda} \|f\|_{1,w}.
\]

Therefore,

\[
m_w\left( \{ x \in \mathbb{R}^n : |T_N f(x)| > \lambda \} \right) \leq \frac{C}{\lambda} \|f\|_{1,w} \tag{3.3}
\]

with a constant independent of \( N, f \), and \( \lambda \). If \( f \in \mathcal{S} \), \( f = f^+ - f^- \) where \( f^+ \) and \( f^- \) and nonnegative and in \( L^1 \cap L^1_w \), so that (3.3) holds for \( f \in \mathcal{S} \). Then
\[ m_w\left( \{ x \in \mathbb{R}^n: |Tf(x)| > \lambda \} \right) \\
\leq m_w\left( \{ x \in \mathbb{R}^n: |T_Nf(x)| + |Tf(x) - T_Nf(x)| > \lambda \} \right) \\
\leq m_w\left( \left\{ x \in \mathbb{R}^n: |T_Nf(x)| > \frac{\lambda}{2} \right\} \right) \\
+ m_w\left( \left\{ x \in \mathbb{R}^n: |Tf(x) - T_Nf(x)| > \frac{\lambda}{2} \right\} \right).\]

Since \( T_Nf \) converges uniformly to \( Tf \) for \( f \in \mathcal{S} \), choosing \( N \) large enough the second term on the right is zero. By (3.3),
\[ m_w\left( \{ x \in \mathbb{R}^n: |Tf(x)| > \lambda \} \right) \leq \frac{C}{\lambda} \|f\|_{1,w} \quad \text{for} \quad f \in \mathcal{S}, \]
which extends to \( L_w^1 \).

To complete the proof of Theorem 1 we need to show
\[
\int_{x \in \mathcal{Q}_k^*} |K_N(x - y) - K_N(x - y_k)|w(x)\,dx \leq C \text{ess inf } w \quad \text{if } y \in \mathcal{Q}_k,
\]
with \( C \) independent of \( k \) and \( N \). Choose \( r < s \) so that \( n/r < l < n/r + 1 \) and \( w' \in A_1 \). Then, using Lemma 1 with \( p = r' \) and \( t = r \) and noting that \( x \in \mathcal{Q}_k^* \) implies \( |x - y_k| > 2\delta_k \), we have for \( y \in \mathcal{Q}_k \) that
\[
\int_{|x - y_k| > 2\delta_k} |K_N(x - y) - K_N(x - y_k)|w(x)\,dx \\
= \sum_{j=1}^{\infty} \int_{2^{j-1}\delta_k < |x - y_k| < 2^j\delta_k} |K_N(x - y) - K_N(x - y_k)|w(x)\,dx \\
\leq \sum_{j=1}^{\infty} \left( \int_{2^{j-1}\delta_k < |x - y_k| < 2^j\delta_k} |K_N(x - y) - K_N(x - y_k)|^{r'}\,dx \right)^{1/r'} \\
\cdot \left( \int_{|x - y_k| < 2^j\delta_k} w'(x)\,dx \right)^{1/r} \\
\leq C \sum_{j=1}^{\infty} \left( \delta_k \right)^{-l(n/r - 1)} \left( \frac{2^j\delta_k}{\delta_k} \right)^{n/r} \left( \frac{2^{j+1}\delta_k}{\delta_k} \right)^{-n} \int_{|x - y_k| < 2^j\delta_k} w'(x)\,dx \right)^{1/r}.\]

Thus, since \( w' \in A_1 \),
\[
\int_{|x - y_k| > 2\delta_k} |K_N(x - y) - K_N(x - y_k)|w(x)\,dx \\
\leq C \sum_{j=1}^{\infty} (2^j)^{n/r - l} \text{ess inf } w(x)_{|x - y_k| < 2^j\delta_k} \\
\leq C \text{ ess inf } w(x) \sum_{j=1}^{\infty} (2^j)^{n/r - l} \leq C \text{ ess inf } w_{|x - y_k| < \delta_k}.
\]
with $C$ independent of $k$ and $N$. This completes the proof of Theorem 1.

We will derive Theorem 2 from Theorem 1 by using Lemma 4 and a characterization of $A_p$ functions proved by P. Jones [9]. He has shown that if $w \in A_p$ then there are $A_1$ weights $u$ and $v$ such that $w = uv^{1-p}$.

Fix $p$, $1 < p < \infty$, and $w$ so that $w^{n/l} \in A_p$. We have $w^{n/l} = uv^{1-p}$, $u, v \in A_1$, or $w = u^{n/l}v^{(1-p)/n}$. Next, write this as

$$w = u^{l/n}v^{(1-p)/n} = (u^\alpha v^\beta)^{t} (u^\gamma v^\delta)^{1-t} = w_0^tw_1^{1-t}.$$

For this to make sense, we need

$$\alpha t + \gamma(1-t) = \frac{l}{n}, \quad (3.4)$$

$$\beta t + \delta(1-t) = \frac{l}{n}(1-p). \quad (3.5)$$

Then, in order to use Lemma 4 for weights which satisfy Theorem 1, we require

$$w_0^{-(r-1)} \in A_{r/l}, \quad 1 < r < \min\left\{\left(\frac{n}{l}\right)^r, p\right\}, \quad (3.6)$$

$$w_1 \in A_{q/l}, \quad q > \max\left\{\frac{n}{l}, p\right\}, \quad (3.7)$$

$$t = \frac{q-p}{q-r}. \quad (3.8)$$

Recall that $u \in A_1$ (similarly $v \in A_1$) implies

$$\frac{1}{|Q|} \int_Q u(y) \, dy < Cu(x) \quad \text{for almost all } x \in Q.$$

Therefore, if $\alpha > 0$ and $\beta < 0$, letting $s = r'/n$, we have

$$\left(\frac{1}{|Q|} \int_Q w_0(x)^{-1/(r-1)} \, dx\right)^{s-1} \left(\frac{1}{|Q|} \int_Q w_0(x)^{(1/(r-1))(1/(s-1))} \, dx\right)^{s-1}$$

$$= \left(\frac{1}{|Q|} \int_Q u(x)^{-\alpha/(r-1)} v(x)^{-\beta/(r-1)} \, dx \right)$$

$$\cdot \left(\frac{1}{|Q|} \int_Q u(x)^{(\alpha/(r-1))(1/(s-1))} v(x)^{(\beta/(r-1))(1/(s-1))} \, dx \right)^{s-1}$$

$$\leq C \left(\frac{1}{|Q|} \int_Q u(x) \, dx\right)^{-\alpha/(r-1)} \left(\frac{1}{|Q|} \int_Q v(x)^{-\beta/(r-1)} \, dx \right)^{s-1}$$

$$\cdot \left(\frac{1}{|Q|} \int_Q v(x) \, dx\right)^{\beta/(r-1)} \left(\frac{1}{|Q|} \int_Q u(x)^{(\alpha/(r-1))(1/(s-1))} \, dx \right)^{s-1}$$

$$= C,$$
if
\[ \alpha = (r - 1) \left( \frac{r' l}{n} - 1 \right) = \frac{r' l}{n} - r + 1 \quad \text{and} \quad \beta = - (r - 1); \]
that is \( w_0^{1/(r - 1)} \in A_{r' l/n} \) for these values of \( \alpha \) and \( \beta \). Similarly, we can show \( w_1 \in A_{q l/n} \) if \( \gamma = 1 \) and \( \delta = - ((q l/n) - 1) \). Using these values of \( \alpha \) and \( \gamma \), we have (3.4) if \( t = 1/r \). Next, solving (3.5) for \( q \), we get \( q = r'(p - 1) \). This value of \( q \) also satisfies (3.8). Therefore, if we choose \( r < \min \{ (n/l)', p \} \) so close to 1 that \( q = r'(p - 1) > \max \{ n/l, p \} \), we can satisfy (3.4)-(3.8), proving Theorem 2.

Before proving Theorem 3, notice that \( -n \geq -lp \) if \( n/l \leq p \), and \( n(p - 1) \leq lp \) if \( p \leq (n/l)' \). Therefore, for \( l < n \) the conclusion of Theorem 3 can be divided into three cases:

1. \( 1 < p < \frac{n}{l} \) and \( -lp < \beta < n(p - 1) \), \hspace{1cm} (3.9)
2. \( \frac{n}{l} < p < \left( \frac{n}{l} \right)' \) and \( -n < \beta < n(p - 1) \), \hspace{1cm} (3.10)
3. \( \left( \frac{n}{l} \right)' < p < \infty \) and \( -n < \beta < lp \). \hspace{1cm} (3.11)

Since (3.11) is the dual of (3.9), we need only concern ourselves with (3.9) and (3.10).

Next, let us interpret Theorem 1 when \( w(x) \) is a power of \( |x| \). Because \( |x|^\beta \in A_p \) if and only if \( -n < \beta < n(p - 1) \), we have \( (l < n) \) that \( T \) is bounded on \( L^p_{|x|^\beta} \) if

\[ \frac{n}{l} < p < \infty \quad \text{and} \quad -n < \beta < pl - n, \]
\[ 1 < p < \left( \frac{n}{l} \right)' \quad \text{and} \quad -n + p(n - l) < \beta < n(p - 1). \] \hspace{1cm} (3.12) (3.13)

However, combining (3.12) and (3.13), we have (3.10) and are left with only proving (3.9).

Let \( q = n/l \) and \( r < n/l \); then also \( r < (n/l)' \). By (3.13) and (3.10), \( T \) is bounded on \( L^p_{|x|^\beta_0} \) and \( L^q_{|x|^\beta_1} \) for \( -n + r(n - l) < \beta_0 < n(r - 1) \) and \( -n < \beta_1 < n(q - 1) \). Using Lemma 4, if \( r < p < q \) we see that \( T \) is bounded on \( L^p_{|x|^\beta} \) for

\[ \beta = \beta_0 \left( \frac{q - p}{q - r} \right) + \beta_1 \left( \frac{p - r}{q - r} \right). \]

Thus \( \beta \) satisfies

\[ \{ -n + r(n - l) \} \left( \frac{q - p}{q - r} \right) - n \left( \frac{p - r}{q - r} \right) \]
\[ < \beta < n(r - 1) \left( \frac{q - p}{q - r} \right) + n(q - 1) \left( \frac{p - r}{q - r} \right). \]
Simplifying and using the fact that \( q = n/l \), we get
\[
\frac{n^2(r-1)}{n-lr} + \frac{plr(l-n)}{n-lr} < \beta < n(p-1).
\] (3.14)

But, as \( r \to 1 \), the left-hand side of (3.14) approaches \(-lp\). So, taking \( r \) sufficiently close to 1 allows us to choose any \( \beta \) satisfying \(-lp < \beta < n(p-1)\).

When \( l = n \), the restriction in Theorem 3 is \(-n < \beta < n(p-1)\) for \(1 < p < \infty\). But, when \( l = n \) in Theorem 1, we require \( w \in A_p \), and \(|x|^\beta \in A_p\) if \(-n < \beta < n(p-1)\).

4. The proof of Theorem 4 is based on an analogue of Lemma 1.

**Lemma 5.** Let \( \Omega \in L'(\Sigma) \) and satisfy the \( L' \)-Dini condition. Set \( K(x) = \Omega(x')/|x|^n \). There exists a constant \( \alpha_0 > 0 \) such that if \(|y| < \alpha_0 R\), then
\[
\left( \int_{R < |x| < 2R} |K(x-y) - K(x)|^r \, dx \right)^{1/r} \leq CR^{n/r-n} \left\{ \frac{|y|}{R} + \int_{\frac{|y|}{2R} < |y|/R} \omega_r(\delta) \frac{d\delta}{\delta} \right\}
\]

**Proof.** We may choose \( \alpha_0 < \frac{1}{2} \); then, since \(|x| > R, |x-y|\) is equivalent to \(|x|\). Therefore,
\[
|K(x-y) - K(x)| = \left| \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(x)}{|x|^n} \right| \\
\leq C \left\{ |\Omega(x)| \frac{|y|}{|x|^{n+1}} + \frac{|\Omega(x-y) - \Omega(x)|}{|x|^n} \right\}.
\]

It follows that
\[
\left( \int_{R < |x| < 2R} |K(x-y) - K(x)|^r \, dx \right)^{1/r} \leq C \left( \int_{R < |x| < 2R} |\Omega(x)|^{r} \frac{|y|^r}{|x|^{n+1}} \, dx \right)^{1/r} \\
+ C \left( \int_{R < |x| < 2R} \frac{|\Omega(x-y) - \Omega(x)|^r}{|x|^n} \, dx \right)^{1/r}.
\] (4.1)

The first term on the right side of (4.1) is bounded by
\[
C\|\Omega\|_{L'(\Sigma)} |y| R^{-(n+1)} R^{n/r} = CR^{n/r-n} \left( \frac{|y|}{R} \right).
\]
Changing to polar coordinates, we see the second term equals
\[
C \left( \int_R^{2R} t^{-n+r-n-1} \left( \int_\Sigma |\Omega(tx' - y) - \Omega(tx')|' \, d\sigma_{x'} \right) \frac{dt}{t} \right)^{1/r} \leq CR^{n/r-n} \left( \int_R^{2R} \left( \int_\Sigma \left| \Omega \left( \frac{x' - \alpha}{|x' - \alpha|} \right) - \Omega(x') \right|' \, d\sigma_{x'} \right) \frac{dt}{t} \right)^{1/r},
\]
where \( \alpha = y/t \). Arguing as in Calderón, Weiss, and Zygmund [1, pp. 65–72], we see the inner integral is bounded by
\[
C \sup_{|\alpha| < |\alpha|_0} \int_\Sigma |\Omega(\rho x') - \Omega(x')|' \, d\sigma_{x'} = C \omega_r \left( \frac{|y|}{t} \right)
\]
as long as \( |\alpha| = |y|/t < \alpha_0 \). Thus, the second term is bounded by
\[
CR^{n/r-n} \left( \int_R^{2R} \omega_r \left( \frac{|y|}{t} \right) \frac{dt}{t} \right)^{1/r} = CR^{n/r-n} \left( \int_{|y|/2R}^{R} \omega_r(\delta) \frac{d\delta}{\delta} \right)^{1/r}
\]
\[
< CR^{n/r-n} \left( \int_{|y|/2R<\delta<|y|/R} \omega_r(\delta) \frac{d\delta}{\delta} \right),
\]
since \( \omega_r \) is essentially constant on intervals of the form \((a, 2a), a > 0 \). Lemma 5 is now proved.

Notice that when \( R = 2^j |y|, \) with a \( j \) such that \( 1/\alpha_0 < 2^j \), we get
\[
\left( \int_{2^j |y| < |x| < 2^{j+1} |y|} |K(x - y) - K(x)|' \, dx \right)^{1/r} \leq C (2^j |y|)^{n/r-n} \left\{ \frac{1}{2^j} + \int_{2^{-j}}^{2^{j+1}} \omega_r(\delta) \frac{d\delta}{\delta} \right\}.
\]

Theorem 4 is proved in exactly the same manner as Theorem 1. Using Lemma 5, we show
\[
(K \ast f)^\#(x) \leq C f^\#(x),
\]
which proves the result for \( p > r' \). The only change necessary is in the decomposition \( f = f_0 + \Sigma f_j \). For Theorem 4,
\[
f_0(y) = f(y) \chi \left( \left\{ y \in \mathbb{R}^n : |x - y| < \frac{1}{\alpha_0} \delta \right\} \right)
\]
and the sum of \( f_j \)'s is over \( j \geq \log_2(1/\alpha_0) \). We get the case \( p = r' \) by interpolation, and \( 1 < p \leq r \) follows by duality. In the weak-type \((1, 1)\) proof, we may have to replace the weak-type \((2, 2)\) result for the good function by a weak-type \((r', r')\) result.

5. We conclude by showing that Theorem 3 is best possible, except for endpoint equalities for \( \beta \). We prove the result for \( p > (n/l)' \); the case \( p < n/l \) follows by duality. For \( n/l < p < (n/l)' \), the Riesz transforms and
an argument like that in [8] show the range of $\beta$ is best possible.

Let $1 < s < 2$, $n/s < l < n$, $(n/l)' < p$ and $\beta > lp$. Define a multiplier $m$ by

$$m(x) = e^{ix\cdot \eta}(1 + |x|^2)^{-1/2}$$

for a fixed $\eta$ of length 1. Note that $\hat{m}(x) = G_l(x - \eta)$ (the Bessel kernel of order $l$) and that $|D\alpha m(x)| \leq C_{\alpha}/(1 + |x|)^l$, so $m \in M(s, l)$, $1 \lesssim s \lesssim \infty$. Moreover, $G_l > 0$ and there exist $c, \mu > 0$ such that $G_l(x) > c|x|^l - n$ if $|x| < \mu$ (see Stein [19, p. 132] for details).

Set

$$f(x) = |x|^{-(n + \beta)/p}|\log|x||^{-\beta} \chi(x \in \mathbb{R}^n : |x| < \mu).$$

If $\delta p > 1$, $f \in L_{|x|^\delta}^p(\mathbb{R}^n)$. Since $Tf(x) = (G_l(\cdot - \eta) \ast f)(x)$,

$$Tf(x) = \int_{|y| < \mu} |y|^{-(n + \beta)/p}|\log|y| |^{-\beta} G_l(x - y - \eta) \, dy$$

$$= \int_{|x - \eta - z| < \mu} \frac{|x - \eta - y|^{-(n + \beta)/p}|\log|x - \eta - z|}{|z|^l} dz$$

by setting $z = x - \eta - y$. Now, if we restrict the integration to $|z| < \frac{1}{2}|x - \eta|$, $|x - \eta - z|/|x - \eta|$ and, if $|x - \eta| < \mu/2$,

$$Tf(x) \gtrsim C|x - \eta|^{-(n + \beta)/p}|\log|x - \eta||^{-\beta} \int_{|z| < \frac{1}{2}|x - \eta|} \frac{dz}{|z|^l}$$

$$\sim C|x - \eta|^{l-(n + \beta)/p}|\log|x - \eta||^{-\delta}.$$ 

Therefore, $Tf \not\in L_{|x|^\delta}^p(\mathbb{R}^n)$ if $(l - (n + \beta)/p)p < -n$; i.e., if $\beta > lp$.

REFERENCES


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