We write numbers today in what is called base 10 notation. The position of a digit determines the value of what that digit represents. For example, in the number 727, the right-most 7 represents 7 units, but the left-most 7 represents 7 hundreds. In other words, 

\[ 727 = 700 + 20 + 7 \]

\[ = 7 \times 100 + 2 \times 10 + 7 \]

\[ = 7 \times 10^2 + 2 \times 10^1 + 7. \]

The reason that this notation is called base 10 is because the place values are powers of 10. Moving from right to left, place values represent \( 1, 10, 100, 1000, 10000, \) and so on, or \( 1, 10, 10^2, 10^3, 10^4, \ldots, \) powers of 10. To get a better understanding of base 10, we will explore other bases in this unit. One analogy between base 10 and another base is to think about a base as a language, and so changing from base 10 changes to another language. The same idea can be made in two languages, but the words and letters used are different. The same is true in using different bases; the same number is represented differently in base 10 than in another base. By working with another base, essentially an unfamiliar language, we will have to focus on the meaning of base. Other bases have been used in the past, notably base 60 and base 12. Base 60 shows up in our way of telling time, and base 12 shows up in some English words, such as dozen and gross.

By the way, another common system of writing numbers, Roman Numerals, is totally different and does not use the idea of place values. The base systems won out over Roman Numerals because they turned out to be more effective ways to write and manipulate numbers.

Quarters, Nickels, and Pennies

To start the activity, get 4 pennies, 4 nickels, and 4 quarters. For each amount of money between 1¢ and $1.24, determine how many pennies, nickels, and quarters you need to represent the amount. Enter the numbers onto the table below in the following manner: write down a three digit number starting with the number of quarters, then the number of
nickels, then the number of pennies. For example, since 32¢ is made from 1 quarter, 1 nickel, and 2 pennies, you would enter 112.

As you do this activity, you should convince yourself that there is only one way to get each amount of money with the coins you have.

<table>
<thead>
<tr>
<th>1¢</th>
<th>2¢</th>
<th>3¢</th>
<th>4¢</th>
<th>5¢</th>
<th>6¢</th>
<th>7¢</th>
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<th>11¢</th>
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<td>120¢</td>
<td>121¢</td>
<td>122¢</td>
<td>123¢</td>
<td>124¢</td>
<td>125¢</td>
</tr>
</tbody>
</table>

**Base 5 Arithmetic**

The mathematical point of the activity *Quarters, Nickels, and Pennies* was to represent numbers in base 5. Recall that

\[
234 = 200 + 30 + 4 = 2 \times 100 + 3 \times 10 + 4 = 2 \times 10^2 + 3 \times 10^1 + 4 \times 1.
\]
This is the usual base 10 representation of this number. We call it base 10 because moving a digit one place to the left represents multiplication by 10. In other words, the 4 in 234 represents 4, while the 4 in 843 represents 40. There is nothing magical in using 10 as a base in representing numbers. In this assignment we will see how to do arithmetic in base 5. Only the digits 0, 1, 2, 3, and 4 are used in base 5. The meaning of a number written in base 5 by writing a sequence of digits, such as 2103, is, for example,

$$2103 = 2 \times 5^3 + 1 \times 5^2 + 0 \times 5^1 + 3 \times 1.$$ 

In Problems 3, 4, 5, and 7, explain how you perform the operations by using the addition and multiplication tables you produced in Problem 2. Do not convert to base 10 to do them. You should strive to see that these tables, together with the usual algorithms for doing arithmetic, suffice for doing the computations.

In the next seven problems all the numbers are written in base 5. One good way to work Problems 3 through 5 is to interpret the numbers written in base 5 as so many quarters, nickels, and pennies, and then physically manipulate coins to perform the computations. Actual coins would be the best; however, if you don’t have coins you can use the pictures of coins at the end of this unit. To handle some of the problems we need a coin representing $5^3 = 125$, and one is drawn in the picture.

1. Write, in increasing order and in base 5, the first twenty-five positive whole numbers. What pattern shows up? How would you continue listing whole numbers past twenty-five?

2. Fill in the following addition and multiplication tables.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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</tbody>
</table>

<table>
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<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
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<tr>
<td>4</td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

3. Perform the following sums in base 5.

(a) $12 + 23$
(b) $32 + 14$
(c) $441 + 304$
(d) $1432 + 44$

4. Perform the following subtractions in base 5.

(a) $34 - 23$
5. Perform the following multiplications in base 5.

(a) $13 \times 4$
(b) $314 \times 3$
(c) $23 \times 40$
(d) $302 \times 214$

6. If you have a number written in base 5 and you multiply it by 10, explain how you can write the product without having to do any calculation.

7. Perform the following divisions in base 5. Write the answer as a quotient plus a remainder.

(a) $42/4$
(b) $33/14$
(c) $213/12$

**Dividing Blocks**

You are given the following blocks:

- 1 blue block
- 2 orange blocks
- 3 green blocks
- 4 red blocks

The different colored blocks have different “values”; they can be exchanged according to the following chart.
In other words, 5 blue blocks have the same value as 1 red block, 5 red blocks have the
same value as 1 green block, 5 green blocks have the same value as 1 yellow block, and 5
yellow blocks have the same value as 1 orange block. How you do divide the blocks among
three children so that each has the same value of blocks? Explain how you do the dividing.
Furthermore, phrase this as a division problem in base 5.

**Base 2**

In this assignment we will consider another example of a different base. In base 2 there are
only two digits, 0 and 1. The addition and multiplication tables for base 2 are very simple.
They are

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>×</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Writing a number in base 2 essentially means writing it as a sum of powers of 2. The
powers of 2, in increasing order, are 1, 2, 4, 8, 16, 32, 64, 128, and so on. For example, \(37 = 32 + 4 + 1\). Filling in the "missing" terms,

\[
37 = 32 + 4 + 1 = 1 	imes 2^5 + 0 	imes 2^4 + 0 	imes 2^3 + 1 	imes 2^2 + 0 	imes 2^1 + 1.
\]

Therefore, the base 2 representation of 37 is 100101.

Here is a method to get base 2 representations that can be used by hand or with a
calculator. Start with a number. If it is odd, write 1 then subtract 1 and divide by 2. If it is
even, write 0 and divide by 2. Repeat this process on the resulting number, writing the next
digit to the left of the first digit. Repeat this process until you reach 1, and write a final 1.

1. Find the binary representation of the following numbers.

   (a) 7
2. Perform the following additions in base 2.

(a) 101 + 110
(b) 1111 + 11
(c) 100 + 101
(d) 11011 + 1010

3. Perform the following multiplications in base 2.

(a) 11 × 11
(b) 101 × 11
(c) 1101 × 101

A Base 2 Game

|   | 1 | 3 | 5 | 7 | 2 | 3 | 6 | 7 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 16 | 17 | 18 | 19 |
|   | 9 | 11| 13| 15|10 |11 |14 |15 |12 |13 |14 |15 |12 |13 |14 |15 |20 |21 |22 |23 |
|   |17 |19 |21 |23 |18 |19 |22 |23 |20 |21 |22 |23 |24 |25 |26 |27 |24 |25 |26 |27 |
|   |25 |27 |29 |31 |26 |27 |30 |31 |28 |29 |30 |31 |28 |29 |30 |31 |28 |29 |30 |31 |

To play this game you have five sheets of paper, each of a different color. Each sheet contains numbers between 1 and 31. Ask someone to think of a number between 1 and 31 but not to tell you what is the number. Ask them to tell you on which sheets does their number appear. You can then tell them what is their number. How do we do this, and how can we build the game?

Here is the secret of how to determine somebody’s number. Add the numbers in the top left corner of each sheet of paper on which the mystery number appears, and that will give you their number. For example, if, reading from left to right, a person’s number appears on the first, second, and fourth sheets listed above, then their number is $1 + 2 + 4 = 7$. We will see why this works and how to construct the sheets of paper in this assignment.

The key to this game is the binary, or base 2, representation of a number. As we saw in the assignment Base 2, every number can be written in base 2 as a series of 0’s and 1’s, and this represents writing a number a sum of powers of 2. For example,
19 = 16 + 2 + 1 = 10011 in base 2

and

28 = 16 + 8 + 4 = 11100 in base 2

Numbers from 1 to 31 can be written in base 2 with at most 5 symbols; that is, we need at most five place values to represent these numbers. Determining if a place is to have a 0 or a 1 corresponds to determining whether or not the corresponding power of 2 is needed to get the number. So, we need five pieces of information to determine the base 2 representation of a number between 1 and 31. You get five pieces of information when somebody tells you which sheets on which their number occurs; they have to answer yes it is on or no it is not on for each of the five pieces of paper.

To construct the game, first write each of the numbers from 1 to 31 as a sum of powers of 2. Next, on five sheets of paper of different colors, draw a large square and subdivide it into a 4 by 4 grid of squares. On the first page, place a 1 in the top left corner. Fill in the remaining squares with all of the numbers between 1 and 31 that have a 1 in the first (right most) digit of the base 2 representation of the number, or that need a 1 to write the number as a sum of powers of 2. On the next sheet of paper, place a 2 in the top left corner. Fill in the remaining squares with those numbers that have a 1 in the 2’s place in the base 2 representation of the number, or that need a 2 to write the number as a sum of powers of 2. Do the same thing with the remaining three sheets of paper, placing 4, 8, and 16 in the top left corners, respectively. For instance, for the number 19, since 19 = 16 + 2 + 1, you would place 19 on the first, second, and fifth sheets, corresponding to 1, 2, and 16. Likewise, since 28 = 16 + 8 + 4, you would place 28 on the third, fourth, and fifth sheets, which correspond to 4, 8, 16. So, to determine the person’s number, you add the numbers in the top left squares of each sheet on which their number appears.

You may place the numbers in any order on the sheets. However, the more systematic you place the numbers the easier it is for somebody to tell you on which sheets their number is located.

As practice in playing the game and doing arithmetic calculations in your head, working in pairs, have one person pick a number and the other person determine what is the number. Trade off so that each person gets a chance to determine the other person’s numbers.

Problem. Describe how to guess a person’s number from the information they give you. Next, write each whole number from 1 to 31 as a sum of powers of 2. Finally, describe how to use this information to build the five sheets; explain this by saying on which sheets the numbers 13 and 26 are to be placed upon.
Computer screens and printers make colors by mixing red, green, and blue together in appropriate quantities. In this assignment we will into the alphanumeric code assigned to a color. If you look at the chart\(^1\) above, you will see a code made up of six letters or numbers that corresponds to a given color. Similarly, if you wish to use a color in a web page, you use the command color = rrggbb, where rrggbb is replaced by the appropriate code. The first two symbols of the code represent red, the next two represent green, and the final two represent blue. These codes represent numbers in base 16. In base 16 we need 16 symbols to represent digits. The standard convention is to use the following symbols, to represent, in order, the digits:

\[
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & A & B & C & D & E & F \\
\end{array}
\]

The base 10 equivalent of these digits is given in the following table.

<table>
<thead>
<tr>
<th>Base 16 Digit</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base 10 Equivalent</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
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<td>12</td>
<td>13</td>
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<td>15</td>
</tr>
</tbody>
</table>

While some situations use the base 16 representation of colors, other situations use base 10. We then need to work with both. The meaning of base 16 is similar to the meaning of base 2 and base 5. A two digit number \(UV\) in base 16 represents, in base 10, the number \(16U + V\). For example, 24 written in base 16 represents \(2 \times 16 + 4 = 36\). Similarly, \(CC\) represents \(C \times 16 + C = 12 \times 16 + 12 = 204\). The largest two digit base 16 number is \(FF\), which represents \(15 \times 16 + 15 = 255\) in base 10.

Here is a procedure to produce the two digit base 16 representation of a number. Take the number, divide by 16, and keep the whole number part. This is the left digit of the base 16 number. Next, take the original number and subtract 16 times the digit you just found. What you get is the right digit. For example, if you start with 232, dividing by 16 gives 14.5. The left digit is then \(14 = D\). Then computing \(232 - 14 \times 16\), we get 8, which is the second digit. So, 232 in base 10 is \(D8\) in base 16.

1. Determine the base 10 equivalent of the red, green, and blue values in each of the following color codes:

(a) \(FF6633\)

\(^1\)The RGB color chart above was created by Douglas Jacobson.
2. You wish to make a color whose red, green, and blue values, in base 10, are 207, 42, and 190, respectively. What is the corresponding 6 digit code for this color?

3. I have a book on writing HTML pages that says the following: CC means 80% brightness, 99 means 60% brightness, 66 means 40% brightness, and 33 means 20% brightness. Check if these statements are exactly true, approximately true, or not true by finding the ratio of the base 10 number represented by each of these to the maximum two digit number in base 16, which is 255.

4. What do you think are the 6 digit codes for each of the three pure colors red, blue, and green?

5. If white is represented by the code FFFFFF, what do you think is the code for black? Explain why your answer should be correct. You should think about what is the relation between white and black to help answer this question.