In these problems, suppose that $G$ is the symmetry group of a wallpaper pattern, and that the translation subgroup has the form $T = \{(I, nt_1 + mt_2) : n, m \in \mathbb{Z}\}$ for some vectors $t_1, t_2$. Let $V = \{nt_1 + mt_2 : n, m \in \mathbb{Z}\}$, the translation lattice. Thus, $T = \{(I, v) : v \in V\}$.

Let $G_0$ be the point group of $G$.

**Problem 1.** Let $(A, w) \in G$. Prove that $(A, w)$ commutes with each translation in $G$ if and only if $A = I$. Use this to prove that $G$ is Abelian if and only if $G_0 = \{I\}$.

**Solution.** Suppose that $(A, w) \in G$ commutes with each translation in $G$. Let $(I, v) \in T$. Then $(A, w)(I, v) = (I, v)(A, w)$. From our formula for composing isometries, this reduces to $(A, w + Av) = (A, w + v)$. Thus, $w + Av = w + v$, or $Av = v$. This holds for all $v \in V$. Therefore, $At_1 = t_1$ and $At_2 = t_2$. Since $\{t_1, t_2\}$ form a basis for $\mathbb{R}^2$, we have seen that this forces $A = I$. Conversely, if $A = I$, then $(I, w) \in T$. Since $T$ is Abelian, any two translations commute, so $(I, w)$ commutes with each translation.

Now, suppose that $G$ is Abelian. Then each $(A, w)$ commutes with everything in $G$, so with each translation in $G$. By the previous paragraph, this forces $A = I$ for each $(A, w) \in G$. This means $G_0 = \{I\}$. Conversely, if $G_0 = \{I\}$, then each element of $G$ is a translation, by the definition of $G_0 = \{A : (A, w) \in G \text{ for some } w \in \mathbb{R}^2\}$. Then $G = T$ is Abelian.

**Problem 2.** Let $G$ be a wallpaper group. If $G = \{(A, v) : A \in G_0, v \in V\}$, prove that $G_0$ is isomorphic to a subgroup of $G$, and that $G$ is isomorphic to the semidirect product $T \times_{\varphi} G_0$, where $\varphi : G_0 \to \text{Aut}(T)$ is the usual action of $G_0$ on $T$, given by $\varphi(A)(v) = Av$.

**Solution.** Let $H = \{(A, 0) : A \in G_0\}$. Then $H$ is a subset of $G$ by the hypothesis that $G = \{(A, v) : A \in G_0, v \in V\}$. It is nonempty since $(I, 0) \in H$, as $I \in G_0$. Next, if $(A, 0), (B, 0) \in H$, then $(A, 0)(B, 0) = (AB, 0)$, and since $AB \in G_0$ (as $A, B \in G_0$ and $G_0$ is a group), we get $(AB, 0) \in H$. Also, $(A, 0)^{-1} = (A^{-1}, 0) \in H$ since $A^{-1} \in G_0$. Thus, $H$ is a subgroup of $G$. The map $(A, 0) \mapsto A$ is a group isomorphism from $H$ to $G_0$; that is is a homomorphism follows from the equation $(A, 0)(B, 0) = (AB, 0)$, it is onto by definition of $H$, and it is 1-1 since $(A, 0) = (I, 0)$ if and only if $A = I$.

Next, define $F : T \times_{\varphi} G_0 \to G$ by $F((I, v), A) = (A, v)$. This is a group homomorphism, since if $(I, v), (I, w) \in T$ and $A, B \in G_0$, we have

\[
\]
The homomorphism $F$ is onto, since if $(A,v) \in G$, then $A \in G_0$ and $v \in V$, so $(I,v) \in T$, and $(A,v) = F((I,v), A)$. Finally, $F$ is 1-1, since
\[
\ker(F) = \{(I,v), A) : F((I,v), A) = (I,0)\}
= \{(I,v), A) : (A,v) = (I,0)\}
= \{(I,0), I)\}
\]
is the identity subgroup of $T \times \varphi G_0$. Thus, $F$ is an isomorphism.

Note that $T \cap H = \{(I,0)\}$ and $G = TH$. This means $G$ is the internal semidirect product of $T$ and $H$. Then $G \cong T \times_\theta H$, where $\theta : H \to \text{Aut}(T)$ is given by $\theta((A,0)) = (A,0)(I,v)(A,0)^{-1} = (I, Av)$. In other words, by identifying $H$ with $G_0$ by the isomorphism given above, $\theta$ is the same action as $\varphi$. This gives another way to see that $G \cong T \times \varphi G_0$.

**Problem 3.** Suppose that a wallpaper group $G$ contains no reflections, but that $g = (f, w) \in G$ is a glide reflection such that the reflection line of $f$ is parallel to $t_2$ and perpendicular to $t_1$. Prove that $g^2 = (I, u)$ for some vector parallel to $t_2$, and that $w \notin V$.

(Hint: Write $w$ as a linear combination of $t_1$ and $t_2$. For the last part, try composing $g$ with a translation in $G$.)

**Solution.** We have $g^2 = (f, w)(f, w) = (f^2, f(w) + w) = (I, f(w) + w)$ since $f$ is a reflection. Since $f$ is linear, $f(f(w) + w) = f(f(w)) + f(w) = w + f(w)$, so the vector $u := f(w) + w$ is on the reflection line of $f$. Thus, it is parallel to $t_2$. If $w \in V$, then $(I, -w)(f, w) \in G$, but this element is equal to $(f, 0)$. This forces $G$ to contain a reflection, which is false. Thus, $w \notin V$.

**Problem 4.** For the attached hexagon grid, determine $t_1$ and $t_2$, and draw all rotation centers on or inside the parallelogram formed by $t_1$ and $t_2$. Identify which rotation centers are for $60^\circ$ rotations, which are for $120^\circ$ rotations, and which are for $180^\circ$.

1. (If a point is a $60^\circ$ rotation center, it is also a $120^\circ$ and a $180^\circ$ rotation center. Only mark it as a $60^\circ$ rotation center.)

**Solution.** One choice of $t_1$ and $t_2$ is shown in the picture below.
Another choice would be \( s_1 = t_1 \) and \( s_2 = t_1 + t_2 \). Let \( r \) be the 60° rotation about the origin; this is a symmetry of the grid. We then have \( r(t_1) = t_1 + t_2 \) and \( r(t_2) = -t_1 \). With respect to the basis \( \{ t_1, t_2 \} \), \( r \) is represented by the matrix

\[
A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.
\]

A point \( \alpha t_1 + \beta t_2 \) is a center of a 60° rotation center if

\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (I - A)^{-1} \begin{pmatrix} n \\ m \end{pmatrix}
\]

for some \( n, m \); this follows from Lemma 2.1. The corresponding rotation is \( (r, nt_1 + mt_2) \). We see that \( (\alpha, \beta) = (n - m, n) \). Thus, \( (\alpha, \beta) \) can be any pair of integers, and these points correspond to the centers of the hexagons. Next, consider \( r^2 \), a 120° rotation. The matrix representing \( r^2 \) with respect to the basis \( \{ t_1, t_2 \} \) is

\[
A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},
\]

and \( \alpha t_1 + \beta t_2 \) is a 120° rotation center if

\[
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (I - A^2)^{-1} \begin{pmatrix} n \\ m \end{pmatrix},
\]

which yields \( (\alpha, \beta) = \frac{1}{3}(2n - m, n + m) \). From a previous homework problem, we see that there are two points inside the parallelogram, \( \frac{2}{3} t_1 + \frac{1}{3} t_2 \) and \( \frac{1}{3} t_1 + \frac{2}{3} t_2 \). Finally, for \( r^3 = -I \), a 180° rotation, the centers, from previous work we have done, are \( \{ \frac{1}{2} v : v \in V \} \). We then get the points \( \frac{1}{2} t_1, \frac{1}{2} t_2, \frac{1}{2} t_1 + t_2, t_1 + \frac{1}{2} t_2, \frac{1}{2} t_1 + \frac{1}{2} t_2 \), which are not 60° rotation centers. The picture of all these centers is the following:
Problem 5. For the attached square grid, determine $t_1$ and $t_2$ and find the point group $G_0$.

Solution. Here is one choice for $t_1$ and $t_2$:

We have a $90^\circ$ rotation $r$ about the origin in $G$, and we have 4 linear reflections $f_1, f_2, f_3, f_4$, about a horizontal line, a vertical line, and the two lines making a $45^\circ$ angle with the $x$-axis and one making a $-45^\circ$ angle. We then have 8 linear isometries $\{I, r, r^2, r^3, f_1, f_2, f_3, f_4\}$ in $G$, and the corresponding matrices lie in $G_0$. Thus, $|G_0| \geq 8$. The only possibilities are then $G_0 = D_4$ or $G_0 = D_6$. However, $D_6$ does not contain a $90^\circ$ rotation. Thus, $G_0 = D_4$. 