Math 526
Take-Home Exam 1 Solutions

In these problems, suppose that $G$ is the symmetry group of a wallpaper pattern, and that the translation subgroup has the form $T = \{(I, nt_1 + mt_2) : n, m \in \mathbb{Z}\}$ for some vectors $t_1, t_2$. Let $V = \{nt_1 + mt_2 : n, m \in \mathbb{Z}\}$, the translation lattice. Thus, $T = \{(I, v) : v \in V\}$.

Let $G_0$ be the point group of $G$.

**Problem 1.** Let $(A, w) \in G$. Prove that $(A, w)$ commutes with each translation in $G$ if and only if $A = I$. Use this to prove that $G$ is Abelian if and only if $G_0 = \{I\}$.

**Solution.** Suppose that $(A, w) \in G$ commutes with each translation in $G$. Let $(I, v) \in T$. Then $(A, w)(I, v) = (I, v)(A, w)$. From our formula for composing isometries, this reduces to $(A, w + Av) = (A, w + v)$. Thus, $w + Av = w + v$, or $Av = v$. This holds for all $v \in V$. Therefore, $At_1 = t_1$ and $At_2 = t_2$. Since $\{t_1, t_2\}$ form a basis for $\mathbb{R}^2$, we have seen that this forces $A = I$. Conversely, if $A = I$, then $(I, w) \in T$. Since $T$ is Abelian, any two translations commute, so $(I, w)$ commutes with each translation.

Now, suppose that $G$ is Abelian. Then each $(A, w)$ commutes with everything in $G$, so with each translation in $G$. By the previous paragraph, this forces $A = I$ for each $(A, w) \in G$. This means $G_0 = \{I\}$. Conversely, if $G_0 = \{I\}$, then each element of $G$ is a translation, by the definition of $G_0 = \{A : (A, w) \in G \text{ for some } w \in \mathbb{R}^2\}$. Then $G = T$ is Abelian.

**Problem 2.** Let $G$ be a wallpaper group. If $G = \{(A, v) : A \in G_0, v \in V\}$, prove that $G_0$ is isomorphic to a subgroup of $G$, and that $G$ is isomorphic to the semidirect product $T \times_\varphi G_0$, where $\varphi : G_0 \to \text{Aut}(T)$ is the usual action of $G_0$ on $T$, given by $\varphi(A)(v) = Av$.

**Solution.** Let $H = \{(A, 0) : A \in G_0\}$. Then $H$ is a subset of $G$ by the hypothesis that $G = \{(A, v) : A \in G_0, v \in V\}$. It is nonempty since $(I, 0) \in H$, as $I \in G_0$. Next, if $(A, 0), (B, 0) \in H$, then $(A, 0)(B, 0) = (AB, 0)$, and since $AB \in G_0$ (as $A, B \in G_0$ and $G_0$ is a group), we get $(AB, 0) \in H$. Also, $(A, 0)^{-1} = (A^{-1}, 0) \in H$ since $A^{-1} \in G_0$. Thus, $H$ is a subgroup of $G$. The map $(A, 0) \mapsto A$ is a group isomorphism from $H$ to $G_0$; that is is a homomorphism follows from the equation $(A, 0)(B, 0) = (AB, 0)$, it is onto by definition of $H$, and it is 1-1 since $(A, 0) = (I, 0)$ if and only if $A = I$.

Next, define $F : T \times_\varphi G_0 \to G$ by $F((I, v), A) = (A, v)$. This is a group homomorphism, since if $(I, v), (I, w) \in T$ and $A, B \in G_0$, we have

$$F((I, v), (A, w))(B, w) = F((I, v) \varphi(A)(I, w), AB)$$
$$= F((I, v)(I, Aw), AB)$$
$$= F((I, v + Aw, AB) = (AB, v + Aw)$$
$$= (A, v)(B, w) = F((I, v), A)F((I, w), B).$$
The homomorphism $F$ is onto, since if $(A, v) \in G$, then $A \in G_0$ and $v \in V$, so $(I, v) \in T$, and $(A, v) = F((I, v), A)$. Finally, $F$ is 1-1, since

$$\ker(F) = \{(I, v), A) : F((I, v), A) = (I, 0)\}$$

$$= \{(I, v), A) : (A, v) = (I, 0)\}$$

$$= \{(I, 0), A)\}$$

is the identity subgroup of $T \times \varphi G_0$. Thus, $F$ is an isomorphism.

Note that $T \cap H = \{(I, 0)\}$ and $G = TH$. This means $G$ is the internal semidirect product of $T$ and $H$. Then $G \cong T \times_\theta H$, where $\theta : H \rightarrow \text{Aut}(T)$ is given by $\theta((A, 0)) = (A, 0)(I, v)(A, 0)^{-1} = (I, Av)$. In other words, by identifying $H$ with $G_0$ by the isomorphism given above, $\theta$ is the same action as $\varphi$. This gives another way to see that $G \cong T \times_\varphi G_0$.

**Problem 3.** Suppose that a wallpaper group $G$ contains no reflections, but that $g = (f, w) \in G$ is a glide reflection such that the reflection line of $f$ is parallel to $t_2$ and perpendicular to $t_1$. Prove that $g^2 = (I, u)$ for some vector parallel to $t_2$, and that $w \notin V$. Prove that there is a glide reflection $h$ in $G$ for which $h^2 = (I, t_2)$.

(Hint: Write $w$ as a linear combination of $t_1$ and $t_2$. For the last part, try composing $g$ with a translation in $G$.)

**Solution.** We have $g^2 = (f, w)(f, w) = (f^2, f(w) + w) = (I, f(w) + w)$ since $f$ is a reflection. Since $f$ is linear, $f(f(w) + w) = f(f(w)) + f(w) = w + f(w)$, so the vector $u := f(w) + w$ is on the reflection line of $f$. Thus, it is parallel to $t_2$. If $w \in V$, then $(I, -w)(f, w) \in G$, but this element is equal to $(f, 0)$. This forces $G$ to contain a reflection, which is false. Thus, $w \notin V$. For the last part, write $w = \alpha t_1 + \beta t_2$ for some $\alpha, \beta \in \mathbb{R}$. Then, since $f(t_1) = -t_1$ and $f(t_2) = t_2$, by the geometric description in the hypotheses, we have $f(w) = -\alpha t_1 + \beta t_2$. Thus, $u = 2\beta t_2$. Since $u \in V$, we must have $2\beta$ is an integer. If $\beta$ is an integer, then $\beta t_2 \in V$, and so $(I, -\beta t_2)(f, w) = (f, \alpha t_1) \in G$. However, since $\alpha t_1$ is perpendicular to the reflection line of $f$, the isometry $(f, \alpha t_1)$ is a reflection in $G$. This is impossible. So, $\beta$ is not an integer. Writing $\beta = n + r$ with $n$ an integer and $0 < r < 1$, we see that $2\beta = 2n + 2r$ being an integer forces $r = 1/2$. Using this, we set $h = (I, -nt_2)(f, w) = (f, \alpha t_1 + \frac{1}{2} t_2)$. Then $h^2 = (I, t_2)$.

**Problem 4.** For the attached hexagon grid, determine $t_1$ and $t_2$, and draw all rotation centers on or inside the parallelogram formed by $t_1$ and $t_2$. Identify which rotation centers are for $60^\circ$ rotations, which are for $120^\circ$ rotations, and which are for $180^\circ$.

1. (If a point is a $60^\circ$ rotation center, it is also a $120^\circ$ and a $180^\circ$ rotation center. Only mark it as a $60^\circ$ rotation center.)

**Solution.** One choice of $t_1$ and $t_2$ is shown in the picture below.
Another choice would be $s_1 = t_1$ and $s_2 = t_1 + t_2$. Let $r$ be the 60° rotation about the origin; this is a symmetry of the grid. We then have $r(t_1) = t_1 + t_2$ and $r(t_2) = -t_1$. With respect to the basis \{t_1, t_2\}, $r$ is represented by the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$  

A point $\alpha t_1 + \beta t_2$ is a center of a 60° rotation center if

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (I - A)^{-1} \begin{pmatrix} n \\ m \end{pmatrix}$$

for some $n, m$; this follows from Lemma 2.1. The corresponding rotation is $(r, nt_1 + mt_2)$. We see that $(\alpha, \beta) = (n - m, n)$. Thus, $(\alpha, \beta)$ can be any pair of integers, and these points correspond to the centers of the hexagons. Next, consider $r^2$, a 120° rotation. The matrix representing $r^2$ with respect to the basis \{t_1, t_2\} is

$$A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

and $\alpha t_1 + \beta t_2$ is a 120° rotation center if

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (I - A^2)^{-1} \begin{pmatrix} n \\ m \end{pmatrix},$$

which yields $(\alpha, \beta) = \frac{1}{3}(2n - m, n + m)$. From a previous homework problem, we see that there are two points inside the parallelogram, $\frac{2}{3}t_1 + \frac{1}{3}t_2$ and $\frac{1}{3}t_1 + \frac{2}{3}t_2$. Finally, for $r^3 = -I$, a 180° rotation, the centers, from previous work we have done, are $\{\frac{1}{3}v : v \in V\}$. We then get the points $\frac{1}{2}t_1, \frac{1}{2}t_2, \frac{1}{2}t_1 + t_2, t_1 + \frac{1}{2}t_2, \frac{1}{2}t_1 + \frac{1}{2}t_2$, which are not 60° rotation centers. The picture of all these centers is the following:
Problem 5. Let $f$ and $g$ be isometries of the plane. Suppose that there are three non-collinear points $P_1, P_2, P_3$ for which $f(P_i) = g(P_i)$ for each $i$. Prove that $f = g$.

Solution. There are orthogonal matrices $A, B$ and vectors $v, w$ with $f = (A, v)$ and $g = (B, w)$. The equations $f(P_i) = g(P_i)$ then can be written as $AP_i + v = BP_i = w$. Subtracting, we get $A(P_1 - P_2) = B(P_1 - P_2)$ and $A(P_1 - P_3) = B(P_1 - P_3)$. Now, since $P_1, P_2, P_3$ are not collinear, $P_1 - P_2$ and $P_1 - P_3$ are nonzero, nonparallel vectors. Thus, they form a basis for $\mathbb{R}^2$. This forces $A = B$. Then, from $f(P_1) = g(P_1)$, we get $AP_1 + v = AP_2 + w$, which yields $v = w$. Thus, $f = (A, v) = (B, w) = g$. 