Problem 1. Let \( \{t_1, t_2\} \) be an integral basis for \( V \). Prove that \( \{t_1, t_1 + t_2\} \) is another integral basis for \( V \).

Solution. Because the set \( \{t_1, t_1 + t_2\} \) has two elements, if it spans \( V \), then it is linearly independent, and so is a basis. Let \( t \in V \). Then \( t = nt_1 + mt_2 \) for some \( n, m \in \mathbb{Z} \). We then have \( t = nt_1 - mt_1 + m(t_1 + t_2) = (n - m)t_1 + m(t_1 + t_2) \). Since both \( n - m \) and \( m \) are integers, each \( t \in V \) is an integral linear combination of \( t_1, t_1 + t_2 \). Thus, \( \{t_1, t_1 + t_2\} \) is an integral basis for \( V \).

Problem 2. Let \( \{t_1, t_2\} \) be an integral basis for \( V \). Determine, with proof, whether or not \( \{t_1, t_1 + 2t_2\} \) is an integral basis for \( V \).

Solution. The set \( \{t_1, t_2 + 2t_2\} \) is not an integral basis for \( V \); for if it were, then we could write \( t_2 = nt_1 + m(t_1 + 2t_2) \) for some \( n, m \in \mathbb{Z} \). This would say \( t_2 = (n + m)t_1 + 2mt_2 \). By equating coefficients, this forces \( 0 = n + m \) and \( 1 = 2m \). This is using the fact that \( \{t_1, t_2\} \) is a basis for \( \mathbb{R}^2 \), and so each vector in \( \mathbb{R}^2 \) has a unique representation as a linear combination of them. These equations force \( m = 1/2 \) and \( n = -1/2 \). Since these are not integers, the original assumption is false.

Problem 3. Let \( \{t_1, t_2\} \) be an integral basis for \( V \). If \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Gl}_2(\mathbb{Z}) \), prove that \( \{at_1 + bt_2, ct_1 + dt_2\} \) is an integral basis for \( V \).

Solution. Let \( t \in V \). Then \( t = nt_1 + mt_2 \) for some \( n, m \in \mathbb{Z} \). We need to show that \( t \) is an integral linear combination of \( at_1 + bt_2 \) and \( ct_1 + dt_2 \). This amounts to showing that we can find \( x, y \in \mathbb{Z} \) with \( nt_1 + mt_2 = x(at_1 + bt_2) + y(ct_1 + dt_2) \). Expanding the right-hand side and equating coefficients yields

\[
\begin{align*}
n &= ax + cy \\
m &= bx + dy,
\end{align*}
\]

or, the single matrix equation

\[
\begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},
\]

which yields

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} \begin{pmatrix} n \\ m \end{pmatrix}.
\]

The \( 2 \times 2 \) matrix in the first equation is in \( \text{Gl}_2(\mathbb{Z}) \), so its inverse is also in \( \text{Gl}_2(\mathbb{Z}) \). Thus, \( x, y \) are both integers, which is what we needed.
Problem 4. If $G_0 = D_6$, determine the matrix representations of the $60^\circ$ rotation and the horizontal reflection if you use the basis $\{t_1, t_1 + t_2\}$ rather than $\{t_1, t_2\}$.

The vectors $t_1, t_2$ have the same length, and are separated by a $120^\circ$ angle; see the picture in Problem 5. Thus, $r(t_1) = t_1 + t_2$ and $r(t_2) = t_1$. Also, $f(t_1) = t_1$ and $f(t_2) = -t_1 - t_2$. Writing in terms of the new basis, we have $r(t_1 + t_2) = t_2 = -t_1 + t_1 + t_2$ and $f(t_1 + t_2) = -t_2 = t_1 - (t_1 + t_2)$. Thus, the matrices in question are

$$
\begin{align*}
r & \leftrightarrow 
\begin{pmatrix}
0 & -1 \\
1 & 1 
\end{pmatrix}, \\
f & \leftrightarrow 
\begin{pmatrix}
1 & 1 \\
0 & -1 
\end{pmatrix}.
\end{align*}
$$

Problem 5. Suppose that $G_0 = D_{3,1}$ or $G_0 = D_6$. Suppose that the 3 reflections in $D_{3,1}$ (respectively the 6 reflections in $D_6$) are actually elements of the wallpaper group $G$. (We will see that this always happens.) Show that the long diagonal in the basic parallelogram determined by $\{t_1, t_2\}$ is a reflection line of a reflection in $G$. You might want to look back at the picture of the reflection lines on Page 38 of the textbook. Lemma 2.2 may prove useful.

Solution. Let $f$ be the reflection through the origin parallel to the long diagonal. By assumption, $(f, 0) \in G$. Now, the long diagonal is obtained by shifting the reflection line of $f$ by $\frac{1}{2}(t_1 + t_2)$. Thus, by Lemma 2.2, we see that $\tau_{t_1 + t_2} \circ f = (f, t_1 + t_2)$ is the reflection across the diagonal. Since $(f, t_1 + t_2) = (I, t_1 + t_2) \circ (f, 0)$ is the composition of two elements of $G$, the result is also an element of $G$. Thus, reflection across this line is a symmetry of the corresponding wallpaper pattern.

Problem 6. Suppose that $G_0 = D_4$, and that the vertical and horizontal reflections through the origin are elements in $G$. (This does not always happen.) Show that the lines in the following diagram are all reflection lines of reflections in the group $G$. 

![Diagram showing reflection lines](image-url)
Solution. Let $h$ and $v$ be horizontal and vertical reflections (fixing the origin), respectively. Then the horizontal lines are reflection lines of the reflections $(h, 0), (h, t_2), (h, 2t_2)$, and the vertical lines are reflection lines of $(v, 0), (v, t_1), (v, 2t_1)$; this follows from Lemma 2.2. Since $(v, 0), (h, 0) \in G$, all six of these reflections are in $G$, by the same reasoning as in Problem 5.