Symmetry Groups:
The Classification of Wallpaper Patterns
Mathematics 482/526

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# Contents

Preface vii

Original Preface ix

1 Introduction 1

2 Symmetry Groups 3
   2.1 Isometries of the Plane . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
   2.2 Examples of Symmetry Groups . . . . . . . . . . . . . . . . . . . . . . . . . 7
   2.3 Background in Group Theory . . . . . . . . . . . . . . . . . . . . . . . . . . 11
   2.4 The Group Structure of Isom(\(\mathbb{R}^2\)) . . . . . . . . . . . . . . . . . . 17
   2.5 Structure of \(O_2(\mathbb{R})\) . . . . . . . . . . . . . . . . . . . . . . . . . . 20
   2.6 Symmetry Groups of Bounded Figures . . . . . . . . . . . . . . . . . . . . . . 22
   2.7 Wallpaper Patterns . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24

3 The Point Group of a Wallpaper Pattern 29
   3.1 Definition and Main Properties . . . . . . . . . . . . . . . . . . . . . . . . . 29
      3.1.1 Exercises . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 34
   3.2 The Five Lattice Types of Wallpaper Patterns . . . . . . . . . . . . . . . . . 34
      3.2.1 Exercises . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 43

4 The Wallpaper Groups 45
   4.1 Notational Description of the Wallpaper Groups . . . . . . . . . . . . . . . . 45
   4.2 Classification of Wallpaper Patterns . . . . . . . . . . . . . . . . . . . . . . . 47
      4.2.1 Exercises . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 54
   4.3 Description of the Wallpaper Groups . . . . . . . . . . . . . . . . . . . . . . . 55
   4.4 Fundamental Domains . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 90
   4.5 Arithmetic Aspects of the Wallpaper Groups . . . . . . . . . . . . . . . . . . 90

A Group Cohomology 101
   A.1 Group Extensions . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 101
   A.2 Group Extensions of \(T\) by \(G_0\) . . . . . . . . . . . . . . . . . . . . . . . 105
   A.3 Higher Cohomology Groups . . . . . . . . . . . . . . . . . . . . . . . . . . . 107
A.4 Calculation of the Groups $H^2(G_0, T)$ .............................. 108

List of Symbols ............................... 115

Bibliography .................................. 117

Index ......................................... 119
List of Figures

4.1 Wallpaper patterns with groups $pm$ and $pg$ ................................. 51
4.2 Wallpaper patterns with groups $p4m$ and $p4g$ .............................. 52
4.3 Wallpaper patterns with groups $pmm$, $pmg$, and $pgg$ .................. 54
4.4 $p2$ rotation centers ............................................................... 91
4.5 $p6/p6m$ rotation centers and reflection lines ............................... 94
4.6 $p3$ rotation centers ............................................................... 96
4.7 Wallpaper patterns with groups $p3m1$ and $p31m$ ......................... 96
4.8 $p3m1$ and $p31m$ reflection lines ............................................. 97
4.9 $p4$ rotation centers ............................................................... 98
4.10 $p4m$ and $p4g$ reflection lines ................................................ 98
4.11 Wallpaper patterns with groups $pmm$, $pmg$, and $pgg$ ............... 100
Preface

The current version of this manuscript was revised to be used as a textbook for Mathematics 482/526 in Spring 2007, a topics course in symmetry groups. The original manuscript, written after I gave a series of lectures in the algebra seminar, was intended for professional mathematicians and graduate students as a description of how to obtain the classification of wallpaper patterns through group cohomology. After deciding to teach a course in symmetry groups, I added some background material, moved the cohomology stuff to an appendix, and made a number of assorted changes in order to use the manuscript as a textbook.

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Original Preface

I first started to think seriously about symmetry groups while teaching a senior-level abstract algebra course in Fall 1999, during which we determined symmetry groups of the Platonic solids. Soon afterward I worked out for myself the classification of the symmetry groups of bounded plane figures and of strip patterns, neither of which is not very hard. From that I decided to try to understand wallpaper patterns. A big step was becoming aware of Mackiw’s book [3], which contains two chapters on wallpaper patterns, I began to read through the second of these chapters, which does the classification. Fairly soon I realized that a wallpaper group is a group extension of its subgroup of translations, a group isomorphic to \( \mathbb{Z}^2 \), and its point group. At this point I started to think about group cohomology. This is an important tool in the theory of finite-dimensional division algebras, my primary area of research, and it is used to classify group extensions. It seemed to me likely that the classification of wallpaper groups should be the same, or nearly so, as the classification of group extensions, even though two group extensions \( 1 \to T \to G \to G_0 \to 1 \) and \( 1 \to T \to G' \to G_0 \to 1 \) can be inequivalent even if \( G \) and \( G' \) are isomorphic. In order to carry out the classification of group extensions, I needed to calculate several cohomology groups. After a little bit of thought, I realized that I needed to use spectral sequences to do this. At this point I decided it was time to finally buckle down and learn about these; I had some familiarity with spectral sequences from listening to various research talks, but I had never studied them in detail. After working through the spectral sequence chapter in Weibel [7], I found that calculating the cohomology groups that arise from wallpaper groups was very easy; it only took quite simple applications of the Lyndon-Hochschild-Serre spectral sequence. Doing so gives 18 inequivalent group extensions; this shows that there is very little difference between group extensions and wallpaper groups. A little more work then showed that only two inequivalent group extensions corresponded to isomorphic symmetry groups.

Armed with this cohomological classification of wallpaper groups, I proceeded to present these ideas in a series of lectures in our algebra seminar (Spring 2000). Because my enjoyment of working through this material and discussions with colleagues, I decided to write this up in a monograph. I knew that nothing I had done was original. However, I was a bit surprised that any details of the classification is not so well known, even though many people know about it. Many books on symmetry and Escher’s tessellations point out the classification without proving it. Furthermore, I liked very much how tightly related the classification of the wallpaper groups is to the classification of the corresponding group
extensions. I thought that doing the classification via group cohomology was a very nice application of this abstract and technical area of algebra. In fact, at the end of a graduate level course in group cohomology I taught in Fall 2000, I gave several examples of computing cohomology groups via spectral sequences by considering wallpaper groups. However, to make this monograph more accessible, I have included a section classifying the wallpaper groups without resorting to any cohomology. By ignoring the chapter on group cohomology, a well-prepared undergraduate can follow the classification given here.

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Chapter 1

Introduction

It is often said that group theory is the study of symmetry. In this book we will use group theory, geometry, and linear algebra to classify the symmetry of certain two-dimensional figures called wallpaper patterns. The way one classifies symmetry of geometric objects is to associate to the object a group, called its symmetry group, and then to classify the possible symmetry groups. The study of symmetry groups of wallpaper patterns began in the nineteenth century by people studying crystals, which exhibit a repeating structure in three dimensions. By the end of the nineteenth century, the classification of the so-called crystallography groups in dimensions 2 and 3 was completed by Fedorov, Schoenflies, and Barlow, building on work by several others.

While many mathematicians know that there are exactly 17, up to isomorphism, symmetry groups of wallpaper patterns, most do not know why this is true. The purpose of this book is to obtain the classification and at the same time see a nice theory which ties group theory, geometry, and linear algebra together. Another purpose of the book is to make a connection between artistic aspects of drawing tessellations and group-theoretic concepts.

The classification and organization of this book begins with the definition of wallpaper patterns, their translation lattices, and their symmetry groups, which are groups of isometries. In Chapter 2 we describe the different types of isometries of the plane and the structure of the group of isometries of the plane. Chapter 3 begins the classification in earnest; we define the point group of a wallpaper pattern and use it to describe the five lattice types of wallpaper patterns. We finish the classification in Chapter 4, and then discuss some further connections between geometry and the descriptions of the groups. We also illustrate similarities and differences between the groups, illustrating these points with examples of wallpaper patterns. In Appendix A, we give a classification of the wallpaper groups through group cohomology.

To illustrate the symmetry groups, we will look closely at many of Escher’s tessellations. These beautiful works of art illustrate very nicely group-theoretic aspects of the symmetry groups. Moreover, studying the symmetry groups helps to understand the geometric restrictions Escher had to discover in order to create his tessellations. Escher gave us a large body
of art with which we can use to illustrate the mathematical ideas involved in describing symmetry groups of wallpaper patterns.
Chapter 2

Symmetry Groups

An isometry of $\mathbb{R}^2$ is a distance-preserving bijection. Let $\text{Isom}(\mathbb{R}^2)$ be the set of isometries of $\mathbb{R}^2$. A simple argument will show that the composition of two isometries is an isometry and that the inverse of an isometry is an isometry. Therefore, $\text{Isom}(\mathbb{R}^2)$ is a group under composition of functions. If $W$ is a subset of $\mathbb{R}^2$, then the symmetry group of $W$ is defined to be

$$\text{Sym}(W) = \{ \varphi \in \text{Isom}(\mathbb{R}^2) : \varphi(W) = W \}.$$ 

In the next section we describe four types of isometries. Later, we will describe geometrically all isometries of the plane.

2.1 Isometries of the Plane

We now describe four classes of isometries of the plane. We will see in Section 2.5 that every isometry is one of these types.

Translations.

If $v \in \mathbb{R}^2$, then we will refer to the map $\tau_v$, given by $\tau_v(x) = x + v$ for all $x \in \mathbb{R}^2$, as the translation by $v$. It is elementary to see that $\tau_v$ is an isometry, for if $x, y \in \mathbb{R}^2$, then

$$\|\tau_v(x) - \tau_v(y)\| = \|(x + v) - (y + v)\| = \|x - y\|.$$ 

Note that $\tau_v \circ \tau_w = \tau_{v+w}$, and that the inverse of $\tau_v$ is $\tau_{-v}$.
Reflections.

Let $\ell$ be a line in $\mathbb{R}^2$. Then the reflection across $\ell$ is an isometry, which one can see by a purely geometric argument, although we give a more algebraic argument below.

If $\ell$ is the line through the origin parallel to a vector $w$, then the reflection across $\ell$ is given by

$$f(x) = 2\left(\frac{x \cdot w}{w \cdot w}\right)w - x.$$  

This comes from the formula for the projection of one vector onto another, which one sees in multivariable calculus. From it a straightforward calculation will show that $f$ is an isometry. This formula also will show that $f$ is a linear transformation. For an arbitrary reflection $g$, let $\tau$ be a translation that sends some fixed point on the reflection line $\ell$ of $g$ to the origin. Then $\tau$ sends $\ell$ to a line $\ell'$ through the origin. If $f$ is the reflection about $\ell'$, then $g = \tau^{-1} \circ f \circ \tau$, and so $g$ is an isometry.

Rotations.

If $\theta$ is an angle, then the rotation $r$ by an angle $\theta$ about the origin is given in coordinates by

$$r\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array}\right)\left(\begin{array}{c} x \\ y \end{array}\right) = \left(\begin{array}{c} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{array}\right).$$

From this formula one can see that a rotation about the origin is an isometry and is a linear transformation. We can use it to describe a rotation about any point. If $r'$ is the rotation by $\theta$ about a point $P \in \mathbb{R}^2$, and if $\tau$ is translation by $P$, then $r' = \tau \circ r \circ \tau^{-1}$. As a consequence, any rotation is an isometry. If a rotation $r$ is not the identity map, then we say that $r$ is a nontrivial rotation.
Glide Reflections.

We can produce new isometries by composition. For example, we may compose a reflection and a translation. The result may be another reflection; see Lemma 2.2 below, although it may be another type of isometry. We will call a composition of a reflection and a translation a *glide reflection*. If a glide reflection is not a reflection, then we say it is non-trivial.

We will see in Section 2.5 below that any isometry of \( \mathbb{R}^2 \) is a composition of a translation with either a rotation or a reflection. Therefore, we have accounted for all types of isometries of the plane.

Exercises

1. From the formula given above, show that a reflection across a line passing through the origin is both a linear transformation and an isometry.

2. Let \( f \) be the reflection across a line \( \ell \). If \( \tau \) is a translation, and \( \tau(\ell) = \ell' \), show that \( \tau^{-1} \circ f \circ \tau \) is the reflection across \( \ell' \).

   (You may use the fact that if an isometry fixed each point on a line, then it is the reflection across the line.)

3. From the formula given above, show that a rotation about the origin is both a linear transformation and an isometry.

4. If \( r \) is the rotation about the origin by an angle \( \theta \), and \( v = \overrightarrow{OP} \) for some point \( P \), show that \( \tau_v \circ r \circ \tau_{v}^{-1} \) is the rotation about \( P \) by the angle \( \theta \).

5. Use Geometer’s Sketchpad to illustrate the facts in Problems 2 and 4.

6. Let \( \ell \) be a line through the origin, and let \( f \) be the reflection across \( \ell \). Suppose that \( v \in \mathbb{R}^2 \), and write \( v = v_1 + v_2 \) with \( v_1 \in \ell \) and \( v_2 \) perpendicular to \( \ell \). If \( g = \tau_v \circ f \), prove that \( g \) is a reflection if and only if \( v_1 = 0 \).
Some Arithmetic Facts

We point out some properties of these classes of isometries. We start with some of the most simple properties. A translation has no fixed points, a rotation has a unique fixed point, and a reflection has a fixed line. A nontrivial translation has infinite order; that is, if \( \tau \neq \text{id} \) is a translation, then \( \tau^n \neq \text{id} \) for all integers \( n > 0 \). A reflection has order 2; thus, a reflection is its own inverse. If \( r \) is a rotation about a point \( P \) by an angle \( \theta \), then \( r^{-1} \) is rotation by \(-\theta\) about the origin. Moreover, if \( \theta = 2\pi/n \) for some integer \( n \), then \( r^n = \text{id} \). Finally, if \( g \) is a glide reflection that is not a reflection, then we claim that \( g^2 \) is a nontrivial translation.

To help us prove this, we point out a couple of facts about reflections. Let \( f \) be a reflection about a line \( \ell \) through the origin. A vector \( x \) lies on the line through the origin that is perpendicular to \( \ell \) if and only if \( f(x) = x \), and \( x \in \ell \) if and only if \( f(x) = -x \). By working with an appropriate basis, every vector in \( \mathbb{R}^2 \) may be written in the form \( u + v \) with \( u \in \ell \) and \( v \) perpendicular to \( \ell \).

We now prove two lemmas that describe various compositions of isometries. While these results are interesting in their own right, we will use them primarily to determine when two symmetry groups are not isomorphic.

**Lemma 2.1.** \(^1\) If \( r \) is a nontrivial rotation about the origin by an angle \( \theta \), and if \( v \) is a vector, then \( \tau_v \circ r \) is a rotation about \(-r(1-v)(1)\) by the angle \( \theta \).

**Proof.** We take for granted that each isometry is either a translation, reflection, rotation, or glide reflection; we will prove this later. Rotations are then distinguished among all isometries in that they have a unique fixed point. Suppose that \( \tau_v \circ r \) has a fixed point \( x \in \mathbb{R}^2 \). Then \( r(x) + v = x \), so \( v = x - r(x) = (\text{id} - r)(x) \). However, since \( r \) is a nontrivial rotation about the origin, it is a linear transformation. From this fact and the representation of a rotation by a matrix by using the standard basis for \( \mathbb{R}^2 \), we see that \( \text{id} - r \) is invertible since \( \det(\text{id} - r) \neq 0 \). Thus, \( x = (\text{id} - r)^{-1}(v) \) is the unique fixed point of \( r \), which means that it is the center of the rotation. To see that the rotation angle for \( \tau_v \circ r \) is \( \theta \),

![Diagram](image)

we note that the picture above has the vectors starting at the origin. If they start where the vector \( v \) terminates, then this picture will represent \( \tau_v \circ r \), and so this rotation also has angle \( \theta \).

**Lemma 2.2.** Let \( f(x) \) be a reflection about a line \( \ell \) passing through the origin, and let \( v \in \mathbb{R}^2 \). If \( g \) is the glide reflection \( g = \tau_v \circ f \), then \( g \) is a reflection if and only if \( v \) is perpendicular to \( \ell \).

\(^1\)fix
to \( \ell \). When this occurs, the reflection line of \( g \) is \( \ell + \frac{1}{2}v \). If \( v \) is not perpendicular to \( \ell \), then \( g \) is a non-trivial glide reflection and \( g^2 \) is translation by \( v + f(v) \), a vector on the line \( \ell \).

Proof. Recall that \( f \) is a linear transformation since \( \ell \) contains the origin. The glide \( g = \tau_v \circ f \) is a reflection if and only if it fixes a vector \( w \). If \( f(w) + v = w \), then \( v = w - f(w) \). However, this forces

\[
f(v) = f(w - f(w)) = f(w) - f^2(w) = f(w) - w
\]

\[
= -v.
\]

Therefore, if \( g \) is a reflection, then \( v \) is perpendicular to \( \ell \). Conversely, if \( v \) is perpendicular to \( \ell \), then \( f(v) = -v \), so \( g(\frac{1}{2}v) = \frac{1}{2}v \). This forces \( g \) to be a reflection; it fixes the line \( \ell + \frac{1}{2}v \). Thus, \( g \) is a non-trivial glide reflection if and only if \( v \) is not perpendicular to \( \ell \). Next, we consider \( g^2 \). If \( x \in \mathbb{R}^2 \), then

\[
g^2(x) = f(f(x) + v) + v = f^2(x) + f(v) + v = x + f(v) + v
\]

since \( f \) is linear. Therefore, \( g^2 \) is translation by \( f(v) + v \). Since \( f(f(v) + v)) = f^2(v) + f(v) = v + f(v) \), this translation vector is fixed by \( f \), so it is on the line \( \ell \). 

\[\square\]

2.2 Examples of Symmetry Groups

The examples we give in this section will help us to understand the definition of symmetry group and to introduce as symmetry groups two important classes of groups, the cyclic and dihedral groups.

Example 2.3. We calculate the symmetry group of an equilateral triangle \( T \). For convenience, we view the origin as the center of the triangle.

The identity function is always a symmetry of any figure. Also, the rotations about the origin by an angle of 120° or 240° are elements of Sym(\( T \)). Furthermore, the three reflections across the three lines of the following picture are also isometries.
We have so far found six isometries of the triangle. We claim that \(\text{Sym}(T)\) consists precisely of these six isometries. To see this, first note that any isometry must send the set \(\{A, B, C\}\) of vertices to itself. There are only six 1-1 functions that send \(\{A, B, C\}\) to itself, and a homework problem shows that any isometry is determined by what it does to three non-collinear points, so we have found all symmetries of the triangle.

If \(r\) is the rotation by 120° and \(f\) is any of the reflections, then \(f, rf, r^2f\) are the three reflections (see Problem 1 below). Moreover, \(r^2\) is the 240° rotation. Therefore, \(\text{Sym}(T) = \{e, r, r^2, f, rf, r^2f\}\). The elements \(r\) and \(f\) satisfy \(r^3 = e\) and \(f^2 = e\). Another exercise will show that \(r\) and \(f\) are related via the relation \(fr = r^2f\). Alternatively, this equation may be rewritten as \(frf = r^{-1}\), since \(f^{-1} = f\) and \(r^2 = r^{-1}\).

**Example 2.4.** Now let us determine the symmetry group of a square, centered at the origin for convenience. As with the triangle, we see that any isometry that preserves the square must permute the four vertices. There are 24 permutations of the vertices. However, not all come from isometries. First, the rotations of 0°, 90°, 180°, and 270° about the origin are symmetries of the square. Also, the reflections about the four dotted lines in the picture below are also symmetries.

We now have eight symmetries of the square and we claim that this is all. To see this, we give a counting argument. There are four choices for where vertex \(A\) can be sent since it must go to one of the vertices. Once a choice has been made, there are just two choices for where \(B\) is sent since it must go to a vertex adjacent to the image of \(A\); this is forced upon us since isometries preserve distance. After images for \(A\) and \(B\) have been chosen, the images of the other two vertices are then fixed; \(C\) is sent to the vertex across from the image of \(A\), and \(D\) is sent to the vertex across from the image of \(B\). So, there is a total of \(4 \cdot 2 = 8\) possible isometries. Since we have found eight, we have them all.
If \( r \) is the rotation by \( 90^\circ \) and \( f \) is any reflection, then the four reflections are \( f, rf, r^2f, r^3f \); this can be seen by an exercise similar to that needed in the previous example. The four rotations are \( r, r^2, r^3, r^4 = e \). Thus, the symmetry group of the square is \( \{e, r, r^2, r^3, f, rf, r^2f, r^3f\} \). We have \( r^4 = e \) and \( f^2 = e \), and an exercise shows that \( fr = r^3f \), or \( frf = r^{-1} \). This is the same relation that holds for the corresponding elements in the previous example.

**Example 2.5.** If we generalize the previous two examples by considering the symmetry group of a regular \( n \)-gon, then we would find that the symmetry group has \( 2n \) elements. If \( r \) is a rotation by \( 360^\circ / n \) and if \( f \) is a reflection, then the rotations in the group are the powers \( r, r^2, \ldots, r^{n-1}, r^n = e \) of \( r \), and the reflections are \( f, rf, \ldots, r^{n-1}f \). Thus, half of the elements are rotations and half are reflections. The elements \( r \) and \( f \) satisfy the relations \( r^n = e \), \( f^2 = e \), and \( frf = r^{-1} \). This group is called the Dihedral group, and is denoted \( D_n \). We also consider the subgroup of all rotations in \( D_n \). This is the cyclic subgroup \( \langle r \rangle = \{e, r, \ldots, r^{n-1}\} \). We denote this group by \( C_n \). This group also arises as a symmetry group. We show how \( C_4 \) arises in the following example.

**Example 2.6.** The following figure has only rotations in its symmetry group; because it has rotations only of \( 0^\circ, 90^\circ, 180^\circ, \) and \( 270^\circ \), its symmetry group is \( C_4 \).

By drawing similar but more complicated pictures, we can represent \( C_n \) as a symmetry group of some plane figure.

**Example 2.7.** The Zia symbol has symmetry group \( D_4 \); we see that besides rotations by \( 0^\circ, 90^\circ, 180^\circ, \) and \( 270^\circ \), it has horizontal, vertical, and diagonal reflections in its symmetry group, and no other symmetry.
Example 2.8. Consider a circle of radius 1 centered at the origin \(0\). Any isometry of the circle must map the center to itself. Thus, any such isometry is linear, by Proposition 2.22. Conversely, if \(\varphi\) is any linear isometry, and if \(P\) is any point with \(d(P,0) = 1\), then \(d(\varphi(P), \varphi(0)) = d(\varphi(P), 0) = 1\). Thus, \(\varphi\) sends the circle to itself. Therefore, the symmetry group of the circle is the full subgroup of all linear isometries, which we will see is \(O_2(\mathbb{R})\).

Example 2.9. We investigate the symmetry group of the following wallpaper pattern, Escher’s “Horsemen”. In the picture below we have drawn in white two vectors \(t_1, t_2\) which together form a basis for the translation lattice.

We set \(t_1\) to be the horizontal translation and \(t_2\) to be the vertical translation. We set the origin of the plane where the two vectors start. The translation lattice \(T\) of this pattern is then \(T = \{nt_1 + mt_2 : n, m \in \mathbb{Z}\}\). Let \(f\) be a reflection about the vertical line through \(t_2\). This is not a symmetry of the pattern. However, if \(v\) is the vector drawn in blue, then the glide reflection \(g := \tau_v \circ f\) is a symmetry of the pattern. Note that \(g^2 = \tau_{t_2}\). The vertical component of \(v\) is \(\frac{1}{2}t_2\). One mathematical fact Escher had to discover is that the vertical component of \(v\) had to be \(\frac{1}{2}t_2\) in order for the glide \(g\) to be a symmetry of the figure, while the horizontal component of \(v\) has no restriction; we will explore this facts later in the course.

If \(\varphi\) is a symmetry of the pattern, the image of the white horseman whose chin is at the origin must go to a white or a brown horseman under \(\varphi\). If it goes to a white horseman, then \(\varphi\) is a translation, and so \(\varphi = \tau_t\) for some \(t \in T\). If \(\varphi\) sends him to a brown horseman, by composing \(\varphi\) with an appropriate translation \(\tau_t\), then \(\tau_t \circ \varphi\) sends him to the brown horseman whose chin is at the end of \(v\). Therefore, \(\tau_t \circ \varphi = g\). Thus \(\varphi = \tau_{-t} \circ g = \tau_{v-t} \circ f\). Therefore, the only symmetries of this figure are translations \(\tau_t\) for some \(t \in T\), or \(\tau_{v-t} \circ f\) for some \(t \in T\). Because \(T\) is a group, we may write the symmetry group as

\[ G = \{\tau_t : t \in T\} \cup \{\tau_{v+t} \circ f : t \in T\}. \]
The linear isometries \( h \) for which \( \tau_w \circ h \in G \) for some vector \( w \) form the set \( \{\text{id}, f\} \). This is a subgroup of \( O_2(\mathbb{R}) \). We will see later in the course the significance of this group. It is not a subgroup of \( G \), an important fact about the art of the tessellation, another idea we will explore in more detail later.

**Exercises**

1. Let \( G \) be the symmetry group of an equilateral triangle. If \( r \) is a 120° rotation and \( f \) is a reflection, show that \( f, rf \), and \( r^2f \) are the three reflections of \( G \).

2. Show that there are exactly six 1-1 functions from a set of 3 elements to itself.

3. Let \( G \) be the symmetry group of an equilateral triangle. If \( r \) is a 120° rotation and \( f \) is a reflection, so that \( fr = r^2f \). Use this to show that \( frf = r^{-1} \).

4. Let \( G \) be the symmetry group of a square. If \( r \) is a 90° rotation and \( f \) a reflection, show that the four reflections of \( G \) are \( f, rf, r^2f, \) and \( r^3f \).

5. Let \( G \) be the symmetry group of a square. If \( r \) is a 90° rotation and \( f \) a reflection, show that \( fr = r^3f \). Use this to show that \( frf = r^{-1} \).

6. Prove that the composition of two reflections about lines intersecting at a point \( P \) is a rotation about \( P \).

7. Prove that the composition of two rotations about a common point \( P \) is another rotation about \( P \).

8. If \( r \) and \( s \) are rotations about different centers, prove that \( rsr^{-1}s^{-1} \) is a nontrivial translation.

9. If \( r \) is a rotation about \( P \) and \( f \) is a reflection not fixing \( P \), prove that \( rf \) is a glide reflection.

10. Prove that any isometry is determined by what it does to three non-collinear points.

### 2.3 Background in Group Theory

Define normal subgroups, group homomorphisms, isomorphism, kernel, examples such as \( O_2(\mathbb{R}) \). In this section we give a quick review of some group theory notions and discuss several examples of groups. While we typically denote a group by a letter, such as \( G \), more formally we should denote it by a pair \((G, \cdot)\), where \( \cdot \) is the operation on the group.

Let \( G \) be a group. A nonempty subset \( H \) of \( G \) is said to be a subgroup of \( G \) if \( H \) forms a group under the induced operation from \( G \). That is, if \( \cdot \) is the operation on \( G \), if \( (H, \cdot|_{H \times H}) \) is a group, then we say it is a subgroup of \( G \). This is equivalent to (i) \( xy \in H \) for each
$x, y \in H$, and (ii) $x^{-1} \in H$ for each $x \in H$. The first of these conditions says that restricting $\cdot$ to $H$ yields an operation on $H$. If $g \in G$, then $\langle g \rangle := \{g^n : n \in \mathbb{Z}\}$ is the smallest subgroup of $G$ containing $g$. It is called the cyclic subgroup of $G$ generated by $g$. The order of $g$ is defined to be $|\langle g \rangle|$. If this order is finite, then the order of $g$ can be described as the smallest positive integer $m$ satisfying $g^m = 1$. When the order is infinite, $g^m = 1$ forces $m = 0$.

Let $G, G'$ be groups. We write 1 for the identity of a group, unless we denote the operations by $+$, in which case we write 0. A homomorphism from $G$ to $G'$ is a function $\varphi : G \to G'$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for each $x, y \in G$. The homomorphism $\varphi$ is said to be an isomorphism if $\varphi$ is a bijection. We write $G \cong G'$ in this case. The kernel of $\varphi$ is defined to be $\ker(\varphi) := \{g \in G : \varphi(g) = 1\}$. It is easy to see that $\ker(\varphi)$ is a subgroup of $G$. It is even more, however. Note that if $x \in \ker(\varphi)$ and $g \in G$, then

$$\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} = \varphi(g)1\varphi(g)^{-1} = 1.$$ 

Thus, $gxg^{-1} \in \ker(\varphi)$. To abstract this condition, let $N$ be a subgroup of $G$. If $gxg^{-1} \in N$ for each $g \in G$ and each $x \in N$, then $N$ is said to be a normal subgroup of $G$. Thus, $\ker(\varphi)$ is a normal subgroup of $G$.

Let $G$ be a group and let $H$ be a subgroup of $G$. If $g \in G$, then the right coset $Hg$ is the set $Hg = \{hg : h \in H\}$. We let $G/H$ denote the set of all right cosets of $H$ in $G$. If we define

$$Hx \cdot Hy = Hxy,$$

we may not get an operation on $G/H$, since this need not be well-defined. However, this is well-defined if and only if $H$ is a normal subgroup of $G$. In that case, $G/H$ is a group under this operation, with identity $H = H1$ and with $(Hg)^{-1} = H(g^{-1})$. Furthermore, the map $\pi : G \to G/H$, given by $\pi(g) = Hg$ is a surjective group homomorphism with kernel $H$. Furthermore, we have the so-called fundamental homomorphism theorem, which says that if $\varphi : G \to G'$ is a homomorphism, then there is a well-defined injective homomorphism $\overline{\varphi} : G/\ker(\varphi) \to G'$, given by $\overline{\varphi}(Hg) = \varphi(g)$. Moreover, $\im(\overline{\varphi}) = \im(\varphi)$, and so $G/\ker(\varphi) \cong \im(\varphi)$. We see that kernels of homomorphisms and normal subgroups are the same thing; if $N$ is a normal subgroup, then it is the kernel of the canonical map $G \to G/N$, and if $K = \ker(\varphi)$, then $K$ is a normal subgroup, as we saw earlier.

We now consider some examples of groups.

**Example 2.10.** If $A$ is a nonempty set, we denote by $\text{Perm}(A)$ the set of all permutations of $A$. This is a group under composition of functions. If $A = \{1, 2, \ldots, n\}$, we usually write $S_n$ in place of $\text{Perm}(A)$. The group $S_n$ is called the symmetric group on $n$ letters.

**Example 2.11.** If $n \geq 1$ is an integer, let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices over $\mathbb{R}$. Set

$$\text{Gl}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 1\}.$$
Then \( \text{Gl}_n(\mathbb{R}) \) is the set of all invertible \( n \times n \) matrices, and it is a group under matrix multiplication. It has some important subgroups. Let 

\[
\text{Sl}_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) : \det(A) = 1 \}, \\
\text{O}_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) : A^T A = I_n \}, \\
\text{SO}_n(\mathbb{R}) = \text{O}_n(\mathbb{R}) \cap \text{Sl}_n(\mathbb{R}) = \{ A \in \text{O}_n(\mathbb{R}) : \det(A) = 1 \}.
\]

Then each of these is subgroups of \( \text{Gl}_n(\mathbb{R}) \). They are the special linear group, the orthogonal group, and the special orthogonal group, respectively. We will be interested primarily in \( \text{O}_n(\mathbb{R}) \) for the purposes of classifying symmetry groups, although the subgroup \( \text{SO}_n(\mathbb{R}) \) will also be important for us. In fact, we will see that \( \text{O}_2(\mathbb{R}) \) is isomorphic to the group of all linear isometries of the plane.

**Example 2.12.** Let \( n \geq 1 \) be an integer. Let \( \theta = 2\pi/n \), and set

\[
r = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

An elementary, but somewhat tedious, calculation shows that \( \{ r^i f^j : 0 \leq i \leq n-1, 0 \leq j \leq 1 \} \) is a subgroup of \( \text{O}_2(\mathbb{R}) \) and has order \( 2n \). This group is called the Dihedral group \( D_n \). The elements \( r \) and \( f \) satisfy the relations \( r^n = f^2 = I_2 \) and \( frf = r^{-1} \). The subgroup \( C_n := \langle r \rangle \) is a cyclic group of order \( n \).

**Example 2.13.** Let \( \text{O}_2(\mathbb{Z}) = \{ A \in M_2(\mathbb{Z}) : A^T A = I_2 \} \). We see that \( \text{O}_2(\mathbb{Z}) \) is a subgroup of \( \text{Gl}_2(\mathbb{R}) \). Closure under matrix multiplication is the same argument as for \( \text{O}_n(\mathbb{R}) \). If \( A \in \text{O}_2(\mathbb{Z}) \), then the condition \( A^T A = I_2 \) implies that \( \deg(A)^2 = 1 \), so \( \det(A) = \pm 1 \). Thus, \( A^{-1} \) also has integer entries, so \( A^{-1} \in \text{O}_2(\mathbb{Z}) \).

Consider the group \( D_3 = \langle r, t \rangle \), interpreted as a subgroup of \( \text{O}_2(\mathbb{R}) \) as in the previous example. The rotation \( r \) does not have integer entries. However, we use a different basis. Set \( t_1 = (1, 0) \) and \( t_2 = (-1/2, \sqrt{3}/2) = r(t_1) \). Then \( r(t_2) = r^2(t_1) = (-1/2, -\sqrt{3}/2) = -t_1 - t_2 \). Therefore, the rotation \( r \) is represented by the matrix

\[
\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}
\]

with respect to the basis \( \{ t_1, t_2 \} \). If we let \( f \) be the reflection about the \( y \)-axis, then \( f(t_1) = -t_1 \) and \( f(t_2) = t_1 + t_2 \). Therefore, the matrix representing \( f \) with respect to the basis is

\[
\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Consequently, \( D_3 \) is isomorphic to the subgroup of \( \text{O}_2(\mathbb{Z}) \) generated by these two matrices.
Example 2.14. Let $\mathbb{C}^*$ be the set of nonzero complex numbers, and set $S^1 = \{x \in \mathbb{C}^* : |x| = 1\}$. Then $S^1$ is a subgroup of $\mathbb{C}^*$ under multiplication. We recall that $S^1 = \{e^{i\theta} : \theta \in \mathbb{R}\}$, and that the laws of exponents imply that the map $\varphi : (\mathbb{R}, +) \to S^1$ given by $\varphi(\theta) = e^{i\theta}$ is a group homomorphism. It is onto by the description of $S^1$, and the kernel of $\varphi$ is \{2\pi n : n \in \mathbb{Z}\} := 2\pi\mathbb{Z}.

Example 2.15. We shall see later that the map $\psi : \mathbb{R} \to SO_2(\mathbb{R})$ given by $\psi(\theta) = \left(\begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array}\right)$ is a group homomorphism (this can be seen by using the addition formulas for sine and cosine), and its kernel is $2\pi\mathbb{Z}$. Consequently, $SO_2(\mathbb{R}) \cong S^1$, since both are isomorphic to $\mathbb{R}/2\pi\mathbb{Z}$.

Example 2.16. We now consider the main example of this book. Let Isom(\mathbb{R}^2) denote the set of all bijections of $\mathbb{R}^2$ which are isometries; that is, all permutations $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ for which $||\varphi(x) - \varphi(y)|| = ||x - y||$ for all $x, y \in \mathbb{R}^2$. It is not hard to show that the composition of isometries is an isometry, the identity is an isometry, and the inverse of an isometry is an isometry. Thus, Isom(\mathbb{R}^2) is a group under composition, as it is a subgroup of Perm(\mathbb{R}^2).

Example 2.17. Let $G, H$ be groups. Then the direct product $G \times H$ is the Cartesian product of $G$ and $H$, with the operation

$$(g, h) \cdot (g', h') = (gg', hh'),$$

is a group. The sets $G' := \{(g, 1) : g \in G\}$ and $H' := \{(1, h) : h \in H\}$ are normal subgroups of $G \times H$ with $G \cong G'$ and $H \cong H'$. Furthermore, $(G \times H)/G' \cong H$. Moreover, $G' \cap H' = 1$ and $G \times H = G'H' = \{xy : x \in G', y \in H'\}$. Conversely, if $E$ is a group containing normal subgroups $N_1$ and $N_2$ satisfying $N_1N_2 = E$ and $N_1 \cap N_2 = 1$, then $E$ is isomorphic to $N_1 \times N_2$ via the map $(n_1, n_2) \mapsto n_1n_2$.

Exercises

1. Let $N$ be a normal subgroup of a group $G$. If $H$ is a subgroup of $G$, prove that $N \cap H$ is a normal subgroup of $H$.

2. If $G, H$ are groups, set $G' := \{(g, 1) \in G \times H : g \in G\}$ and $H' := \{(1, h) \in G \times H : h \in H\}$. Prove that $G'$ and $H'$ are normal subgroups of $G \times H$, that $G' \cong G$ and $H' \cong H$, and that $(G \times H)/G' \cong H$. 


2.3. Background in Group Theory

Group Actions and Semidirect Products

Let $S$ be a set and $G$ a group. We say that $G$ acts on $S$ if there is a function $G \times S \to S$, denoted by $(g, s) \mapsto gs$, satisfying the axioms (i) $1s = s$ for each $s \in S$, (ii) $g(hs) = (gh)s$ for all $g, h \in G$ and $s \in S$. Alternatively, we call this a group action of the group $G$ on the set $S$. If we have a group action of $G$ on $S$, then we have an induced homomorphism $\varphi : G \to \text{Perm}(S)$, given by $\varphi(g)(s) = gs$. In other words, $\varphi(g)$ is the permutation of $S$ that sends $s$ to $gs$. Conversely, if we have a homomorphism $\psi : G \to \text{Perm}(S)$, we can define a group action of $G$ on $S$ by $gs = \psi(g)(s)$.

Example 2.18. The group $\text{Gl}_2(\mathbb{R})$ acts on $\mathbb{R}^2$ via matrix multiplication. That is, it acts by the formula $A \cdot v = Av$, where we interpret $v$ as a column vector, and $Av$ is ordinary matrix multiplication. This action restricts to yield a group action of $\text{O}_2(\mathbb{R})$ on $\mathbb{R}^2$. In terms of the homomorphism interpretation of group actions, this action comes from the natural identification of $\text{Gl}_2(\mathbb{R})$ as a subgroup of $\text{Perm}(\mathbb{R}^2)$.

Example 2.19. An important part of our classification of wallpaper groups will be to consider group actions of subgroups of $\text{O}_2(\mathbb{R})$ on subgroups of $\mathbb{R}^2$. Let $\{v_1, v_2\}$ be a basis for $\mathbb{R}^2$, and let $T = \{n_1v_1 + n_2v_2 : n_1, n_2 \in \mathbb{Z}\}$, the subgroup of all integer linear combinations of $v_1$ and $v_2$. The natural action of $\text{O}_2(\mathbb{R})$ may or may not yield an action on $T$, because $Av_1$ or $Av_2$ may not be an integer linear combination of $v_1, v_2$ for some $A \in \text{O}_2(\mathbb{R})$. To look at some specific examples, consider $D_4 = \langle r, f \rangle$, where

$$r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

If $v_1 = (1, 0)$ and $v_2 = (0, 1)$, then $rv_1 = v_2$ and $rv_2 = -v_1$. Also, $fv_1 = -v_1$ and $fv_2 = v_2$. Thus, any element of $D_4$ induces a permutation of $T$. On the other hand, if $v_1 = (1, 0)$ and $v_2 = (1/2, \sqrt{3}/2)$. Then $rv_1 = (-\sqrt{3}/2, 1/2)$ is not an integer linear combination of $v_1$ and $v_2$.

Let $N$ and $H$ be groups, and suppose that $H$ acts on the set $N$ by group automorphisms. That is, the action induces a homomorphism from $H$ to $\text{Aut}(N)$, the group of all automorphisms of $N$, which is a subgroup of $\text{Perm}(N)$. If we write the group operation on $N$ as $+$, then the statement that the action is via group automorphisms translates into the axiom $h(n_1 + n_2) = hn_1 + hn_2$ for all $h \in H$ and $n_1, n_2 \in N$. However, we will consider this idea for groups written multiplicatively.

Example 2.20. Let $G$ be a group containing an Abelian normal subgroup $N$. We write $H = G/N$. Define a map $\varphi : H \to \text{Aut}(N)$ as follows. For $h = Ng \in H$, set $\varphi(Ng)$ to be the automorphism of $N$ given by $\varphi(Ng)(x) = gxg^{-1}$. This is well defined, since if we write $Ng = N\tilde{g}$ for some $\tilde{g} \in G$, then $g^{-1}\tilde{g} \in N$. Therefore, as $N$ is Abelian,

$$x = (g^{-1}\tilde{g})x(g^{-1}\tilde{g})^{-1} = g^{-1}\tilde{g}x(\tilde{g})^{-1}g.$$
By conjugating both sides by \(g\), we have \(gxg^{-1} = g' x(g')^{-1}\). Here is another way to look at this example. The map that sends \(g\) to the inner automorphism of \(G\) induced by \(g\) is a group homomorphism \(G \to \text{Aut}(G)\), and since \(N\) is normal in \(G\), restriction yields a homomorphism \(G \to \text{Aut}(N)\). Since \(N\) is Abelian, \(gxg^{-1} = x\) for all \(g, x \in N\). Thus, \(N \subseteq \ker(\varphi)\). Thus, there is an induced homomorphism \(G/N \to \text{Aut}(N)\).

We can generalize this a tiny bit by supposing that \(\psi : G \to G_0\) is a surjective group homomorphism with kernel \(N\). Then \(G_0 \cong G/N\). We make \(G_0\) act on \(N\) by defining \(g \cdot n = tnt^{-1}\) for any \(t \in G\) with \(\psi(t) = g\).

Suppose that \(N\) and \(H\) are groups, and let \(\varphi : H \to \text{Aut}(N)\) be a group homomorphism. The semidirect product \(N \times_{\varphi} H\) is defined to be the Cartesian product as a set, with operation

\[(n, h)(n', h') = (n\varphi(h)(n'), hh').\]

A straightforward argument shows that \(N \times_{\varphi} H\) is a group containing the subgroups \(N' = \{(n, 1) : n \in N\} \cong N\) and \(H' = \{(1, h) : h \in H\} \cong H\) such that \(N'\) is normal, \(N'H' = N \times_{\varphi} H\), and \(N' \cap H' = 1\). Conversely, if \(G\) is a group containing subgroups \(N\) and \(H\) with \(N\) normal in \(G\), \(G = NH\), and \(N \cap H = 1\), then \(G\) is isomorphic to the semidirect product \(N \times_{\varphi} H\), via the map \((n, h) \mapsto nh\), where \(\varphi : H \to \text{Aut}(N)\) is given by \(\varphi(h)(n) = hnh^{-1}\).

**Exercises**

1. Let \(G\) be a group containing a normal subgroup \(N\), a subgroup \(H\), such that \(N \cap H = \{1\}\) and \(NH = G\). Prove that \(G \cong N \times_{\varphi} H\), where \(\varphi : H \to \text{Aut}(N)\) is given by \(\varphi(h)(n) = hnh^{-1}\).

2. Let \(\psi : G \to G_0\) be a surjective group homomorphism with kernel \(T\). Prove that \(G\) is isomorphic to the semidirect product of \(G_0\) and \(T\), under the action of \(G_0\) on \(T\) described in Example 2.20, if and only if there is a group homomorphism \(\sigma : G_0 \to G\) with \(\psi \circ \sigma = \text{id}\).

3. Let \(N\) be an Abelian normal subgroup of a group \(G\), and set \(G_0 = G/N\). For each coset \(\sigma \in G_0\) choose an element \(x_\sigma \in G\) with \(Nx_\sigma = \sigma\). The element \(x_\sigma\) is not uniquely determined, but it is unique up to multiplication by an element of \(N\).

   (a) If \(\sigma, \tau \in G_0\), show that \(x_\sigma x_\tau x_{\sigma\tau}^{-1} \in N\).

   (b) Show that the action of \(G_0\) on \(N\) described in Example 2.20 above satisfies \(\varphi(\sigma)(n) = x_\sigma nx_\sigma^{-1}\).

   (c) Define \(f : G_0 \times G_0 \to N\) by \(f(\sigma, \tau) = x_\sigma x_\tau x_{\sigma\tau}^{-1}\). Show that \(f\) satisfies the formula \(f(\sigma, \tau)f(\sigma\tau, \rho) = \sigma(f(\tau, \rho))f(\sigma, \tau\rho)\) for all \(\sigma, \tau, \rho \in G_0\).
2.4 The Group Structure of Isom(\(\mathbb{R}^2\))

In this section we give a group-theoretic decomposition of the group Isom(\(\mathbb{R}^2\)). We only need to consider Isom(\(\mathbb{R}^2\)) in our study of wallpaper groups. However, our analysis works for \(\mathbb{R}^n\) for any \(n\). We will use the description of Isom(\(\mathbb{R}^2\)) obtained here to help us classify wallpaper groups. There are two special subgroups we will consider. One is the subgroup \(T\) of all translations of \(\mathbb{R}^2\). The second is the orthogonal group \(O_2(\mathbb{R})\), the set of all isometries that are also linear transformations of the vector space \(\mathbb{R}^2\). An alternative description, which we will prove below, is that this is the group of all isometries that fix the origin. By coordinatizing the plane, we may consider points in the plane as vectors. Following usual notation, we will write \(\|u\|\) for the length of a vector \(u\). The distance between two vectors \(u\) and \(v\) is then \(\|u - v\|\). With this notation, we see that a bijection \(f\) of the plane is an isometry if \(\|f(u) - f(v)\| = \|u - v\|\) for all vectors \(u, v\).

Let \(g\) be an isometry with \(g(0) = 0\). From this condition we see for any \(u \in \mathbb{R}^2\) that

\[
\|g(u)\| = \|g(u) - g(0)\| = \|u - 0\| = \|u\|.
\]

In other words, \(g\) preserves the length of a vector. Recall that if \(u\) and \(v\) are vectors, then there is a unique angle \(\theta\) with \(0 \leq \theta \leq \pi\) such that

\[
\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta.
\]

This fact is a restatement of the Cauchy-Schwartz inequality. A consequence of this inequality is that the dot product is given by the formula \(u \cdot v = \|u\| \|v\| \cos \theta\).

**Lemma 2.21.** If \(g\) is an isometry of \(\mathbb{R}^2\) with \(g(0) = 0\), then \(g\) preserves angles and dot products. That is, for any \(u, v \in \mathbb{R}^2\), the angle between \(g(u)\) and \(g(v)\) is the same as the angle between \(u\) and \(v\), and \(g(u) \cdot g(v) = u \cdot v\).

**Proof.** Let \(g\) be an isometry with \(g(0) = 0\). Recall from above that this implies \(\|g(u)\| = \|u\|\) for all vectors \(u\). If \(\theta\) is the angle between two vectors \(u\) and \(v\), then

\[
\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta.
\]

If \(\theta'\) is the angle between \(g(u)\) and \(g(v)\), then

\[
\|g(u) - g(v)\|^2 = \|g(u)\|^2 + \|g(v)\|^2 - 2\|g(u)\|\|g(v)\| \cos \theta'.
\]

However, since \(\|g(u)\| = \|u\|\) and \(\|g(v)\| = \|v\|\), we get

\[
\|u - v\|^2 = \|g(u) - g(v)\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta'.
\]

This forces \(\cos \theta' = \cos \theta\). Since \(0 \leq \theta, \theta' \leq \pi\), we conclude that \(\theta' = \theta\).

To see that \(g\) preserves dot products, we have \(u \cdot v = \|u\|\|v\| \cos \theta\). By the previous paragraph, \(\theta\) is also the angle between \(g(u)\) and \(g(v)\). Therefore, \(g(u) \cdot g(v) = \|g(u)\|\|g(v)\| \cos \theta\).

Since \(\|g(u)\| = \|u\|\) and \(\|g(v)\| = \|v\|\), this yields \(g(u) \cdot g(v) = u \cdot v\). \(\square\)
Proposition 2.22. If $g$ is an isometry of $\mathbb{R}^2$ with $g(0) = 0$, then $g$ is a linear transformation.

Proof. Let $\{v_1, v_2\}$ be an orthonormal basis of $\mathbb{R}^2$, and set $w_i = g(u_i)$. First of all, $\|w_i\| = \|v_i\| = 1$ since $g$ preserves length. Next, by Lemma 2.21, the angle between $w_1$ and $w_2$ is equal to the angle between $v_1$ and $v_2$, which is $\pi/2$. Therefore, $\{w_1, w_2\}$ is an orthonormal basis of $\mathbb{R}^2$. Recall that if $u = \alpha_1 v_1 + \alpha_2 v_2$, then the coefficients $\alpha_i$ are determined by the formula $\alpha_i = u \cdot v_i$. So, we have $\alpha_i = g(u) \cdot g(v_i) = g(u) \cdot w_i$ by Lemma 2.21. However, $g(u) = \alpha_1 w_1 + \alpha_2 w_2$ since $\{w_1, w_2\}$ is an orthonormal basis, so the coefficient of $w_i$ is $g(u) \cdot w_i$. We conclude that $g(u) = (g(u) \cdot v_1) w_1 + (g(u) \cdot v_2) w_2$. From this formula we show that $g$ is a linear transformation. Let $u, v \in \mathbb{R}^2$. Then

$$g(u + v) = \sum_i ((u + v) \cdot w_i) w_i = \sum_i (u \cdot w_i) w_i + \sum_i (v \cdot w_i) w_i$$

and if $\gamma$ is any scalar, then

$$g(\gamma u) = \sum_i (\gamma u \cdot w_i) w_i = \sum_i \gamma (u \cdot w_i) w_i = \gamma \sum_i (u \cdot w_i) w_i$$

This proves that $g$ is a linear transformation. \qed

Corollary 2.23. Let $f$ be an isometry of $\mathbb{R}^2$. Then $f(x) = g(x) + b$ for some linear isometry $g$ and some $b \in \mathbb{R}^2$.

Proof. Let $b = f(0)$ and set $g(x) = f(x) - b$. Then $g$ is the composition of $f$ and the translation $\tau_{-b}$, so $g$ is an isometry. Since $g(0) = f(0) - b = b - b = 0$, Proposition 2.22 shows that $g$ is a linear transformation. \qed

If $g$ is a linear transformation on $\mathbb{R}^2$, by viewing the elements of $\mathbb{R}^2$ as column matrices we may write $g(u) = Au$ for some $n \times n$ matrix $A$. The matrix an isometry $g$ is not arbitrary. We get a restriction on $A$ by knowing that $g$ preserves dot products. Let $\{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$. If $g(x) = Ax$, then the $(i, j)$-entry of $A^T A$ is $g(e_i) \cdot g(e_j) = e_i \cdot e_j$, which is 1 for $i = j$ and 0 otherwise. This shows us that $A^T A = I_2$, the $2 \times 2$ identity matrix. Conversely, if $A$ is a matrix with $A^T A = I_2$, we claim that the linear map $g$ defined by $g(x) = Ax$ is an isometry. For,

$$g(u) \cdot g(v) = (Au) \cdot (Av) = (Au)^T (Av)$$

$$= u^T (A^T A)v = u^T v = u \cdot v.$$

Therefore, by setting $v = u$, we have $\|g(u)\| = \|u\|$. Finally, since $g$ is linear, $\|g(u) - g(w)\| = \|g(u - w)\| = \|u - w\|.$

Recall that the set of matrices that satisfy the condition $A^T A = I_2$ is called the orthogonal group, and is denoted $O_2(\mathbb{R})$. 
Corollary 2.24. If $f$ is an isometry of $\mathbb{R}^2$, then $f(x) = Ax + b$ for some $b \in \mathbb{R}^2$ and some $2 \times 2$ matrix $A \in O_2(\mathbb{R})$.

We point out that we obtained this description of isometries only from the assumption that an isometry preserves distance, not using that it is a bijection. It shows that a distance-preserving map of $\mathbb{R}^2$ is automatically a bijection.

Because of the connection between linear transformations and matrices, we get the following connection between $O_2(\mathbb{R})$ and $\text{Isom}(\mathbb{R}^2)$.

Proposition 2.25. Let $H$ be the subgroup of isometries of $\mathbb{R}^2$ that preserve the origin. Then $H \cong O_2(\mathbb{R})$.

Proof. We define a map $\sigma : O_2(\mathbb{R}) \to H$ by $\sigma(A)$ is the isometry $x \mapsto Ax$. In other words, $\sigma(A)(x) = Ax$. We have

$$\sigma(AB)(x) = (AB)x = A(Bx) = \sigma(A)(Bx) = \sigma(A)(\sigma(B)(x)) = (\sigma(A)\sigma(B))(x).$$

Therefore, $\sigma(AB) = \sigma(A)\sigma(B)$. So, $\sigma$ is a group homomorphism. If $\sigma(A)$ is the identity function, then $\sigma(A)(x) = x$ for all $x$. Then $Ax = x$ for all $x$. But then the matrix $A$ defines the identity linear transformation, so $A = I_2$. Therefore, $\sigma$ is injective. Finally, if $g \in H$, then $g(x) = Ax$ for some matrix $A$ by Corollary 2.23. Corollary 2.24 shows that $A^T A = I_2$, so $A \in O_2(\mathbb{R})$. This proves that $g = \sigma(A)$, so $\sigma$ is surjective. Therefore, $\sigma$ is a group isomorphism. \qed

\[\] We now show how $\text{Isom}(\mathbb{R}^2)$ can be constructed from the group $\mathbb{T}$ of all translations and $O_2(\mathbb{R})$. We view $O_2(\mathbb{R})$ as both the group of linear isometries and the group of all orthogonal matrices. We write $G = \text{Isom}(\mathbb{R}^2)$ and $H = O_2(\mathbb{R})$ for ease of notation. We proved above that every isometry is a composition of a linear isometry and a translation. Therefore, $G = \mathbb{T}H$.

We next note that $\mathbb{T}$ is a normal subgroup of $G$. To see this, the equation $G = \mathbb{T}H$ shows that it is enough to prove that if $f$ is a linear isometry and $\tau$ is a translation, then $f \circ \tau \circ f^{-1}$ is a translation. Suppose that $\tau(x) = x + v$. Then $\tau(f^{-1}(x)) = f^{-1}(x) + v$. So,

$$f \tau f^{-1}(x) = f(f^{-1}(x) + v) = x + f(v)$$

since $f$ is linear. Therefore, $f \circ \tau \circ f^{-1}$ is translation by $f(v)$. In particular, this yields $f \circ \tau_v \circ f^{-1} = \tau_{f(v)}$. It is clear that $\mathbb{T} \cap H = \{\text{id}\}$ since any nontrivial translation fixes no point. We therefore have written $G = \mathbb{T}H$ with $\mathbb{T}$ a normal subgroup, $H$ a subgroup, and $\mathbb{T} \cap H = \{\text{id}\}$. This means $G$ is the semidirect product of $\mathbb{T}$ and $H$. By recalling the construction of semidirect products we can be a little more precise about the structure of

\[\]do this as a theorem?
G. If we denote by $\varphi_h$ the restriction to $T$ of the inner automorphism $\tau \mapsto h\tau h^{-1}$, then multiplication on $G = TH$ is given by

$$(\tau_u h)(\tau_v h') = (\tau_u h\tau_v h^{-1})(hh') = (\tau_u \varphi_h(\tau_v))(hh')$$

$$= (\tau_u \tau_{h(v)})(hh') = (\tau_{u+h(v)})(hh').$$

Using the isomorphism $T \cong \mathbb{R}^2$ and viewing elements of $H$ in terms of matrices, the map $\varphi : H \to \text{Aut}(T)$ is more concretely given as

$$\varphi(A)(v) = Av,$$

where $A$ is an orthogonal matrix and $v \in \mathbb{R}^2$. There is a surjective group homomorphism $G \to H$ that sends $th$ to $h$; the kernel of this map is $T$. Furthermore, $G$ is then isomorphic to $(O_2(\mathbb{R}) \times \mathbb{R}^2, \cdot)$, where the operation $\cdot$ on this Cartesian product is

$$(A, u) \cdot (B, v) = (AB, u + Av).$$

We will use the notation $(f, u)$ to represent the isometry $\tau_u \circ f$ for a linear isometry. If we wish to use a matrix $A$ rather than the corresponding linear map, we will write $(A, u)$ for the resulting isometry.

### 2.5 Structure of $O_2(\mathbb{R})$

As we saw in the previous section, the group $\text{Isom}(\mathbb{R}^2)$ is built from the subgroup of translations and the subgroup $O_2(\mathbb{R})$ of linear isometries. We point out some facts about $O_2(\mathbb{R})$. If we use the standard basis for $\mathbb{R}^2$, then an element of $O_2(\mathbb{R})$ can be represented by a $2 \times 2$ matrix. Furthermore, any such matrix $A$ satisfies the condition $A^T A = I_2$. Taking determinants, we obtain $\det(A) = \pm 1$. Since the determinant function is a group homomorphism from $O_2(\mathbb{R})$ to the nonzero real numbers, its kernel is a normal subgroup of $O_2(\mathbb{R})$. This subgroup is called the special orthogonal group, and is denoted $SO_2(\mathbb{R})$. Thus,

$$SO_2(\mathbb{R}) = \{ A \in O_2(\mathbb{R}) : \det(A) = 1 \}.$$

Note that $[O_2(\mathbb{R}) : SO_2(\mathbb{R})] = 2$, since $\det$ is a group homomorphism from $O_2(\mathbb{R})$ onto the group $\{ \pm 1 \}$ of two elements, and the kernel is $SO_2(\mathbb{R})$.

Let $A$ be the matrix with respect to the standard basis for an element in $O_2(\mathbb{R})$. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the condition $A^T A = I_2$ gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\]
This yields \( a^2 + c^2 = 1 \) and \( b^2 + d^2 = 1 \). Therefore, there is an angle \( \theta \) with \( a = \cos \theta \) and \( c = \sin \theta \). Furthermore, the condition \( ab + cd = 0 \) says that the vector \((b, d)\) is orthogonal to \((a, c)\). Since \( b^2 + d^2 = 1 \), this forces \((b, d) = (-\sin \theta, \cos \theta)\) or \((b, d) = (\sin \theta, -\cos \theta)\). The first choice gives a matrix with determinant 1 and the second choice gives a matrix of determinant \(-1\). From this we see that if \( A \in \text{SO}_2(\mathbb{R}) \), then we may write

\[
A = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

for some angle \( \theta \). In other words, \( A \) is a rotation about the origin by an angle \( \theta \). On the other hand, if \( A \notin \text{SO}_2(\mathbb{R}) \), then

\[
A = \begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{pmatrix}.
\]

From the formula given earlier for a reflection across a line through the origin, we can see that \( A \) is the reflection across the line \( y = (\tan \theta/2)x \).

To summarize, elements of \( \text{SO}_2(\mathbb{R}) \) are rotations and elements of \( \text{O}_2(\mathbb{R}) \setminus \text{SO}_2(\mathbb{R}) \) are reflections. Since every element of \( \text{Isom}(\mathbb{R}^2) \) is the composition of a translation with an element of \( \text{O}_2(\mathbb{R}) \), the following is an immediate corollary of this description of elements of the orthogonal group.

**Corollary 2.26.** Every isometry of the plane is either a translation, a rotation, a reflection, or a glide reflection.

Furthermore, if \( r \in \text{SO}_2(\mathbb{R}) \) and \( f \notin \text{SO}_2(\mathbb{R}) \), then \( rf \notin \text{SO}_2(\mathbb{R}) \), so \( rf \) is a reflection. Thus, \((rf)^2 = 1\), which is equivalent to \( frf = r^{-1} \). Recall that the dihedral group \( D_n \) is the group of symmetries of a regular \( n \)-gon. It is given by generators and relations as \( D_n = \langle r, f \rangle \) with

\[
r^n = f^2 = 1, \\
frf = r^{-1}.
\]

We can identify \( D_n \) as a subgroup of \( \text{O}_2(\mathbb{R}) \) by setting \( r \) to be the rotation by an angle of \( 2\pi/n \) and \( f \) any reflection. As a converse to this example, we have the following property of finite subgroups of \( \text{O}_2(\mathbb{R}) \).

**Proposition 2.27.** Let \( G \) be a finite subgroup of \( \text{O}_2(\mathbb{R}) \). Then \( G \) is isomorphic to either a cyclic group of order \( n \) or a dihedral group of order \( 2n \), for some integer \( n \).

**Proof.** Let \( N = G \cap \text{SO}_2(\mathbb{R}) \), a normal subgroup of \( G \). Since \( [\text{O}_2(\mathbb{R}) : \text{SO}_2(\mathbb{R})] = 2 \) and \( G/N \) is isomorphic to a subgroup of \( \text{O}_2(\mathbb{R})/\text{SO}_2(\mathbb{R}) \), we get \( |G : N| \leq 2 \). If \( N = \{1\} \), then either \( G = \{1\} \) is cyclic, or \( G = \langle f \rangle \) for some reflection \( f \), so \( G \cong D_1 \). Therefore, assume that \( N \neq \{1\} \). The group \( N \) consists of rotations. Since it is finite, there is a nontrivial
rotation \( r \in N \) of minimal possible angle \( \theta \). If \( r' \) is any other nontrivial rotation in \( N \), and if \( r' \) is a rotation by \( \phi \), then there is an integer \( m \) with \( m\phi \leq \theta < (m+1)\phi \). The rotation \( r(r')^{-m} \) is a rotation by the angle \( 0 \leq \theta - m\phi < \theta \). Minimality of \( \theta \) then forces \( m\phi = \theta \). In other words, \( r' \in \langle r \rangle \). This proves that \( N = \langle r \rangle \) is cyclic. If \( G = N \), then \( G \) is cyclic. If \( G \neq N \), then \( [G : N] = 2 \). If \( f \in G \setminus N \), then as pointed out before, \( frf = r^{-1} \). If \( n = |N| \), then \( |G| = 2n \), and \( G \) is generated by \( r \) and \( f \), and satisfies the relations \( r^n = f^2 = 1 \) and \( frf = r^{-1} \). Therefore, \( G \cong D_n \).

### 2.6 Symmetry Groups of Bounded Figures

In this section we classify the symmetry groups of a bounded plane figure. We will see that any such symmetry group is isomorphic to a subgroup of \( O_2(\mathbb{Z}) \). Moreover, if the figure has a “smallest” rotation, then we will see that its symmetry group is either \( C_n \) or \( D_n \).

To start, we will be specific about what type of figures we consider. A subset \( S \) of \( \mathbb{R}^2 \) is said to be bounded if there is a positive number \( r \) such that \( S \) is a subset of the closed disk \( \{ P \in \mathbb{R}^2 : |P| \leq r \} \) of radius \( r \). We begin with a simple but important lemma restricting the type of group that can arise as the symmetry group of a bounded figure.

**Lemma 2.28.** Let \( S \) be a bounded figure. Then \( \text{Sym}(S) \) does not contain a nontrivial translation.

**Proof.** Set \( G = \text{Sym}(S) \). Suppose that \( \tau \in G \) is a translation, and suppose that \( \tau \) is translation by the vector \( b \). Then, since \( G \) is a group, \( \tau^n \in G \) for every positive integer \( n \). Thus, translation by \( nb \) is a symmetry of \( S \) for every \( n \). Since \( S \) is bounded, there is an \( r \) such that \( |P| \leq r \) for every \( P \in S \). But then \( \tau^n(P) = P + nb \in S \) for every \( n \). The triangle inequality implies that \( |nb| \leq |P| + |P + nb| \leq 2r \). This yields \( n |b| \leq 2r \). However, since this is true for every \( n \), we must have \( |b| = 0 \); thus, \( b = 0 \). Therefore, \( \tau \) is the identity map. □

To prove the main result about symmetry groups of bounded figures, we need some facts about compositions of isometries, whose proofs we gave as homework problems.

1. The composition of two reflections about lines intersecting at a point \( P \) is a rotation about \( P \).
2. The composition of two rotations about a common point \( P \) is another rotation about \( P \).
3. If \( r \) and \( s \) are rotations about different centers, then \( rsr^{-1}s^{-1} \) is a nontrivial translation.
4. If \( r \) is a rotation about \( P \) and \( f \) is a reflection not fixing \( P \), then \( rf \) is a glide reflection.

**Proposition 2.29.** Let \( S \) be a bounded figure, and set \( G = \text{Sym}(S) \). Then \( G \) is a subgroup of \( O_2(\mathbb{R}) \). In particular, every isometry of \( S \) is linear.
Proof. By the lemma there are no nontrivial translations in \( G \). As a consequence, there are no nontrivial glide reflections, since the square of a nontrivial glide is a nontrivial translation by Lemma 2.2. Thus, \( G \) must consist solely of rotations and/or reflections. If \( G \) is either the trivial group or is generated by a single reflection, then the result is clear (by defining the origin to be a point on the reflection line). We may then assume that \( G \) contains a nontrivial rotation or two reflections. In either case \( G \) contains a nontrivial rotation since the composition of two reflections is a rotation. If \( G \) contains two rotations \( r \) and \( s \), then \( rs \) is a rotation if \( r \) and \( s \) share a common rotation center. However, if they have different centers, then \( r s r^{-1} s^{-1} \) is a nontrivial translation. By the lemma, this cannot happen. Therefore, every rotation in \( G \) has the same center, which we will view as the origin. If \( r \in G \) and if \( f \in G \) is a reflection, then \( fr \) is a glide reflection if \( f \) does preserve the rotation center of \( r \). Since \( G \) does not contain a glide reflection, \( f \) must preserve the origin. Therefore, all symmetries of \( S \) preserve the origin, and so \( G \subseteq O_2(R) \) is comprised of linear isometries. \( \square \)

Let \( S \) be a bounded figure. By the proposition, \( G = \text{Sym}(S) \) is a subgroup of \( O_2(\mathbb{R}) \). We can then consider the subgroup \( R \) of rotations of \( S \). We consider the case where \( G \) contains a nontrivial rotation of smallest possible angle. We first note that \( |G : R| \leq 2 \). To see why, since \( G \) is a subgroup of \( O_2(\mathbb{R}) \), and \( SO_2(\mathbb{R}) \) is the subgroup of all rotations, \( R = G \cap SO_2(\mathbb{R}) \). Thus, the inclusion \( G \subseteq O_2(\mathbb{R}) \) induces an injective homomorphism \( G/R \to O_2(\mathbb{R})/SO_2(\mathbb{R}) \), which is a group of order 2. Thus, \( |G/R| \leq 2 \).

**Corollary 2.30.** If \( S \) is a bounded figure such that \( \text{Sym}(S) \) contains a rotation of smallest possible angle, then \( \text{Sym}(G) \) is either isomorphic to \( C_n \) or to \( D_n \) for some \( n \).

Proof. Let \( r \) be a rotation in \( G = \text{Sym}(S) \) of smallest possible nonzero angle \( \theta \). Recall that \( \theta \) is only unique up to multiples of \( 2\pi \). We may assume that \( 0 < \theta \leq \pi \), since if \( G \) contains a rotation by \( \theta \), then it contains a rotation by \( -\theta \), and if \( \pi < \theta < 2\pi \), then \( 0 < 2\pi - \theta < \pi \), and \( r^{-1} \) is rotation by \( 2\pi - \theta \). If \( s \) is another rotation in \( G \), and if \( s \) is rotation by \( \phi \), then there is an integer \( m \) with \( m\theta \leq \phi < (m+1)\theta \). The symmetry \( sr^{-m} \) has rotation angle \( \phi - m\theta \), which by the inequality is smaller than \( \theta \). By choice of \( \theta \), we must have \( \phi - m\theta = 0 \), so \( \phi = m\theta \). Thus, \( s = r^m \). Therefore, the group \( R \) of rotations of \( S \) is the cyclic group generated by \( r \). Moreover, we claim that \( \theta = 2\pi/n \) for some \( n \). To prove this, let \( n \) be the smallest integer with \( 2\pi/\theta \leq n \). Then \( n - 1 < 2\pi/\theta \). Multiplying these inequalities by \( \theta \) and rewriting a little gives

\[
2\pi \leq n\theta < 2 + \theta,
\]

so \( 0 \leq n\theta - 2\pi < \theta \). However, the rotation angle in the interval \([0,2\pi]\) for \( r^n \) is \( n\theta - 2\pi \). Since \( r^n \in G \), minimality of \( \theta \) forces \( n\theta - 2\pi = 0 \), giving \( \theta = 2\pi/n \), as desired. Since we have shown that \( R = \langle r \rangle \), and \( r^n = \text{id} \) but \( r^m \neq \text{id} \) for \( 0 < m < n \), we see that \( |R| = n \), and so \( |R| < \infty \). If \( G = R \), then \( G = C_n \). If not, then \( G \) contains a reflection \( f \). Therefore, \( [G : R] = 2 \) by Lemma ???. In any case, \( |G : R| < \infty \). Since \( |G| = |G : R| \cdot |R| \), the group \( G \) is finite. By Proposition 2.27, we see that \( G \) is \( C_n \) or \( D_n \) for some \( n \).\( \square \)
Exercises

1. Prove that the composition of two reflections about lines intersecting at a point $P$ is a rotation about $P$. Moreover, if $\theta$ is the angle between the two reflection lines, prove that the composition is a rotation by $\theta/2$.

2. Prove that the composition of two rotations about a common point $P$ is another rotation about $P$. Moreover, if the rotations are by angles $\theta$ and $\phi$, prove that the composition is rotation by $\theta + \phi$.

2.7 Wallpaper Patterns

We will give a formal definition shortly, but intuitively a wallpaper pattern is a design used for making wallpaper. One can make such a pattern by taking a pattern inside of a rectangle and repeating it horizontally and vertically. The example below was made by taking a figure consisting of a single $\Gamma$, and translating it horizontally and vertically.

There are several other ways to repeat a basic figure to get different wallpaper patterns. For instance, one could repeat the figure in two other ways to obtain the following two patterns.
2.7. Wallpaper Patterns

In the first figure, we used 180° rotations to repeat the pattern vertically. In the second, one column is a mirror image of an adjacent column.

We now become more precise. If \( W \) is a subset of \( \mathbb{R}^2 \), recall that the symmetry group of \( W \) is

\[
\text{Sym}(W) = \{ \varphi \in \text{Isom}(\mathbb{R}^2) : \varphi(W) = W \}.
\]

Recall also that for \( v \in \mathbb{R}^2 \) we denote the translation by \( v \) map by \( \tau_v \). Furthermore, if \( v, w \in \mathbb{R}^2 \), then \( \tau_v \circ \tau_w = \tau_{v+w} \) and \( \tau_{v}^{-1} = \tau_{-v} \). Moreover, if \( \mathbf{0} \) is the zero vector, then \( \tau_0 \) is the identity map. These facts show that the set of translations \( T \) forms a subgroup of \( \text{Isom}(\mathbb{R}^2) \). Therefore,

\[
\text{Sym}(W) \cap T = \{ \tau \in \text{Sym}(W) : \tau \text{ is a translation} \}.
\]

is a subgroup of \( \text{Sym}(W) \), and we call it the translation subgroup of \( \text{Sym}(W) \).

In each of the pictures above, the symmetry group contains horizontal and vertical translations. Moreover, there is a horizontal and a vertical translation of smallest possible length. If \( \tau_1 \) is the smallest horizontal translation and \( \tau_2 \) the smallest vertical translation of one of the patterns, then any translation of the pattern is of the form \( \tau_1^n \tau_2^m \) for some pair of integers \( n, m \). The primary characteristic of wallpaper patterns is that there are always two translations \( \tau_1, \tau_2 \) of the pattern such that any other translation is of the form \( \tau_1^n \tau_2^m \) for some integers \( n, m \); we will soon formalize this in a definition.

**Lemma 2.31.** The group \( \mathbb{R}^2 \) is isomorphic to the subgroup of \( \text{Isom}(\mathbb{R}^2) \) consisting of all translations of \( \mathbb{R}^2 \) via the map \( v \mapsto \tau_v \). Therefore, the translation subgroup of the symmetry group of a figure in \( \mathbb{R}^2 \) is isomorphic to a subgroup of \( \mathbb{R}^2 \).

**Proof.** We show that the map \( \varphi \) given by \( \varphi(v) = \tau_v \) is an isomorphism. We have already pointed out that \( \tau_v \circ \tau_w = \tau_{v+w} \) and \( \tau_v^{-1} = \tau_{-v} \) for any \( v, w \in \mathbb{R}^2 \). Thus, \( \varphi \) is a homomorphism. It is surjective by the definition of a translation. It is injective, since if \( \varphi(v) = \text{id} \), then \( \tau_v(x) = x \) for all \( x \in \mathbb{R}^2 \). However, since \( \tau_v(x) = x + v \), this forces \( v = \mathbf{0} \). Thus, \( \varphi \) is an isomorphism. The final statement is clear, since if \( T \) is the translation subgroup of some symmetry group, then \( \varphi^{-1}(T) \) is a subgroup of \( \mathbb{R}^2 \) isomorphic to \( T \). \( \square \)

Because of this lemma, we will frequently identify a translation \( \tau_v \) with the vector \( v \); this should not cause any confusion.

**Definition 2.32.** A two-dimensional lattice in \( \mathbb{R}^2 \) is a subgroup of the form \( \{ nt_1 + mt_2 : n, m \in \mathbb{Z} \} \) for some basis \( \{ t_1, t_2 \} \) of \( \mathbb{R}^2 \).

We may define a lattice in \( \mathbb{R}^2 \) more generally to be a finitely generated subgroup of \( \mathbb{R}^2 \). It follows from the fundamental theorem for finitely generated Abelian groups [1, Theorem 4.5.1] that every lattice is isomorphic to \( \mathbb{Z}^r \) for some integer \( r \). Therefore, a lattice has a basis as a \( \mathbb{Z} \)-module. The dimension of a lattice is the size of a basis. We will only work with two-dimensional lattices in \( \mathbb{R}^2 \). If \( T \) is such a lattice, say \( T = \{ nt_1 + mt_2 : n, m \in \mathbb{Z} \} \)
for some basis \( \{ t_1, t_2 \} \) of \( \mathbb{R}^2 \). Then every element of \( T \) is an integral linear combination of the basis elements. We will call the set \( \{ t_1, t_2 \} \) an \textit{integral basis} to emphasize that it is a basis of \( T \) and not just a basis of \( \mathbb{R}^2 \). Two simple properties of lattices we will use are that (i) \( T \) contains a nonzero vector of minimal length, and (ii) \( T \) contains only finitely many vectors inside any circle. While there is always a vector \( v \) of minimal length in a lattice, it is not unique since \( v \) and \( -v \) have the same length. These facts can be understood pictorially by drawing \( T \) as a subset of \( \mathbb{R}^2 \) as in the following picture.

We can now give the definition of a wallpaper pattern.

\textbf{Definition 2.33.} A subset \( W \) of \( \mathbb{R}^2 \) is a \textit{wallpaper pattern} if the translation subgroup of the symmetry group \( \text{Sym}(W) \) is a two-dimensional lattice. The symmetry group of a wallpaper pattern is said to be a \textit{wallpaper group}.

Escher’s tessellations are wonderful examples of wallpaper patterns. He drew many pictures with the same symmetry as the following drawing.

The two patterns below have isomorphic symmetry groups; both groups consist only of translations, so each is isomorphic to \( \mathbb{Z}^2 \).
Here are two further examples of wallpaper patterns with isomorphic wallpaper groups.

In both cases the wallpaper groups are generated by the translation lattice and a rotation of $180^\circ$; it is not hard to see that the two groups are isomorphic. In fact, each is generated by three elements $t_1, t_2, r$ and subject to the relations

\[
\begin{align*}
  t_1 t_2 &= t_2 t_1, \\
  r t_1 r^{-1} &= t_1^{-1}, \\
  r t_2 r^{-1} &= t_2^{-1}, \\
  r^2 &= 1.
\end{align*}
\]

In each case \{\(t_1, t_2\}\} is a basis for the translation subgroup and \(r\) is a $180^\circ$ rotation. By choosing a center of a rotation to be the origin, we can then let \(r\) be that rotation.

In the following tessellation of Escher, our symmetry group is generated by two translations \(t_1\) and \(t_2\) and a $90^\circ$ degree rotation \(r\).
In this picture, if we keep track of color, then the rotation is not a symmetry of the pattern, and the symmetry group contains only translations. While Escher viewed color as an important part of the picture, in determining the symmetry group of a wallpaper pattern we will ignore color. One way to use our definition to find the symmetry group of a drawing is to imagine an uncolored outline of a drawing. The symmetry group is then the symmetry group of the set of points occurring in the outline.

Our classification of wallpaper groups will proceed as follows. If $G$ is a wallpaper group, then an easy observation shows that the subgroup $T$ of translations is a normal subgroup of $G$; we point this out at the end of Section 2.4. We will first determine the possible groups that arise as $G/T$. Second, we will determine all the possible ways that $G/T$ can act on $T$; this action is described for any group $G$ and any normal subgroup $T$ in Section A.1. More concretely, for a wallpaper group $G$, we will identify $G/T$ with a subgroup of $O_2(\mathbb{R})$ in Section 3.1, and through this identification, $G/T$ acts on $T$ by viewing $T \subseteq \mathbb{R}^2$, on which $O_2(\mathbb{R})$ acts naturally. Finally, we will determine how to build $G$ from $T$ and $G/T$ with some additional information. Finally, we will use this information to describe explicitly all seventeen wallpaper groups as subgroups of $\text{Isom}(\mathbb{R}^2)$ and give wallpaper patterns for each group.
Chapter 3

The Point Group of a Wallpaper Pattern

Recall that a wallpaper group is a subgroup $G$ of $\text{Isom}(\mathbb{R}^2)$ that contains a two-dimensional lattice $T$ of translations as a normal subgroup. As we indicated in the introduction, the study of the quotient group $G/T$ will be the first step for us to determine all wallpaper groups. By choosing a basis for $T$ we will exhibit $G/T$ as a subgroup of $O_2(\mathbb{R})$ and see how elements of $G/T$ act on this basis. This action will be a key in describing and distinguishing wallpaper groups.

3.1 Definition and Main Properties

Let $G$ be a wallpaper group with translation lattice $T$. In this section we will give an interpretation of $G/T$, and we will determine all possible groups, up to isomorphism, that can occur as $G/T$ for a wallpaper group $G$.

We give a notation for isometries that will prove convenient. If $\varphi$ is an isometry, then $\varphi(x) = A(x) + b$ for some $A \in O_2(\mathbb{R})$ and $b \in \mathbb{R}^2$ by Corollary 2.23. To simplify notation, we write $\varphi = (A, b)$, and note that composition in $\text{Isom}(\mathbb{R}^2)$ translates into the following formula

$$(A, b)(C, d) = (AC, Ad + b).$$

Furthermore, inverses are given by

$$(A, b)^{-1} = (A^{-1}, -A^{-1}b).$$

With this notation, translation by a vector $b$ is $(I, b)$, and an element of $O_2(\mathbb{R})$ is of the form $(A, 0)$. We now define the point group of $G$.

**Definition 3.1.** Let $G$ be a wallpaper group. The point group $G_0$ of $G$ is the set

$$\{ A \in O_2(\mathbb{R}) : (A, b) \in G \text{ for some } b \in \mathbb{R}^2 \}.$$
From the formulas above for composition and inversion, we see that the point group is a subgroup of $O_2(\mathbb{R})$. In fact, we have the following interpretation of the point group.

**Proposition 3.2.** If $G$ is a wallpaper group with translation lattice $T$ and point group $G_0$, then $G_0 \cong G/T$.

**Proof.** If $\varphi$ is the group homomorphism $\text{Isom}(\mathbb{R}^2) \to O_2(\mathbb{R})$ given by $(A, b) \mapsto A$, then $\varphi(G) = G_0$, which shows that $G_0$ is a subgroup, being the image of a group under a group homomorphism. The kernel of $\varphi|_G$ is $T$, since $T$ is the intersection of $G$ and the translation subgroup of $\text{Isom}(\mathbb{R}^2)$, and the translation subgroup is the kernel of $\varphi$. \[\square\]

As we saw in Section 2.3, the group $O_2(\mathbb{R})$ acts on the group $\mathbb{T}$ of translations by conjugation. More precisely, if $t \in \mathbb{R}^2$ and $A \in O_2(\mathbb{R})$, then the equation

$$(A, 0)(I, t)(A, 0)^{-1} = (I, At)$$

shows that under the natural isomorphism $\mathbb{T} \cong \mathbb{R}^2$, this conjugation action is the natural action of $O_2(\mathbb{R})$ on $\mathbb{R}^2$. This action restricts to an action of $G_0$ on $T$, for if $A \in G_0$ and $t \in T$, then there is a $b \in \mathbb{R}^2$ with $(A, b) \in G$. Then $(A, b)(I, t)(A, b)^{-1} \in G$, and is equal to $(I, At)$; consequently, $At \in T$. The presence of this action, together with the group structure of $T$, will allow us to determine the groups that arise as $G_0$ for some wallpaper group $G$. To set some notation, we write $C_n$ for the cyclic group of order $n$. We view $C_n$ as a subgroup of $O_2(\mathbb{R})$ by considering it to be the cyclic group generated by a rotation of $2\pi/n$. Also, let $D_n$ be the dihedral group of order $2n$. This group is generated by two elements $r, f$ and subject to the relations $r^n = f^2 = \text{id}$ and $rf = r^{-1}$. As we noted in Example 2.12, we may view $D_n$ as a subgroup of $O_2(\mathbb{R})$ by letting $r$ be a rotation of $2\pi/n$ and $f$ any reflection. Note that by choosing different $f$ we get different subgroups of $O_2(\mathbb{R})$ which are isomorphic to $D_n$. The following lemma will be used in determining the possible point groups.

**Lemma 3.3.** The point group $G_0$ of a wallpaper group $G$ is finite.

**Proof.** Let $\{t_1, t_2\}$ be a basis of $T$, and let $C$ be a circle centered at the origin that contains $t_1$ and $t_2$ in its interior. We remarked earlier that there are only finitely many elements of $T$ inside $C$. Since $G_0$ is a subgroup of $O_2(\mathbb{R})$, its elements restrict to give permutations of the interior of $C$. There are then only finitely many pairs of elements of $T$ that can occur as the image of $\{t_1, t_2\}$ under an element of $G_0$. However, since $\{t_1, t_2\}$ is a basis of $\mathbb{R}^2$, any element of $G_0$ is determined by its action on $\{t_1, t_2\}$. Therefore, $G_0$ is finite.
We now determine the possible groups that can arise as the point group of a wallpaper group.

**Theorem 3.4.** Let $G_0$ be the point group of a wallpaper group $G$. Then $G_0$ is isomorphic to one of the following ten groups

$$\left\{ C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6 \right\}.$$ 

**Proof.** By Lemma 3.3, $G_0$ is a finite group. From Proposition 2.27, we then know that $G_0$ is isomorphic to $C_n$ or $D_n$ for some $n$. It remains to determine the possible values of $n$. In the proof of Proposition 2.27, we saw that $N = G_0 \cap \text{SO}_2(\mathbb{R})$ is a cyclic group generated by a rotation $r$ of minimal possible angle. Moreover, $|N| = n$, so $r$ has order $n$, which means that $r$ is rotation about the angle $\theta = 2\pi/n$. We represent $r$ by a matrix in two ways. First, with respect to the standard basis, the matrix representing $r$ is

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ 

On the other hand, let $\{t_1, t_2\}$ be an integral basis for $T$. Since $r(T) = T$ and $T = \mathbb{Z}t_1 \oplus \mathbb{Z}t_2$, the matrix for $r$ with respect to this basis is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $a, b, c, d$ integers. Since these two matrices represent the same linear transformation but with respect to different bases, they are conjugate; therefore, they have the same trace. This yields $2\cos \theta = a + d \in \mathbb{Z}$. This means $\cos \theta \in \{-1/2, 0, 1/2\}$. Looking at the graph of the cosine function, we see that when $0 < \theta \leq 2\pi$, we must have

$$\theta = 2\pi/n \in \{\pi/3, \pi/2, 2\pi/3, \pi, 2\pi\} = \{2\pi/6, 2\pi/4, 2\pi/3, 2\pi/2, 2\pi/1\}.$$ 

Therefore, $N = \langle r \rangle$ has order $n \in \{1, 2, 3, 4, 6\}$. Since $G_0$ is isomorphic to either $C_n$ or $D_n$ for $n = |N|$, this completes the proof.
We next verify that \( G_0 \) is uniquely determined by \( G \). To be more precise, we show that if two wallpaper groups are isomorphic, then their point groups are isomorphic. The following lemma will do this for us by identifying the subgroup \( T \) of a wallpaper group \( G \) in a purely group-theoretic way.

**Lemma 3.5.** Let \( G \) be a symmetry group with translation lattice \( T \), and set

\[
G_n = \{ x \in G : xg^n = g^nx \text{ for all } g \in G \}.
\]

Then \( T = G_n \) whenever \( n \) is a multiple of \( [G : T] \). Furthermore, if \( G \) and \( G' \) are wallpaper groups with translation lattices \( T \) and \( T' \), respectively, and if \( \varphi : G \to G' \) is an isomorphism, then \( \varphi(T) = T' \).

**Proof.** By Lemma 3.3, the group \( G/T \cong G_0 \) is finite. Let \( n \) be a multiple of \( [G : T] \). If \( g \in G \), then \( g^n \in T \). Since \( T \) is Abelian, if \( t \in T \), then \( tg^n = g^n t \) for all \( g \in G \), so \( t \in G_n \). Conversely, suppose that \( x \in G_n \). We may write \( x = (A, b) \) with \( b \in \mathbb{R}^2 \). Consider \( g = (I, t) \) with \( t \in T \). Then \( g^n = (I, nt) \). Since \( x \in G_n \), we have \( xg^n x^{-1} = g^n \). However, by our description of the action of \( O_2(\mathbb{R}) \) on \( \mathbb{T} \), we have \( xg^n x^{-1} = (I, A(nt)) \). Consequently, \( A(nt) = nt \) for every \( t \in T \). If \( \{t_1, t_2\} \) is an integral basis for \( T \), then \( \{nt_1, nt_2\} \) is a basis for \( \mathbb{R}^2 \). Since \( A \in O_2(\mathbb{R}) \) is determined by its action on a basis, we must have \( A = I \), so \( x = (I, b) \in T \). We have thus proved that \( G_n = T \).

Now, suppose \( G \) and \( G' \) are isomorphic wallpaper groups, and let \( \varphi : G \to G' \) be an isomorphism. It is elementary to see that \( \varphi(G_n) = G'_n \) for all \( n \). Let \( m = [G : T] \) and \( m' = [G' : T'] \). If we set \( n = mm' \), the previous paragraph shows that \( T = G_n \) and \( T' = G'_n \). Since \( \varphi(G_n) = G'_n \), we get \( \varphi(T) = T' \), as desired.

**Corollary 3.6.** If \( G \) and \( G' \) are isomorphic wallpaper groups, then their point groups \( G_0 \) and \( G'_0 \) are isomorphic.

**Proof.** Let \( \varphi : G \to G' \) be an isomorphism. If \( T \) (resp. \( T' \)) is the translation lattice of \( G \) (resp. \( G' \)), then \( \varphi(T) = T' \) by Lemma 3.5. Therefore, \( \varphi \) induces an isomorphism between \( G/T \) and \( G'/T' \). Since these groups are isomorphic to \( G_0 \) and \( G'_0 \), respectively, the point groups \( G_0 \) and \( G'_0 \) are isomorphic.

By looking more carefully at an isomorphism between wallpaper groups, we can prove a stronger statement. In the next result we give a necessary criterion on the point groups for two wallpaper groups to be isomorphic. We will use this criterion to show that certain wallpaper groups are not isomorphic. To help understand this result, if \( G \) is a wallpaper group with translation lattice \( T \) and point group \( G_0 \), the action of \( G_0 \) on \( T \) yields a group homomorphism \( G_0 \to \text{Aut}(T) \cong \text{Aut}(\mathbb{Z}^2) \). By picking a basis \( \{t_1, t_2\} \) of \( T \), elements of \( \text{Aut}(\mathbb{Z}^2) \) can be represented by \( 2 \times 2 \) matrices with integer entries. This gives an isomorphism \( \text{Aut}(\mathbb{Z}^2) \cong \text{Gl}_2(\mathbb{Z}) \), the group of units of the ring \( M_2(\mathbb{Z}) \) of \( 2 \times 2 \) matrices with integer entries.
3.1. Definition and Main Properties

For such a matrix to be invertible over \( \mathbb{Z} \), its determinant must be a unit in \( \mathbb{Z} \). Therefore, \( \text{Gl}_2(\mathbb{Z}) \) consists of \( 2 \times 2 \) integral matrices with determinant \( \pm 1 \). Thus, by using this action, we can represent \( G_0 \) as a subgroup of \( \text{Gl}_2(\mathbb{Z}) \).

**Proposition 3.7.** Let \( \varphi : G \to G' \) be an isomorphism of the wallpaper groups \( G \) and \( G' \). Let \( T \) (resp. \( T' \)) be the translation lattice and \( G_0 \) (resp. \( G'_0 \)) the point group of \( G \) (resp. \( G' \)). By choosing integral bases for \( T \) and \( T' \), the map \( \varphi|_T \) is a linear isomorphism, given by a matrix \( U \in \text{Gl}_2(\mathbb{Z}) \), and the induced isomorphism \( \overline{\varphi} : G_0 \to G'_0 \) is conjugation by \( U \).

**Proof.** Suppose \( \varphi : G \to G' \) is an isomorphism. By Lemma 3.5, the restriction of \( \varphi \) to \( T \) is an isomorphism from \( T \) to \( T' \). Suppose that \( \{ t_1, t_2 \} \) is a basis for \( T \) and \( \{ s_1, s_2 \} \) is a basis for \( T' \). We then have

\[
\begin{align*}
\varphi(t_1) &= \alpha s_1 + \beta s_2, \\
\varphi(t_2) &= \gamma s_1 + \delta s_2
\end{align*}
\]

for some integers \( \alpha, \beta, \gamma, \delta \). Since \( \varphi|_T \) is a \( \mathbb{Z} \)-module isomorphism, it is determined by what it does to the basis of \( T \) and can be represented by a matrix with integer entries

\[
U = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}.
\]

Also, since \( \varphi^{-1} \) is an isomorphism that sends \( T' \) to \( T \), we see that \( U^{-1} \) also has integer entries. Therefore, \( U \in \text{Gl}_2(\mathbb{Z}) \). Now, take \( (A, b) \in G \), and write \( (C, d) = \varphi(A, b) \). For \( t \in T \) we have \( \varphi(I, t) = (I, Ut) \) by the definition of \( U \). Therefore,

\[
(C, d)(I, Ut)(C, d)^{-1} = \varphi ((A, b)(I, t)(A, b)^{-1}),
\]

or

\[
(I, CUt) = \varphi ((I, At)) = (I, UAt).
\]

In other words, \( CU = UA \), so \( C = UAU^{-1} \). Thus, the induced map \( G_0 \to G'_0 \) is conjugation by \( U \).

**Corollary 3.8.** Let \( G \) and \( G' \) be isomorphic wallpaper groups with point groups \( G_0 \) and \( G'_0 \), respectively. Identifying \( G_0 \) and \( G'_0 \) as subgroups of \( \text{Gl}_2(\mathbb{Z}) \) by choosing bases for the translation lattices of \( G \) and \( G' \), there is a matrix \( U \in \text{Gl}_2(\mathbb{Z}) \) with \( G'_0 = UG_0U^{-1} \).

The converse of this corollary is also true. If the translation lattices of two wallpaper groups are isomorphic via a map \( U \) for which conjugation by \( U \) is an isomorphism between their point groups, then we obtain an isomorphism between the groups via \( (g, t) \mapsto (UgU^{-1}, U(t)) \). This corollary tells us that in order for two wallpaper groups to be isomorphic, their point groups must be conjugate in \( \text{Gl}_2(\mathbb{Z}) \), once we have represented them as subgroups of \( \text{Gl}_2(\mathbb{Z}) \). This is a stronger condition than the point groups being isomorphic.
For example, the point groups $C_2$ and $D_1$ are isomorphic. However, by the matrix representations we obtain for them in Section 3.2 below, we see that they are not conjugate in $\text{Gl}_2(\mathbb{Z})$. Therefore, a wallpaper group with point group $C_2$ is not isomorphic to one with point group $D_1$. This corresponds to the geometric fact that a wallpaper pattern with a 180° rotation symmetry and no reflectional symmetry is “different” from one with a reflectional symmetry and no rotation symmetry.

### 3.1.1 Exercises

1. Let $G$ and $G'$ be wallpaper groups with translation lattice $T$ and $T'$, respectively, and point group $G_0$ and $G'_0$, respectively. Let $U \in \text{Gl}_2(\mathbb{Z})$, and suppose that multiplication by $U$ induces an isomorphism from $T$ to $T'$ (viewing these as subgroups of $\mathbb{R}^2$) and that $G'_0 = UG_0U^{-1}$, when viewing $G_0$ and $G'_0$ as subgroups of $\text{Gl}_2(\mathbb{Z})$, prove that the map $(g, t) \mapsto (UgU^{-1}, Ut)$ is an isomorphism from $G$ to $G'$.

### 3.2 The Five Lattice Types of Wallpaper Patterns

In the previous section we proved that the point group $G_0$ of a wallpaper pattern is isomorphic to one of the ten groups $\{C_n, D_n : n = 1, 2, 3, 4, 6\}$. Note that this set has only nine non-isomorphic groups. However, by considering the action of $G_0$ on $T$, we will see that even though $C_2 \cong D_1$ as abstract groups, they will be distinguished by their actions on $T$. By fixing a basis $\{t_1, t_2\}$ of $T$, we have an isomorphism $T \cong \mathbb{Z}^2$, and using the basis, the action of $G_0$ on $T$ induces a group homomorphism $G_0 \to \text{Aut}(\mathbb{Z}^2) \cong \text{Gl}_2(\mathbb{Z})$. In other words, a choice of basis together with the action of $G_0$ on $T$ gives us a representation of $G$ as a specific subgroup of $\text{Gl}_2(\mathbb{Z})$.

We will see that, viewing lattices geometrically, there are five types of lattices with respect to the $G_0$-action; parallelogram, rectangular, square, rhombus, and hexagonal. We will be specific in what we mean as we look at the action of the ten groups above on $T$. We will always choose the coordinate system to assume that $t_1 = (1, 0)$. 

![point group C2](image1.png)  
point group $C_2$  

![point group D1](image2.png)  
point group $D_1$
The groups $C_1, C_2$: parallelogram lattices.

As mentioned above, we will represent a point group as a subgroup of $\text{GL}_2(\mathbb{Z})$ by choosing a basis $\{t_1, t_2\}$ for $T$. The groups $C_1$ and $C_2$ are very easy to describe and their description does not depend on the basis. If $G_0 = C_1$, then

$$C_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$ 

On the other hand, if $G_0 = C_2$, then the rotation of $180^\circ$ is multiplication by $-1$ on $T$. Therefore,

$$C_2 = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$$ 

The lattice in these cases is called a parallelogram lattice. This is the most general situation; we have not given any restriction on the geometry of the lattice.

![Parallelogram Lattice](image)

We next consider the groups $C_n$ and $D_n$ for $n \geq 3$. The following lemma will help us find a convenient basis for $T$ in these cases.

**Lemma 3.9.** Suppose that $G_0$ contains a rotation $r$ about an angle $2\pi/n$ for $n \geq 3$. If $t$ is a nonzero element of $T$ of minimal length, then $\{t, r(t)\}$ is a basis for $T$.

**Proof.** Let $\{t_1, t_2\}$ be a basis for $T$. Then

$$t = at_1 + bt_2$$

$$r(t) = ct_1 + dt_2$$

for some integers $a, b, c, d$. The set $\{t, r(t)\}$ is linearly independent because $n > 2$, so we can solve for $t_1$ in the two equations above; therefore, $t_1 = \alpha t + \beta r t$ for some rational numbers $\alpha, \beta$. Write $\alpha = \alpha_0 + \epsilon$ and $\beta = \beta_0 + \epsilon'$ with $\alpha_0, \beta_0 \in \mathbb{Z}$ and $|\epsilon|, |\epsilon'| \leq 1/2$. We have $s = \alpha_0 t + \beta_0 r(t) \in T$, so $(t_1 - s) = \epsilon t + \epsilon' r(t) \in T$. Since $t$ and $r(t)$ are not parallel, we see that

$$||t_1 - s|| = ||\epsilon t + \epsilon' r(t)|| < ||\epsilon t|| + ||\epsilon' r(t)|| \leq \frac{1}{2} (||t|| + ||r(t)||)$$

$$= \frac{1}{2} (2 ||t||) = ||t||,$$
a contradiction to the minimality of $\|t\|$, unless $s = t_1$. Therefore, $t_1 = s$ is a $\mathbb{Z}$-linear combination of $t$ and $r(t)$. Similarly, $t_2$ is a $\mathbb{Z}$-linear combination of $t$ and $r(t)$. Since $\{t_1, t_2\}$ is a basis of $T$, the set $\{t, r(t)\}$ is also a basis for $T$. □

The groups $C_4$, $D_4$: square lattices.

Let $r$ be a rotation by $90^\circ$ in $G_0$, which we assume to be either $C_4$ or $D_4$. By Lemma 3.9, if $t = t_1$ is a vector in $T$ of minimal length, then $\{t_1, r(t_1)\}$ is a basis for $T$. The lattice is called a square lattice.

With respect to this basis, we see that if $G_0 = C_4 = \langle r \rangle$, then the representation of $G_0$ by this basis is

$$C_4 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

On the other hand, if $G_0 = D_4$, then $G_0$ contains a reflection $f$. The four elements $f, rf, r^2f, r^3f$ are all the reflections in $G_0$. These reflections must preserve the set of vectors in $T$ of minimal length; four such vectors are $\pm t_1, \pm t_2$. However, a short argument shows that any other point on the circle of radius $\|t_1\|$ centered at the origin is a distance of less than $\|t_1\|$ from one of these four points. Figure 3.2 makes this easy to see visually. The difference of these two vectors would then be a vector in $T$ of length less than $\|t_1\|$. Since this is impossible, we see that the four vectors above are all the vectors of minimal length in $T$. The four lines of reflection are then given in the following picture.
3.2. The Five Lattice Types of Wallpaper Patterns

The vectors of minimal length in $T$ when $G_0 = D_4$

Since $D_4$ is generated by $r$ and any reflection, using the reflection about the line parallel to $t_1$, we obtain the representation

$$D_4 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$$  

The groups $C_3, D_3, C_6, D_6$: hexagonal lattices.

Let $r$ be a rotation by $120^\circ$. If $t_1$ is a vector in $T$ of minimal length, then by setting $t_2 = r(t_1)$, the set $\{t_1, t_2\}$ is a basis for $T$, by Lemma 3.9. The lattice in this case is called a hexagonal lattice.

The group $C_3$ is generated by $r$ and $C_6$ is generated by a $60^\circ$ rotation; thus, we obtain

$$C_3 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle$$

and

$$C_6 = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Figure 3.2 indicates that we have six vectors in $T$ of minimal length.
Any point on the circle above other than the six shown is a distance less than \( \|t_1\| \) from one of these six points. This shows that these six vectors are all the vectors of minimal length in \( T \).

If \( G_0 = D_3 \) or \( D_6 \), then \( G_0 \) contains 6 or 12 reflections, respectively. Any reflection must permute the six vectors in Figure 3.2. For \( G_0 = D_6 \), we then have the following six lines of reflection.

The group \( D_6 \) is generated by \( C_6 \) and any reflection; using the reflection that fixes \( t_1 \), we have

\[
D_6 = \left\{ \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\}.
\]

If \( G_0 = D_3 \), then the point group contains three reflections. The lines of reflection are separated by 60° angles; if \( f \) is a reflection in \( D_3 \), then \( r f \) is a reflection whose line of reflection makes a 60° angle with that of \( f \). The reflection lines for \( D_3 \) must be reflection lines for \( D_6 \) since \( D_3 \) is a subgroup of \( D_6 \). We then have two possibilities: The three lines are the lines that are at angles 30°, 90°, 150° with \( t_1 \) or are the lines at angles 0°, 60°, 120° with \( t_1 \). This indicates that \( D_3 \) can act in two ways with respect to this basis. We write \( D_{3,l} \) and \( D_{3,s} \) to distinguish these two actions; therefore, generating \( D_{3,l} \) and \( D_{3,s} \) with the 120° rotation and with the reflection about the 30° and the 0° reflection lines, respectively, we have

\[
D_{3,l} = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\}
\]
3.2. The Five Lattice Types of Wallpaper Patterns

and

\[ D_{3,s} = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\rangle. \]

To give meaning to this subscript notation, we note that \( l \) and \( s \) stand for long and short, respectively. The vectors \( t_1 \) and \( t_2 \) span a parallelogram which has a long and a short diagonal. The group \( D_{3,s} \) contains a reflection about the 60° line, which is the short diagonal. The group \( D_{3,l} \) has a reflection across the 150° line, which is parallel to the long diagonal.

Both of these cases do occur; in the pictures below, the parallelogram formed by the vectors \( t_1 \) and \( t_2 \) is drawn in the center. The case \( D_{3,l} \) occurs in the left-hand picture, and \( D_{3,s} \) in the right-hand picture.

**Proposition 3.10.** The groups \( D_{3,l} \) and \( D_{3,s} \) are not conjugate in \( \text{Gl}_2(\mathbb{Z}) \).

**Proof.** To prove this, suppose there is a matrix \( U \in \text{Gl}_2(\mathbb{Z}) \) with \( D_{3,l} = UD_{3,s}U^{-1} \). Because conjugation preserves determinants and the determinant of a reflection is \(-1\), the three reflections of \( D_{3,s} \) must be sent to the three reflections of \( D_{3,l} \). We can obtain any reflection in \( D_3 \) from any other reflection by conjugation by \( I \), \( r \), or \( r^2 \). Therefore, we may assume that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

for some \( a, b, c, d \in \mathbb{Z} \) with \( ad - bc = \pm 1 \). Simplifying yields \( d = -a \) and \( c = -b \). Therefore, \( ad - bc = b^2 - a^2 = (b-a)(b+a) \). Since this is \( \pm 1 \), one term is 1 and the other is \(-1\). We then have four cases, \( a = \pm 1 \) and \( b = 0 \) or \( a = 0 \) and \( b = \pm 1 \). Conjugation by \(-I_2\) is the identity; therefore, we may assume that

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
or
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

However, since
\[
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}
\]
and
\[
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},
\]
neither conjugation sends $D_{3,s}$ to $D_{3,l}$ since neither of these results is an element of $D_{3,l}$. The groups $D_{3,l}$ and $D_{3,s}$ are thus not conjugate in $\text{GL}_2(\mathbb{Z})$.

A consequence of the proposition is that two wallpaper groups with point groups $D_{3,l}$ and $D_{3,s}$, respectively, are not isomorphic, by Proposition 3.8.

**The groups $D_1, D_2$: rectangular or rhombic lattices.**

If $G_0 = D_1$ or $D_2$, then $G_0$ does not contain a rotation of order at least 3. Therefore, we cannot apply Lemma 3.9 to obtain a basis for $T$. We produce a basis in another way. In each of these cases we have a nontrivial reflection $f$ in $G_0$. Let $t \in T$ be a nonzero vector not parallel to the line of reflection of $f$. Since $f$ maps $T$ to $T$, the vectors $t + f(t)$ and $t - f(t)$ are elements of $T$, so $T$ contains nonzero vectors both parallel and perpendicular to the line of reflection.

Let $s_1$ and $s_2$ be nonzero vectors of minimal length parallel and perpendicular, respectively, to the reflection line. The discrete nature of $T$ implies that such vectors exist, and that any vector parallel to (resp. perpendicular to) this line is an integer multiple of $s_1$ (resp. $s_2$). Therefore, for any $t \in T$, we have

\[
t + f(t) = m_t s_1,
\]
\[
t - f(t) = n_t s_2
\]
for some $m_t, n_t \in \mathbb{Z}$. Solving for $t$ gives

\[
t = \frac{m_t}{2} s_1 + \frac{n_t}{2} s_2.
\]

If, for every $t \in T$, both integers $m_t, n_t$ are even, the set $\{s_1, s_2\}$ spans $T$, and so is a basis for $T$. On the other hand, if $m_t$ or $n_t$ is odd for some $t$, then both have to be odd, else $\frac{1}{2} s_1$
or $\frac{1}{2}s_2$ is in $T$, a contradiction. If we set $t_1 = \frac{1}{2}(s_1 + s_2)$ and $t_2 = \frac{1}{2}(s_1 - s_2) = f(t_1)$, then $t_1, t_2 \in T$, and

$$t = \frac{m_t}{2}s_1 + \frac{n_t}{2}s_2 = \left(\frac{m_t + n_t}{2}\right)\left(\frac{s_1 + s_2}{2}\right) + \left(\frac{m_t - n_t}{2}\right)\left(\frac{s_1 - s_2}{2}\right)$$

$$= m't_1 + n't_2$$

with $m', n' \in \mathbb{Z}$. Since any $t$ is then an integral linear combination of $t_1$ and $t_2$, the set $\{t_1, t_2\}$ is a basis for $T$.

To summarize these two cases, we either have a basis $\{t_1, t_2\}$ of two orthogonal vectors, one of which is fixed by a reflection in $G_0$,

or we have a basis of vectors of the same length with a reflection that interchanges them.

In the first case we say that $T$ is a **rectangular lattice**

and in the second case that $T$ is a **rhombic lattice.**
Rhombic Lattice

We can now get matrix representations for $D_1$ and $D_2$. For each group there are two possibilities, corresponding to two different actions on $T$. We subscript the group by $p$ for rectangular and $c$ for rhombic. We are using these subscripts to match common notation used for wallpaper groups which will be described in Chapter 4.1. We have

$$D_{1,p} = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$$

and

$$D_{1,c} = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle,$$

while for $D_2$, which contains a rotation of $180^\circ$, we obtain

$$D_{2,p} = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$$

and

$$D_{2,c} = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle.$$

**Proposition 3.11.** The groups $D_{1,p}$ and $D_{1,c}$ are not conjugate in $\text{Gl}_2(\mathbb{Z})$, and neither are the groups $D_{2,p}$ and $D_{2,c}$.

**Proof.** For $D_{1,p}$ and $D_{1,c}$, suppose that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$. Multiplying these and setting the two sides equal yields $d = -b$ and $c = -a$. Then $ad - bc = -2ab$, which is not $\pm 1$ since $a$ and $b$ are integers. Therefore, $D_{1,p}$ and $D_{1,c}$ are not conjugate in $\text{Gl}_2(\mathbb{Z})$. For $D_{2,p}$ and $D_{2,c}$, the previous calculation shows that we need only check if there are $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Similar calculations show that this forces $2ab = \pm 1$, again a contradiction. \[\square\]

This proposition shows that no wallpaper group whose point group is one of the four groups $D_{1,p}, D_{1,c}, D_{2,p}, D_{2,c}$ is isomorphic to a wallpaper group whose point group is another. We finish this chapter by pointing that neither $D_1$ is conjugate to $C_2$, a point we mentioned in the previous section. For, $C_2$ is a central subgroup of $\text{O}_2(\mathbb{R})$, so it is conjugate only to itself. Since neither $D_1$ is equal to $C_2$, neither is conjugate to $C_2$.  


3.2.1 Exercises

1. In the subsection on $C_4$ and $D_4$, show that the four vectors $\pm t_1, \pm t_2$ are all of the vectors in $T$ of minimal length.

2. In the subsection on $D_3$, $D_3$, $C_6$, $D_6$, show that the six vectors $t_1, t_1 + t_2, t_2, -t_1, -t_1 - t_2, -t_2$ are all of the vectors in $T$ of minimal length.

3. In the subsection on $D_1, D_2$, if $\ell$ is a reflection line, prove that there exist nonzero vectors of minimal length parallel and perpendicular, respectively to $\ell$. 
Chapter 4

The Wallpaper Groups

In the previous chapter we determined, up to conjugacy, all possible point groups $G_0$ of a wallpaper pattern. We will now see how these calculations lead us to a determination of all wallpaper groups, up to isomorphism. We also give an explicit description of a wallpaper group in each isomorphism class as a subgroup of $\text{Isom}(\mathbb{R}^2)$ given by generators along with pictures of corresponding wallpaper patterns.

4.1 Notational Description of the Wallpaper Groups

In the previous chapter, by making use of an appropriate basis, we showed that any wallpaper group has a point group which is conjugate in $\text{GL}_2(\mathbb{Z})$ to one of the following thirteen groups:

- $C_1 = \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$
- $D_{1,p} = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$
- $D_{1,c} = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$
- $C_2 = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$
- $D_{2,p} = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$
- $D_{2,p} = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$
- $C_3 = \langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \rangle$
- $D_{3,l} = \langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \rangle$
- $D_{3,s} = \langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$
- $C_4 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$
- $D_4 = \langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$
- $D_6 = \langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \rangle$

In this chapter we will determine all possible wallpaper groups. We will give a description of the groups in this section, and will prove the classification in the next section. For ten of the thirteen possibilities for the point group $G_0$ there is only one wallpaper group for the given point group. For $D_{1,p}$ and $D_4$, we will see that there are two different wallpaper groups for each of these point groups, and for $D_{2,p}$, there are three different wallpaper groups. Therefore, there are seventeen different wallpaper groups. We list them now. To have a notation to refer to them, we will use the standard notation used by crystallographers.
Table 4.1: The 17 Wallpaper Groups

<table>
<thead>
<tr>
<th>Wallpaper Group</th>
<th>Full Name</th>
<th>Point Group</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>p</em>&lt;sub&gt;1&lt;/sub&gt;</td>
<td><em>p</em>111</td>
<td><em>C</em>&lt;sub&gt;1&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>cm</em></td>
<td><em>c</em>1m1</td>
<td><em>D</em>&lt;sub&gt;1,c&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>pm</em></td>
<td><em>p</em>1m1</td>
<td><em>D</em>&lt;sub&gt;1,p&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>pg</em></td>
<td><em>p</em>1g1</td>
<td></td>
</tr>
<tr>
<td><em>p</em>&lt;sup&gt;2&lt;/sup&gt;</td>
<td><em>p</em>211</td>
<td><em>C</em>&lt;sub&gt;2&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>cmm</em></td>
<td><em>c</em>2mm</td>
<td><em>D</em>&lt;sub&gt;2,c&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>pmm</em></td>
<td><em>p</em>2mm</td>
<td></td>
</tr>
<tr>
<td><em>pmg</em></td>
<td><em>p</em>2mg</td>
<td></td>
</tr>
<tr>
<td><em>pgg</em></td>
<td><em>p</em>2gg</td>
<td></td>
</tr>
<tr>
<td><em>p</em>3</td>
<td><em>p</em>311</td>
<td><em>C</em>&lt;sub&gt;3&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>p</em>3m1</td>
<td><em>p</em>3m1</td>
<td><em>D</em>&lt;sub&gt;3,l&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>p</em>31m</td>
<td><em>p</em>31m</td>
<td><em>D</em>&lt;sub&gt;3,s&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>p</em>4</td>
<td><em>p</em>411</td>
<td><em>C</em>&lt;sub&gt;4&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>p</em>4m</td>
<td><em>p</em>4m1</td>
<td><em>D</em></td>
</tr>
<tr>
<td><em>p</em>4g</td>
<td><em>p</em>4g1</td>
<td></td>
</tr>
<tr>
<td><em>p</em>6</td>
<td><em>p</em>611</td>
<td><em>C</em>&lt;sub&gt;6&lt;/sub&gt;</td>
</tr>
<tr>
<td><em>p</em>6m</td>
<td><em>p</em>6m1</td>
<td><em>D</em></td>
</tr>
</tbody>
</table>

We give a brief description of the naming scheme. First of all, we choose a basis \( t_1, t_2 \) for the translation lattice as in Section 3.2. We consider the direction of \( t_1 \) to be the \( x \)-axis. The full name consists of four symbols. The first symbol represents the lattice type; \( p \) for primitive and \( c \) for centered (or rhombic). The second symbol is the largest order of a rotation. The third symbol is either an \( m \), \( g \), or 1. An \( m \) (resp. \( g \)) means there is a reflection line (resp. glide reflection line but not a reflection line) perpendicular to the \( x \)-axis while a 1 means there is no line of either type. Finally, the fourth symbol is also either an \( m \), \( a \), or a 1. In this case an \( m \) (resp. \( g \)) represents a reflection line (resp. glide reflection line) at an angle \( \alpha \) with the \( x \)-axis, the angle depending on the largest order of rotation as follows: \( \alpha = 180^\circ \) for \( n = 1, 2 \); \( \alpha = 60^\circ \) for \( n = 3, 6 \); \( \alpha = 45^\circ \) for \( n = 4 \).

For example, the group name \( p*3m1 \) represents a group with a \( 120^\circ \) rotation, a reflection line perpendicular to the \( x \)-axis, and no reflection or glide line at an angle of \( 60^\circ \) with the \( x \)-axis. However, in the group \( p*31m \), we have the same rotation, but no reflection or glide line perpendicular to the \( x \)-axis, while there is a reflection line at an angle of \( 60^\circ \) with the \( x \)-axis.
4.2 Classification of Wallpaper Patterns

In this section we classify the wallpaper groups. There is a fair amount of overlap with Section 4.1, but we repeat many ideas here to be self contained.

In this section, as in the previous section, we will use systematically the notation \((g, t_g)\) to denote an element of a wallpaper group. The element \(t_g\) need not be an element of \(T\). For example, if we view \(pg\) as the symmetry group of Escher’s Horseman on Page 26, then it contains a glide reflection of the form \((f, \frac{1}{2}t_1)\) with neither \((f,0)\) nor the translation component \((I, \frac{1}{2}t_1)\) an element of \(pg\).

To begin, we prove a simple lemma to enable us to write down elements of a wallpaper group in terms of the point group and certain vectors in \(\mathbb{R}^2\). We will see that it is the determination of these vectors that will determine our wallpaper groups.

**Lemma 4.1.** Let \(G\) be a wallpaper group with point group \(G_0\). For each \(g \in G_0\) there is a vector \(t_g \in \mathbb{R}^2\) with \((g, t_g) \in G\). Furthermore, \(t_g\) is uniquely determined up to addition by an element of \(T\). Moreover, \(G = \{(g, t_g + t) : g \in G_0, t \in T\}\).

**Proof.** Recall from Proposition 3.2 that the map \(\varphi : G \to G_0\) defined by \(\varphi(g, t) = g\) is a surjective homomorphism with kernel \(T\). Therefore, for each \(g \in G_0\), there is a vector \(t_g\) with \((g, t_g) \in G\). If \((g, s_g) \in G\), then \(\varphi(g, s_g) = \varphi(g, t_g)\), so \((g, s_g) \equiv (g, t_g) (\text{mod ker} (\varphi))\). Since \(\text{ker} (\varphi) = T\), there is a \(t \in T\) with \((g, s_g) = (I, t)(g, t_g) = (g, t + t_g)\). Thus, \(s_g = t_g + t\).

In other words, \(t_g\) is uniquely determined up to addition by an element of \(T\). Finally, since \(\varphi\) is a surjection onto \(G_0\) with kernel \(T\), and since \((g, t_g) \in \varphi^{-1}(g)\) and \(G = \bigcup_{g \in G_0} \varphi^{-1}(g)\), we see that \(G = \{(g, t_g + t) : g \in G_0, t \in T\}\). \(\Box\)

We will refer to the vectors \(\{t_g\}_{g \in G_0}\) as the point group vectors for the wallpaper group \(G\). One way to think about them is that \(t_g\) is the unique vector, module addition by an element of \(T\), satisfying \((g, t_g) \in G\). Thus, if for some \(g \in G_0\) we have \((g, s) \in G\) for some \(s \in \mathbb{R}^2\), we may choose \(t_g = s\), or given an earlier choice of \(t_g\), we have \(s = t_g + t\) for some \(t \in T\). From the lemma, it follows that if \(\{t_1, t_2\}\) is a basis for the translation lattice, then \(G\) is generated by the two translations \((I, t_1), (I, t_2)\) together with the elements \(\{(g, t_g) : g \in G_0\}\). All told, this is a finite set of generators. To describe explicitly a wallpaper group \(G\), it is then sufficient to determine the vectors \(\{t_g\}\).

Let \(G\) be a wallpaper group with point group \(G_0\) and translation lattice \(T\). To determine \(G\) we need to know \(G_0\) and to determine the possible choices of the point group vectors. We will accomplish this by determining restrictions on the vectors, and then determining all sets \(\{t_g\}\) satisfying these restrictions.

**Lemma 4.2.** Let \(G\) be a wallpaper group with point group \(G_0\) and translation lattice \(T\). If \(G = \{(g, t_g + t) : g \in G_0, t \in T\}\), then the point group vectors \(t_g \in \mathbb{R}^2\) are unique modulo addition by an element of \(T\) and satisfy

\[
g + g(t_h) - t_{gh} \in T.
\]  \hfill (4.1)
Conversely, if $G_0$ is a subgroup of $O_2(\mathbb{R})$ which acts on a 2-dimension lattice $T \subseteq \mathbb{R}^2$, and if $\{t_g\}_{g \in G_0} \subseteq \mathbb{R}^2$ satisfies Equation 4.1, then $G := \{(g, t_g + t) : g \in G_0, t \in T\}$ is a wallpaper group with point group $G_0$ and translation lattice $T$.

Proof. If $g, h \in G_0$, then
\[(g, t_g)(h, t_h) = (gh, g + g(t_h)).\]
Since $(gh, t_{gh})$ is another element of $G$, this forces $(gh, t_g + g(t_h))(gh, t_{gh})^{-1} \in G$. However,
\[
(gh, t_g + g(t_h))(gh, t_{gh})^{-1} = (gh, t_g + g(t_h))(h^{-1}g^{-1}, -h^{-1}g^{-1}(t_{gh}))
\]
\[
= (I, t_g + g(t_h) - t_{gh}).
\]
Therefore, $t_g + g(t_h) - t_{gh} \in T$.

Conversely, if we have, for each $g \in G_0$, an element $t_g \in \mathbb{R}^2$ such that Equation 4.1 is satisfied, then the set $\{(g, t + t_g) : g \in G_0, t \in T\}$ is a subgroup of $\text{Isom}(\mathbb{R}^2)$; we leave this fact for an exercise. If $\varphi : \text{Isom}(\mathbb{R}^2) \to O_2(\mathbb{R})$ is the usual map which sends $(g, t)$ to $g$, then $\varphi(G) = G_0$ and $\ker(\varphi|_G) = T$. We leave as an exercise that there is a subset $W \subseteq \mathbb{R}^2$ with $G = \text{Sym}(W)$. Thus, since $G \cap T$ is a 2-dimensional lattice, $G$ is a wallpaper group. The conditions $\varphi(G) = G_0$ and $\ker(\varphi|_G) = T$ show that the point group is $G_0$ and the translation lattice is $T$.

For example, if $t_g = 0$ for all $g$, then Equation 4.1 is clearly satisfied, and the corresponding wallpaper group is $G = \{(g, t) : g \in G_0, t \in T\}$. In this case $G_0$ is actually isomorphic to the subgroup $\{(g, 0) : g \in G_0\}$ of $G$. In this situation, $G$ is the semidirect product of $G_0$ and $T$. When this occurs, we refer to $G$ as the split wallpaper group with point group $G_0$ and translation lattice $T$. Since there are thirteen different point groups, we have thirteen different split wallpaper groups.

Next, we consider how we can tell if wallpaper groups are isomorphic by comparing their respective sets of point group vectors.

Lemma 4.3. Suppose that $G$ and $G'$ are wallpaper groups with the same point group $G_0$ and the same translation lattice $T$, and with sets of point group vectors $\{t_g\}_{g \in G_0}$ and $\{t'_g\}_{g \in G'_0}$, respectively. If there is a vector $v \in \mathbb{R}^2$ with $t'_g = t_g + v - g(v)$ for each $g \in G_0$, then $G \cong G'$.

Proof. Suppose there is $v \in \mathbb{R}^2$ satisfying $t'_g = t_g + v - g(v)$ for each $g \in G_0$, and consider the inner automorphism $\sigma : \text{Isom}(\mathbb{R}^2) \to \text{Isom}(\mathbb{R}^2)$ given by $(g, t) \mapsto (I, v)(g, t)(I, v)^{-1}$. This map is given by the formula
\[
(1, v)(g, t)(I, v)^{-1} = (I, v)(g, t)(I, -v) = (g, v + t)(I, -v)
\]
\[
= (g, v + t - g(v)).
\]
Therefore, $\sigma(g, t_g + t) = (g, t_g + v - g(v) + t) = (g, t'_g + t)$ for any $t \in T$. This shows that $\sigma(G) = G'$. Thus, $\sigma|_G$ yields an isomorphism from $G$ to $G'$.
Corollary 4.4. Let $G$ be a wallpaper group with point group vectors $\{t_g\}_{g \in G_0}$. If there is $v \in \mathbb{R}^2$ with $t_g \equiv v - g(v)(\text{mod } T)$ for each $g \in G_0$, then $G$ is a split wallpaper group.

Proof. By hypothesis, there is $s_g \in T$ with $t_g = s_g + v - g(v)$. Because the $t_g$ are unique only up to addition by an element of $T$, we may replace $t_g$ by $t_g - s_g$ to assume that $t_g = v - g(v)$ for each $g \in G_0$. The split wallpaper group $G'$ has point group vectors $t'_g = 0$ for each $g \in G_0$. Since $t'_g = t_g + v - g(v)$ for each $g \in G_0$, the lemma shows that $G \cong G'$, so $G$ is a split wallpaper group.

The converse is somewhat technical, and we leave it for a homework problem.

We have seen that there are thirteen possible point groups; therefore, we have thirteen split wallpaper groups. We will see that there are just four more wallpaper groups once we finish the classification.

Proposition 4.5. Let $G$ be a wallpaper group with point group $C_n$ for some $n$. Then $G$ is isomorphic to a split wallpaper group.

Proof. The argument for the case $n = 1$ is different from the general argument. If $n = 1$, then $G_0 = 0$, so $G = T$ is split. Thus, we only need to consider the case $G_0 = C_n$ with $n > 1$. Let $r$ be a generator of $C_n$. With notation above, for each $i$ with $1 \leq i < n$, there are $t_
u i \in \mathbb{R}^2$ with $(r^i, t_
u i) \in G$. For ease of notation, we set $u = t_r$. Note that, by induction, we have

$$(r, u)^i = (r^i, u + r(u) + \cdots + r^{i-1}(u)).$$

Since $t_
u i$ is uniquely determined up to addition by an element of $T$, we may assume that $t_
u i = u + r(u) + \cdots + r^{i-1}(u)$. Now, since $r$ is a nontrivial rotation, it fixes a unique point. As a consequence, $I - r$ is invertible, so there is a vector $v \in \mathbb{R}^2$ with $u = v - r(v)$. Another simple induction argument yields $t_
u i = v - r^i(v)$. Therefore, by Lemma 4.3, we see that $G$ is split.

This proposition also gives information about wallpaper groups with point group $D_n$.

Corollary 4.6. Let $G$ be a wallpaper group with point group $D_n$, and let $\varphi : G \to G_0 = D_n$ be the canonical homomorphism. Then $\varphi^{-1}(C_n)$ is a split wallpaper group, and if $f \in D_n \setminus C_n$ with $(f, t_f) \in G$, then $G$ is generated by $\varphi^{-1}(C_n)$ and $(f, t_f)$.

Proof. The group $\varphi^{-1}(C_n)$ contains $\ker(\varphi) = T$, so the translation subgroup of $\varphi^{-1}(C_n)$ is $T$, and thus it is a wallpaper group. It is split by Proposition 4.5. Finally, the group generated by $\varphi^{-1}(C_n)$ and $(f, t_f)$ contains $T$ and maps onto $D_n$; thus it must be $G$.

We use this corollary to determine wallpaper groups with point group $D_n$. We write $D_n = (r, f)$ with relations $r^n = e = f^2$ and $frf = r^{-1}$. Let $G$ be a wallpaper group with point group $D_n$, and suppose that its set of point group vectors is $\{t_g\}_{g \in D_n}$. As a consequence of the proof of the corollary, we may choose $t_{fr} = 0$ for all $i$. Set $u = t_f$. Since
(r, 0)(f, u) = (r^if, r^i(u)), we may choose \( t_{r^if} = r^i(u) \). To summarize, we have the following setup:

\[
\begin{align*}
  t_{r_i} &= 0, \\
  t_f &= u, \\
  t_{r^if} &= r^i(u)
\end{align*}
\]

for each \( i \). Therefore, the choice of \( u \) completely determines \( G \). One or two special cases of Equation 4.1 will allow us to determine \( u \). First, if \( g = h = f \), then Equation 4.1 yields

\[
u + f(u) \in T \tag{4.2}
\]
since \( t_f^2 = t_f = 0 \). Next, with \( g = rf \) and \( h = r \), we have

\[
(rf, r(u))(r, 0) = (rfr, r(u)) = (f, r(u)).
\]

Therefore,

\[
r(u) - u \in T. \tag{4.3}
\]

We will use Equations ??, 4.2, and 4.3 to describe all wallpaper groups with a dihedral point group. We break up the classification into cases, one for each possible dihedral point group.

**Proposition 4.7.** There are two nonisomorphic wallpaper groups with point group \( D_{1,p} \).

**Proof.** Consider \( G_0 = D_{1,p} \). Then \( G_0 = \langle f \rangle \), and the only condition we need to consider is \( u + f(u) \in T \). We saw in Section 3.2 that \( T \) has a basis \( \{t_1, t_2\} \) with \( f(t_1) = t_1 \) and \( f(t_2) = -t_2 \). Let \( u = \alpha t_1 + \beta t_2 \) with \( \alpha, \beta \in \mathbb{R} \). Because \( u \) is only determined up to addition by an element of \( T \), we may modify \( \alpha \) and \( \beta \) to assume that \( 0 \leq \alpha, \beta < 1 \). We have \( u + f(u) = 2\alpha t_1 \). For this to be an element of \( T \) we must have \( 2\alpha \in \mathbb{Z} \), so \( \alpha = 0 \) or \( \alpha = \frac{1}{2} \). Therefore, there are two wallpaper groups with point group \( D_{1,p} \); one is the split group and the other corresponds to the choice of \( u = \frac{1}{2}t_1 + \beta t_2 \). The choice of \( \beta \) is irrelevant; if \( G \) corresponds to the choice of \( u = \frac{1}{2}t_1 \) and \( G' \) corresponds to the choice of \( u = \frac{1}{2}t_1 + \beta t_2 \) for any \( \beta \), then \( G' \cong G \) via the isomorphism given by conjugation by \((I, \frac{1}{2} \beta t_2)\), as Lemma 4.3 shows. The choice of \( u = \frac{1}{2}t_1 + \beta t_2 \) does not yield a split wallpaper group, since if it did, then there would be \( t \in T \) such that \( \langle f, u + t \rangle \) is isomorphic to \( G_0 = D_1 \), which in turn means \( (f, u + t)^2 = \text{id} \). However, \((f, u + t)^2 = (I, u + t + f(u) + f(t)) \). Since \( f(u) = u \), we see that \( t_1 + t + f(t) = 0 \). Because \( t + f(t) \) has even \( t_1 \)-coefficient, \( t + f(t) \neq -t_1 \), which shows this choice of \( u \) does indeed result in a non-split group. 

The following pictures illustrate the two groups, \( pm \) and \( pg \), respectively.

**Proposition 4.8.** The only wallpaper group with point group \( D_{1,c} \) is split.
4.2. Classification of Wallpaper Patterns

Proof. Let \( G_0 = D_{1,c} \). The difference between \( D_{1,c} \) and \( D_{1,p} \) is that for \( D_{1,c} \) the lattice \( T \) has a basis \( \{t_1, t_2\} \) with \( f(t_1) = t_2 \) and \( f(t_2) = t_1 \). As above, we write \( u = t_f \). If \( u = \alpha t_1 + \beta t_2 \), then \( u + f(u) = (\alpha + \beta)(t_1 + t_2) \). For this to be an element of \( T \), we must have \( \alpha + \beta \in \mathbb{Z} \). We may choose \( \beta = -\alpha \) since \( u \) is uniquely determined modulo \( T \). Then \( u = \alpha(t_1 - t_2) \). The choice of \( \alpha \) does not affect the group since if \( G \) corresponds to \( \alpha = 0 \), so \( u = 0 \), and if \( G' \) corresponds to another choice of \( \alpha \), then \( G' \cong G \) via the isomorphism given by conjugation by \( (I, \alpha t_1) \). Thus, \( G \) is split, as we see by taking \( \alpha = 0 \), so \( u = 0 \).

Proposition 4.9. The only wallpaper group with point group \( D_{3,l} \) or \( D_{3,s} \) is split.

Proof. We consider \( D_{3,l} \) and \( D_{3,s} \) simultaneously. For both cases \( T \) has a basis \( \{t_1, t_2\} \) such that the rotation \( r \) of \( 120^\circ \) satisfies \( r(t_1) = t_2 \) and \( r(t_2) = -t_1 - t_2 \). If \( f \) is any reflection, Equation 4.3 forces \( r(u) - u \in T \), where \( u = t_f \). If \( u = \alpha t_1 + \beta t_2 \), then

\[
r(u) - u = \alpha t_2 + \beta(-t_1 - t_2) - (\alpha t_1 + \beta t_2) = - (\alpha + \beta)t_1 + (\alpha - 2\beta)t_2.
\]

Thus, \( r(u) - u \in T \) when \( \alpha + \beta \in \mathbb{Z} \) and \( \alpha - 2\beta \in \mathbb{Z} \). If we restrict \( \beta \) to the range \( 0 \leq \beta < 1 \), then these two conditions force \( \beta = 0, \frac{1}{3}, \frac{2}{3} \). Then \( \alpha \) is determined uniquely, modulo \( \mathbb{Z} \), as \( -\beta \). Thus, we have three possible choices for \( u \), either \( 0, \frac{1}{3}(-t_1 + t_2), \) or \( \frac{2}{3}(-t_1 + t_2) \). We wish to show that all three choices produce isomorphic wallpaper groups. To do this we use Equation 4.2. We point out that with \( t_{r} = 0 \) and \( t_{r \cdot f} = r^i(u) \), Equation 4.2 is satisfied if and only if there is a \( v \in \mathbb{R}^2 \) with \( r(v) \equiv v(\text{mod } T) \) and \( u \equiv v - f(v)(\text{mod } T) \). We show that \( v = -u \) satisfies these conditions. Since \( r(u) - u \in T \), we see that \( r(v) \equiv v(\text{mod } T) \). Next, \( v - f(v) = -u + f(u) \equiv 2u(\text{mod } T) \) since Equation 4.2 yields \( f(u) \equiv u(\text{mod } T) \). Thus, \( u - (v - f(v)) \equiv 3u(\text{mod } T) \). In all three possibilities for \( u \), we have \( 3u \equiv 0(\text{mod } T) \). Thus, Equation 4.2 shows that a wallpaper group with point group \( D_{3,l} \) or \( D_{3,s} \) is split.

Proposition 4.10. The only wallpaper group with point group \( D_6 \) is split.
Proof. For \( G_0 = D_6 \) our argument is similar to the previous case. We say in the previous chapter that there is a basis \( \{t_1, t_2\} \) with \( r(t_1) = t_1 + t_2 \) and \( r(t_2) = -t_1 \). Therefore, if \( t_f = u = \alpha t_1 + \beta t_2 \), then

\[
    r(u) - u = \alpha(t_1 + t_2) - \beta t_1 - (\alpha t_1 - \beta t_2) = -\beta t_1 + (\alpha + \beta)t_2.
\]

For \( r(u) - u \in T \), we must have \( \beta \in \mathbb{Z} \) and \( \alpha + \beta \in \mathbb{Z} \). Thus, both \( \alpha, \beta \in \mathbb{Z} \), so any choice of \( \alpha \) and \( \beta \) yields the same group, up to isomorphism, as the choice \( \alpha = \beta = 0 \). Therefore, we only have the split wallpaper group with point group \( D_6 \). \( \square \)

**Proposition 4.11.** There are two nonisomorphic wallpaper groups with point group \( D_4 \).

Proof. Let \( G_0 = D_4 \). There is a basis \( \{t_1, t_2\} \) for \( T \) with \( r(t_1) = t_2 \) and \( r(t_2) = -t_1 \). Set \( u = t_f = \alpha t_1 + \beta t_2 \). The condition \( r(u) - u \in T \) says \( (\alpha t_2 - \beta t_1) - (\alpha t_1 + \beta t_2) \in T \), or \( (-\alpha - \beta)t_1 + (\alpha - \beta)t_2 \in T \). In other words, \( \alpha + \beta \in \mathbb{Z} \) and \( \alpha - \beta \in \mathbb{Z} \). Therefore, with \( 0 \leq \alpha, \beta < 1 \), we have the solutions \( \alpha = \beta = 0 \) and \( \alpha = \beta = \frac{1}{2} \). So, either \( u = 0 \) or \( u = \frac{1}{2}(t_1 + t_2) \). We then have two wallpaper groups with point group \( D_4 \). The proof that the second choice results in a non-split group is similar to the argument for the case \( D_{1,9} \), and we will leave it as an exercise. \( \square \)

The two groups with point group \( D_4 \) are illustrated with the following wallpaper patterns.

![Wallpaper patterns](p4m.png) ![Wallpaper patterns](p4g.png)

Figure 4.2: Wallpaper patterns with groups \( p4m \) and \( p4g \)

**Proposition 4.12.** The only wallpaper group with point group \( D_{2,c} \) is split.

Proof. Let \( G_0 = D_{2,c} \). Then \( T \) has a basis \( \{t_1, t_2\} \) such that \( f(t_1) = t_2 \) and \( f(t_2) = t_1 \). The 180° rotation \( r \) satisfies \( r(t) = -t \) for all \( t \in T \). Let \( u = t_f = \alpha t_1 + \beta t_2 \) as above. The condition \( r(u) - u \in T \) and \( 0 \leq \alpha, \beta < 1 \) forces \( \alpha = 0, \frac{1}{2} \) and \( \beta = 0, \frac{1}{2} \). Since \( f(u) + u = (\alpha + \beta)(t_1 + t_2) \), this says \( \alpha + \beta \in \mathbb{Z} \). Therefore, \( u = 0 \) or \( u = \frac{1}{2}(t_1 + t_2) \). Remembering that we can modify \( u \) by adding an element of \( T \), in the second case we may replace \( u \) by \( u - t_2 \) to assume that \( u = \frac{1}{2}(t_1 - t_2) \). Then the two groups obtained by the two choices of \( u \) are isomorphic, since \( \frac{1}{2}(t_1 - t_2) = (I - f)(\frac{1}{2}t_1) \). Therefore, we have only one wallpaper group, which is the split group. \( \square \)
Proposition 4.13. There are three nonisomorphic wallpaper groups with point group $D_{2p}$.

Proof. Let $G_0 = D_{2p}$. Then $T$ has a basis $\{t_1, t_2\}$ with $f(t_1) = t_1$ and $f(t_2) = -t_2$. As with the previous case, $r(t) = -t$ for all $t \in T$. If $u = \alpha t_1 + \beta t_2$, then $r(t) - t \in T$ says that $\alpha = 0, \frac{1}{2}$ and $\beta = 0, \frac{1}{2}$. The condition $f(u) + u \in T$ gives no further restriction. Therefore, we have four possibilities:

$$
\begin{align*}
  u &= 0, \\
  u &= \frac{1}{2} t_1, \\
  u &= \frac{1}{2} t_2, \\
  u &= \frac{1}{2} (t_1 + t_2).
\end{align*}
$$

We claim that these four cases yield three nonisomorphic groups. We denote the resulting groups corresponding to these four choices of $u$ by $pmm$, $pmg$, $pmg'$, and $pgg$, respectively. We will show that $pmg'$ and $pmg$ are isomorphic, and that no two of $pmm$, $pmg$, and $pgg$ are isomorphic. These groups have the following descriptions

$$
\begin{align*}
  pmm &= \langle t_1, t_2, r, g_1 \rangle, \\
  pmg &= \langle t_1, t_2, r, g_2 \rangle, \\
  pmg' &= \langle t_1, t_2, r, g_3 \rangle, \\
  pgg &= \langle t_1, t_2, r, g_4 \rangle,
\end{align*}
$$

where $T$ is generated by $t_1, t_2$, and the $g_i$ are given by

$$
\begin{align*}
  g_1 &= \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right), \\
  g_2 &= \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} t_1 \right), \\
  g_3 &= \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} t_2 \right), \\
  g_4 &= \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} (t_1 + t_2) \right).
\end{align*}
$$

The groups $pmg$ and $pmg'$ are isomorphic because if

$$
U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

the matrix that represents the reflection about the line $y = x$, then conjugation by $U$ is an isomorphism between $pmg$ and $pmg'$.

We thus have at most three nonisomorphic wallpaper groups in this case, $pmm$, $pmg$, and $pgg$. To see they are pairwise nonisomorphic, we argue first that $pmm$ is not isomorphic to either $pmg$ or $pgg$, and second that $pmg$ and $pgg$ are not isomorphic. First, note that $pmm$ contains a subgroup of order 4, the subgroup generated by $r$ and the horizontal reflection $f$. We claim that $pmg$ does not contain such a subgroup. Lemmas 2.1 and 2.2 imply that the elements of $pmg$ of order 2 are $(r, t)$ for any $t \in T$ and $(rf, nt_2)$ for any $n \in \mathbb{Z}$. However, no product of two of these elements also has order 2. This proves that $pmg$ is not isomorphic to $pmm$. Similarly, the only elements of $pgg$ of order 2 are of the form $(r, t)$ and the product...
of any two has infinite order. So, \(pmm\) is not isomorphic to \(pgg\). Second, if there is an isomorphism \(\varphi\) from \(pmg\) to \(pgg\), Corollary 3.8 shows that there is a matrix \(U \in \text{Gl}_2(\mathbb{Z})\) with \(\varphi(g, t) = (UgU^{-1}, Ut)\) for all \((g, t) \in pmg\). From the description above of elements of order 2 in \(pmg\) and \(pgg\), \((rf, 0)\) is sent to an element of the form \((r, t)\) for some \(t \in T\). This forces \(UrU^{-1} = r\), a contradiction since \(\det(rf) = -1\) and \(\det(r) = 1\). Therefore, \(pmg\) and \(pgg\) are not isomorphic. 

The following three pictures have groups \(pmm\), \(pmg\), and \(pgg\), respectively.

![Wallpaper patterns with groups pmm, pmg, and pgg](image)

These propositions together finish the classification of the wallpaper groups. We have thus seen that there are thirteen split wallpaper groups, one for each of the thirteen point groups. The remaining four groups, \(pg\), \(pmg\), \(pgg\), and \(p4g\) are the non-split groups.

### 4.2.1 Exercises

1. Suppose that \(G_0\) is a subgroup of \(O_2(\mathbb{R})\) acting on a 2-dimensional lattice \(T\), and let \(\{t_g\}_{g \in G_0} \subseteq \mathbb{R}^2\) satisfy Equation 4.1. Prove that \(G := \{(g, t_g + t) : g \in G_0, t \in T\}\) is a subgroup of \(\text{Isom}(\mathbb{R}^2)\).

2. Let \(G\) be the group in the previous problem. Prove that \(G = \text{Sym}(W)\) for some subset \(W\) of \(\mathbb{R}^2\).

3. Let \(G\) be a wallpaper group. Suppose that \(G\) is the semidirect product of \(G_0\) and \(T\). Then \(G\) contains a subgroup \(\{(g, t_g) : g \in G\}\) isomorphic to \(G_0\). By using closure under composition of this subset, show that \(t_g + g(t_h) = t_{gh}\) for each \(g, h \in G_0\). If \(G_0 = C_n\) for some \(n\), use the ideas of the section to show that we may choose \(t_g = 0\) for each \(g \in C_n\). On the other hand, if \(G_0 = D_n\), with the notation of the section, writing
$t_f = u$, show that there is $v \in \mathbb{R}^2$ with $u = v - f(v)$, and that $t_g = v - g(v)$ for each $g \in D_n$.

4. Let $Z$ be the set of all 2-cocycles. Prove that $Z$ is an Abelian group under pointwise addition of functions.

5. Let $B$ be the set of all 2-coboundaries. Prove that $B$ is a subgroup of $Z$.

6. Prove that each wallpaper group

### 4.3 Description of the Wallpaper Groups

In this section we describe each wallpaper group in terms of generators and relations along with giving two wallpaper patterns whose symmetry group is the given group. Throughout we write $\{t_1, t_2\}$ for a basis of the translation lattice $T$, described for each lattice type as in Section 3.2. We recall the labelling of the groups, organized according to the lattice type.

<table>
<thead>
<tr>
<th>Parallelogram</th>
<th>Rectangular</th>
<th>Rhombic</th>
<th>Square</th>
<th>Hexagonal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p1$</td>
<td>$pm$</td>
<td>$cm$</td>
<td>$p4$</td>
<td>$p3$</td>
</tr>
<tr>
<td>$p2$</td>
<td>$pg$</td>
<td>$cmm$</td>
<td>$p4m$</td>
<td>$p3m1$</td>
</tr>
<tr>
<td></td>
<td>$pmm$</td>
<td></td>
<td>$p4g$</td>
<td>$p31m$</td>
</tr>
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<td></td>
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<td>$p6$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$p6m$</td>
</tr>
</tbody>
</table>
$pI$. Generators $t_1, t_2$; point group $C_1$. 

Diagram showing the pattern and transformation elements related to the point group $C_1$. The diagram illustrates the symmetry elements of the point group, including translations along the axes and the orientation of the pattern.
4.3. Description of the Wallpaper Groups
\textit{cm}. Generators $t_1, t_2, \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right)$; point group $D_{1c}$. 
4.3. Description of the Wallpaper Groups
\emph{pm.} Generators $t_1, t_2, \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right)$; point group $D_{1,p}$.
4.3. Description of the Wallpaper Groups
Generators $t_1, t_2, \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} t_1 \right)$; point group $D_{1p}$.
4.3. Description of the Wallpaper Groups
$p2$. Generators $t_1, t_2, \left( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right)$; point group $C_2$. 

\[
\]

\[
\]
4.3. Description of the Wallpaper Groups
Generators $t_1, t_2, \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right), \left( \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, 0 \right)$; point group $D_{2,c}$. 
$pmm$. Generators $t_1, t_2, \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right), 0 \right), \left(\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right), 0 \right)$; point group $D_{2,p}$. 
4.3. Description of the Wallpaper Groups
Generators $t_1, t_2, \left( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right), \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} t_1 \right)$; point group $D_{2p}$. 

\[ \]
4.3. Description of the Wallpaper Groups
pgg. Generators $t_1, t_2, \left( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right), \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}(t_1 + t_2) \right)$; point group $D_{2,p}$.
4.3. Description of the Wallpaper Groups

![Wallpaper Pattern](image)
$p3$. Generators $t_1, t_2, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 0$; point group $C_3$. 
$p3m1$. Generators $t_1, t_2, \left( \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 0 \right), \left( \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, 0 \right)$; point group $D_{3,l}$. 
4.3. Description of the Wallpaper Groups
$p31m$. Generators $t_1, t_2, \left( \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 0 \right), \left( \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, 0 \right)$; point group $D_{3,s}$. 
4.3. Description of the Wallpaper Groups
\( p_4 \). Generators \( t_1, t_2, \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right) \); point group \( C_4 \).
4.3. Description of the Wallpaper Groups
Chapter 4. The Wallpaper Groups

$p4m$. Generators $t_1, t_2, \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right), \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right)$; point group $D_4$. 

[Diagram of p4m pattern with generators and point group $D_4$ highlighted]
\( p4g \) Generators \( t_1, t_2, \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right), \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}(t_1 + t_2) \right) \); point group \( D_4 \).
4.3. Description of the Wallpaper Groups
\textit{p6}. Generators $t_1, t_2, \left( \begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array} \right), 0$; point group $C_6$. 
4.3. Description of the Wallpaper Groups
$p6m$. Generators $t_1, t_2, \left( \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right), \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right)$; point group $D_6$. 
4.3. Description of the Wallpaper Groups
4.4 Fundamental Domains

Throughout this section we continue to write \( \{t_1, t_2\} \) for a basis of the translation lattice \( T \), described for each lattice type as in Section 3.2. In any wallpaper pattern there is a parallelogram (with half of its boundary) such that every point in the plane is uniquely obtained as a translation of one point in the parallelogram. This is a fundamental domain with respect to the translation subgroup. It is not uniquely determined.

For example, for the group \( pmm \), if one considers the bottom left corner of the following fundamental domain to be the origin, then the point group consists of the identity, the 180° rotation about the origin, and horizontal and vertical reflections. If one starts with a triangle in the bottom left and applies the three nonidentity transformations in the point group, then one gets the three unshaded triangles of the following picture. By translating them by \( t_1 \), \( t_2 \), and \( t_1 + t_2 \), respectively, they are placed inside the rectangle.

For different subgroups one can get a different fundamental domain. For instance, for the entire group, one would only need one of the four triangles above; every other triangle can be obtained by some isometry in the group. On the other hand, for the subgroup generated by \( t_1, t_2 \), and the 180° rotation, one would need two triangles. The following pictures indicate fundamental domains for these two groups.

4.5 Arithmetic Aspects of the Wallpaper Groups

In this section we address some particular aspects of some of the symmetry groups, such as illustrating the differences between wallpaper groups with the same point group. The
4.5. Arithmetic Aspects of the Wallpaper Groups

group-theoretic differences are nicely illustrated with examples of wallpaper patterns. For example, we show how, using our description of the wallpaper groups, how to determine centers of rotations and lines of reflections of various wallpaper patterns.

The first point we make is how to determine, by the wallpaper patterns, when a wallpaper group is not split. Recall that if \( G \) has translation lattice \( T \) and point group \( G_0 \), and if \( G \) is split, then \( G = \{(g, t) : g \in G_0, t \in T\} \). In particular, if \((g, t) \in G\), then both \((g, 0) \in G\) and \(t \in T\). Thus, if a wallpaper pattern with symmetry group \( G \) has a glide reflection \((f, t)\) for which \(t \notin T\), then \( G \) is not split. For example, Escher’s *Horseman* on Page 26, the corresponding symmetry group has a nontrivial glide reflection, so it is not a split wallpaper group.

Before we discuss specific groups, we point out two general facts. First, if \( r \in D_n \) is a rotation by an angle \( \theta \) and \( f \in D_n \) is a reflection about the line \( \ell \), then the reflection line of \( rf \) is the line \( \ell' \) obtained by rotating \( \ell \) by \( \theta/2 \). Second, by Lemma 2.2, if a wallpaper group contains a reflection across a line \( \ell \) through the origin, and if \( v \) is a vector perpendicular to \( \ell \), then the group contains reflections with reflection lines \( \{\ell + \frac{n}{2}v : n \in \mathbb{Z}\} \), and that these lines are precisely the reflection lines parallel to \( \ell \).\(^1\)

\[p2.\]

The group \( p2 \) consists of translations and \( 180^\circ \) rotations. What are the centers of rotations? Let \( r \) be the \( 180^\circ \) rotation about the origin. We see that as a consequence of Lemma 2.1 that \((r, t)\) is a \(180^\circ \) rotation about the point \( t/2 \). Therefore, the set of centers of rotations for \(180^\circ \) rotations in \( p2 \) is the lattice \( \{\frac{1}{2}t : t \in T\} \). Restricting to a fundamental domain, we then have the following picture for the rotation centers. Escher’s drawings of tessellations

![Figure 4.4: p2 rotation centers](image)

with symmetry group \( p2 \) often show the rotation centers. For example both Camels and Squirrels, shown on Page 27 show all rotation centers.

The same pattern of rotation centers holds for any symmetry group for whom the rotation subgroup of the point group is of order 2, for exactly the same reason as for \( p2 \). These groups

\(^1\)do we need this fact here?
are $pmm$, $pmg$, $pgg$, and $cmm$.

$pg$.

We next consider the group $pg$. This group is generated by translations and a glide reflection. We may assume that the glide is of the form $(f, \frac{1}{2}t_1 + \alpha t_2) = (f, u)$, where $f$ is a reflection and $\alpha$ is an arbitrary number. We recall from Section 4.1 that the cocycle $c$ representing the group extension for $pg$ satisfies

$$c(f, f) = tf + f(tf) - t_{id}$$
$$= u + f(u) - 0 = t_1;$$

the vector $\alpha t_2$ does not show up in this value of the cocycle $c$. Therefore, it does not affect the equivalence class of the group extension corresponding to $c$. The picture below shows two wallpaper patterns of type $pg$ corresponding to different choices of $\alpha$.

![Wallpaper patterns of type pg](image)

Escher has several pictures illustrating the symmetry type $pg$. In the following two we also see how the translation component of the basic glide reflection is different in the two pictures. The translation component is marked on the pictures. In both cases the glide is a vertical reflection followed by the indicated translation.
Wallpaper patterns with symmetry groups *pm* and *cm* (resp. *pmm* and *cmm*) can be distinguished by considering the relationship between reflection lines and a fundamental domain. In Section 3.2, we distinguished a rectangular lattice from a rhombic lattice by finding a basis \( \{t_1, t_2\} \) such that, for rectangular lattices, there is a reflection fixing \( t_1 \), and in the case of a rhombic lattice, there is a reflection interchanging \( t_1 \) and \( t_2 \). If we draw a fundamental domain appropriately, for *pm* and *pmm* there will be a reflection line parallel to a side of the domain, while for *cm* and *cmm* there will be a reflection line parallel to one of the diagonals of the domain. For example, in the left picture below, with a fundamental domain superimposed, we see that the diagonals of the domain are reflection lines; thus, the symmetry group of this wallpaper pattern is *cmm*. The picture below on the right has reflection lines parallel to the sides of the domain, so the group of this pattern is *pmm*. 

*pm, cm, pmm, and cmm.*
**p6 and p6m.**

The groups p6 and p6m contain 60° rotations, and p6m also has 6 basic reflections. In the picture below, we view the center of the hexagon as the origin. For either group the dark circles are rotation centers of 60° rotations, the hollow circles are centers of 120° rotations, and the remaining circles are rotation centers of 180° rotations. To see why this is so, let $r$...
be a 60° rotation through the origin. As we noted for p2, since \( r^3 \) is a 180° rotation, \((r^3, t)\) is a 180° rotation centered at \( \frac{1}{2} t \). Now, \{\( r^i(t_1) : 1 \leq i \leq 6 \}\) are the six translation vectors of minimal length in \( T \). Thus, \((r^3, r^i(t_1))\) yield the six 180° rotations centered at the six grey points inside the hexagon. For the six grey points on the hexagon’s border, we point out that the vectors \( r^i(t_1) + r^{i+1}(t_1) \) lie at angles 30°, 90°, 150°, 210°, 270°, and 330° to the \( x \)-axis, and each have length \( \sqrt{3}||t_1|| \). Thus, the six vectors \( \frac{1}{2}(r^i(t_1) + r^{i+1}(t_1)) \) yield the grey points on the hexagon’s border, and are rotation centers for \((r^3, r^i(t_1) + r^{i+1}(t_1))\).

To find the 60° rotation centers, we point out that \( P \) is such a center if \( t = P - r(P) \) for some \( t \in T \); the point \( P \) is the rotation center of \((r, t)\). We wish to find those \( P \) with \( ||P|| \leq ||t_1|| \). Now, \( P \) is determined from \( t \) by \( P = -(r - I)^{-1}(t) \). However, \( r \) satisfies its characteristic equation, which seen to be \( x^2 - x + 1 = 0 \), and so \( r^2 = r - I \). Thus, \( P = -r^{-2}(t) = r(t) \) since \( r^3 = -1 \). If we take the six vectors \( r^i(t_1) \) of minimal length in \( T \) and apply \( r \), we obtain the six vertices of the hexagon; these then are all of the 60° rotation centers, other than the origin, inside the hexagon.

Finally, for the 120° rotations, if \( P \) is a rotation center, then \( P = (I - r^2)^{-1}(t) \) for some \( t \in T \). By using the matrix for \( r \), we see that

\[
(I - r^2)^{-1} = \frac{1}{3} \begin{pmatrix} 3/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 3/2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}.
\]

In other words, the linear transformation \((I - r^2)^{-1}\) is rotation by an angle \(-60°\) followed by multiplication by \(1/\sqrt{3}\). If we take the six corners of the hexagon and apply this transformation, we get the six white circles in the picture above.

The reflection lines for \( p6m \) consist of the six lines through the center together with the six other lines that appear inside the hexagon. To explain this, let \( f \) be the reflection that fixes \( t_1 \), the horizontal translation of minimal length. The first six are the reflection lines of \{\( r^i f : 1 \leq i \leq 6 \}\}. To explain the other six reflection lines, we note that \{\( r^i(t_1) : 1 \leq i \leq 6 \}\) are the six translation vectors of minimal length in \( T \). The vector \( r^i(t_1) \) is perpendicular to the reflection line of exactly one of the \( r^i f \). For example, \( t_1 \) is perpendicular to the reflection line of \( r^3 f \). Thus, \((r^3 f, t_1)\) is a reflection whose reflection line is the translate of the reflection line of \( f \) by the vector \( \frac{1}{2} t_1 \). This is the right-most vertical line in the picture above. The other five lines are translates by \( \frac{1}{2} r^i(t_1) \) of the appropriate reflection line passing through the center.

**p3m1 and p31m.**

We next consider the groups whose rotation subgroup of the point group is \( C_3 \). In addition to considering rotation centers, we investigate the difference between \( p3m1 \) and \( p31m \). Perhaps the easiest way to visually distinguish wallpaper patterns with these groups is to consider the relation between lines of reflection and centers of rotation.

We first consider rotation centers. The following wallpaper pattern has symmetry type \( p3 \). The centers of rotations are indicated in the picture; from the analysis for \( p6 \), we can
conclude that centers of 60° or 120° rotations in \( p6 \) are rotation centers for \( p3 \). Note that in any parallelogram that represents a fundamental domain for the translation subgroup, the four corners along with the centers of the two triangular halves are centers of rotation. The parallelogram marked in the picture below is a fundamental domain for \( p3 \).

![Figure 4.6: \( p3 \) rotation centers](image)

As we saw in Section 3.2, the difference between the symmetry groups \( p3m1 \) and \( p31m \) is in the choice of reflection lines. The group \( p3m1 \) corresponds to taking the three reflection lines that make angles of 30°, 90°, and 150° with one of the translation vectors, while \( p31m \) has the three reflection lines at 0°, 60°, and 120° with respect to a translation vector. By taking the picture above and reflecting the picture accordingly, we obtain the figure on the left below, whose symmetry group is \( p3m1 \) and the figure on the right, whose symmetry group is \( p31m \).

![Figure 4.7: Wallpaper patterns with groups \( p3m1 \) and \( p31m \)](image)

As we can see in the pictures below, the centers of rotation are all on lines of reflection for \( p3m1 \), while not all centers are on lines of reflection for \( p31m \). This fact yields a straightforward way to distinguish between figures whose symmetry groups are either \( p3m1 \) or \( p31m \). Another way to distinguish them is to notice that if one draws a basic hexagon,
as below, then the reflection lines of $p3m1$ never pass through corners of the hexagon while the reflection lines for $p31m$ always pass through the corners.

![Reflection lines for $p3m1$ and $p31m$](image)

**Figure 4.8: $p3m1$ and $p31m$ reflection lines**

**$p4$, $p4m$, and $p4g$.**

The groups $p4$, $p4m$ and $p4g$ are those containing a $90^\circ$ rotation. We first look at the rotation centers in a fundamental domain, which is a square. We place the origin at the center of one $90^\circ$ rotation $r$. All other $90^\circ$ rotations are of the form $(r, t)$ for some $t \in T$. The center $P$ of $(r, t)$ satisfies $t = P - r(P) = (I - r)(P)$, so $P = (I - r)^{-1}(t)$. By using the matrix representation for $r$, we have

$$I - r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

whose inverse is

$$(I - r)^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.$$

The point of the final form of $(I - r)^{-1}$ is to see that $(I - r)^{-1}$ is the composition of a $45^\circ$ rotation followed by multiplication by $1/\sqrt{2}$. If we consider the four rotations $(r, \pm t_1)$, $(r, \pm t_2)$, then the rotation centers are the four corners of the fundamental domain, which are the images under $(I - r)^{-1}$ of $\pm t_1, t_2$. Thus, inside a fundamental domain, we have the center and the four corners as centers of $90^\circ$ rotations.

To find the rotation centers of $180^\circ$ rotations, we recall from the investigation of $p2$ that the set of such centers is $\{t/2 : t \in T\}$. If we ignore those centers that are also centers of $90^\circ$ rotations, we are left with the hollow points in the picture above.
The groups $p4m$ and $p4g$, have reflections in addition to 90° rotations. These groups can be distinguished visually by relating lines of reflection to centers of rotation. For $p4m$ the group is generated by the translation subgroup and the point group. Therefore, this group contains reflections parallel to the sides of a fundamental domain, which is a square. All centers of rotations consequently lie on reflection lines. The group $p4g$ does not contain the point group $D_4$. However, it does contain reflections in two perpendicular directions. There are centers of rotations that do not lie on reflection lines. The following pictures indicate centers of rotation and lines of reflection.

We see that for $p4m$, all centers of rotation are on reflection lines, while the same is not true for $p4g$. To be more precise, in patterns with group $p4g$, centers of 90° rotations do not lie on rotation lines. To explain these reflection lines, recall that $p4g$ is generated by the translations, $r$, and $g = (h, \frac{1}{2}(t_1 + t_2)$, where $h$ is the reflection about the $x$-axis. The four reflections in the point group of $p4g$ are $h, rh, r^2h, r^3h$. Therefore, any reflection in $p4g$ must
be of the form $(r^i h, \frac{1}{2}(t_1 + t_2) + t)$ for some $t \in T$. Recall that $(r^i h, v)$ is a reflection if and only if $r^i h(v) = -v$. To determine the reflections in $p4g$, we must find those $t \in T$ for which

$$r^i h \left( \frac{1}{2}(t_1 + t_2) + t \right) = - \left( \frac{1}{2}(t_1 + t_2) + t \right).$$

(4.4)

By writing $t = nt_1 + mt_2$ and recalling that $r(t_1) = t_2$ and $r(t_2) = -t_1$, a short calculation will show that Equation 4.4 holds only when $i = 1$ and $1 + n = -m$, or when $i = 3$ and $n = m$. The reflection line of $rh$ makes a $45^\circ$ with the $x$-axis. In the case $i = 1$, if we set $n = 1$, then we see that the reflection line of $(rh, \frac{1}{2}(t_1 + t_2) - t_2) = (rh, \frac{1}{2}t_1 - \frac{1}{2}t_2)$ is obtained by translating the line $y = x$ by the vector $\frac{1}{2} (\frac{1}{2}t_1 - \frac{1}{2}t_2)$; similarly, with $n = -1$, the reflection line of $(rh, \frac{1}{2}(t_1 + t_2) - t_1)$ is obtained by translating the line $y = x$ by $\frac{1}{2} (-\frac{1}{2}t_1 + \frac{1}{2}t_2)$. These two lines are the two $45^\circ$ lines in the picture above. Similarly, the reflection lines of $(r^3 h, \frac{1}{2}(t_1 + t_2))$ and $(r^3 h, \frac{1}{2}(t_1 + t_2) - (t_1 + t_2))$ are the two $135^\circ$ lines above; these reflections correspond to the choices $n = m = 0$ and $n = m = -1$, respectively.

For example, consider Escher’s picture below, with a fundamental domain indicated as a diamond in the center of the picture.

The horizontal and vertical lines are reflection lines but do not represent translations. So, the lattice of lines drawn by Escher does not show the translation lattice.

**pmm, pmg, and pgg.**

The point group $D_{2r}$ yields three nonisomorphic symmetry groups, *pmm*, *pmg*, and *pgg*. One way to distinguish patterns for these groups is to consider reflections. The group *pmm* contains reflections in two non-parallel directions, while *pmg* contains reflections in only one direction. The group *pgg* does not contain any reflections. For example, the left-most picture below has symmetry group *pmm*, and there are horizontal and vertical reflections of the pattern.

In the center picture, which has symmetry group *pmg*, there are vertical reflections only, so the figure has reflections in only one direction. There is, however, a glide reflection of the
form \((h, \frac{1}{2}t_1)\), where \(h\) is a horizontal reflection, and \(t_1\) is the smallest horizontal translation of the figure. If \(v\) is a vertical reflection in \(D_{2\pi}\), then \(v = hr\), and so \((v, \frac{1}{2}t_1) = (h, \frac{1}{2}t_1)(r, 0) \in G\). Now, since \(\frac{1}{2}t_1\) is perpendicular to the reflection line of \(v\), the isometry \((v, \frac{1}{2}t_1)\) is a vertical reflection.

Finally, in the right-most figure, which has symmetry group \(pgg\), there are no reflections of the pattern. There are glide reflections \((h, \frac{1}{2}(t_1 + t_2))\) and \((v, \frac{1}{2}(t_1 + t_2))\), where \(h\) and \(v\) are horizontal and vertical reflections, respectively, and \(\{t_1, t_2\}\) is a basis consisting of a horizontal and a vertical translation. To see that \(pgg\) does not contain a reflection, we first point out that any reflection in \(pgg\) would have to be of the form \((h, \frac{1}{2}(t_1 + t_2) + t)\) or \((v, \frac{1}{2}(t_1 + t_2) + t)\) for some \(t \in T\) since these are the only elements whose image in the point group \(G_0 = D_2\) is a reflection. If \((h, \frac{1}{2}(t_1 + t_2) + t)\) is a reflection, then \(h \left(\frac{1}{2}(t_1 + t_2) + t\right) = -\left(\frac{1}{2}(t_1 + t_2) + t\right)\). A short calculation shows that there is no value of \(t\) for which this is true. Similar reasoning shows that \((v, \frac{1}{2}(t_1 + t_2) + t)\) is not a reflection for any \(t\).
Appendix A

Group Cohomology

This appendix is the technical heart of the original manuscript. We have seen that given a wallpaper group $G$, we have an Abelian normal subgroup $T$ of translations, the quotient group $G/T \cong G_0$, and an action of $G_0$ on $T$. To understand $G$ we need to see how $G$ is built from $T$ and $G_0$. What we will see is that, given $T$ and $G_0$, an Abelian group describes the possible groups that can be built from these two groups. This is the second cohomology group $H^2(G_0, T)$. In this chapter we describe building a group from a subgroup and a quotient group, we define the group $H^2(G_0, T)$ and see how it is connected with building groups, and we calculate $H^2(G_0, T)$ in all the different cases that arise for wallpaper groups. From these calculations we will be able to write down all of the isomorphism classes of wallpaper groups.

A.1 Group Extensions

To help us describe wallpaper groups, and to understand better how they are built from $T$ and $G_0$, we discuss the concept of a group extension. A wallpaper group contains the Abelian normal subgroup $T$ of translations, and $G/T$ is a finite group isomorphic to the point group $G_0$. Phrasing this in another way, there is an exact sequence

$$1 \to T \to G \to G_0 \to 1$$

with $T$ Abelian. We discuss this situation in the abstract.

Let $G_0$ and $T$ be fixed groups with $T$ Abelian. A group extension of $T$ by $G_0$ is an exact sequence

$$1 \to T \to G \xrightarrow{\pi} G_0 \to 1.$$ 

For simplicity we view $T$ as a subgroup of $G$. We will describe such a sequence by two additional pieces of information, an action of $G_0$ on $T$ and a 2-cocycle. First, for the action, for each $g \in G_0$ choose an $x_g \in G$ with $\pi(x_g) = g$. The inner automorphism $y \mapsto x_g y x_g^{-1}$ restricts to an automorphism of $T$ since $T$ is normal in $G$. We define the action of $g$ on $T$ by $g(t) = x_g t x_g^{-1}$ for all $t \in T$. The first thing to note is that this is well-defined: if $\pi(y_g) = g$, then $y_g^{-1} x_g \in \ker(\pi) = T$, since $T$ is Abelian we get $y_g^{-1} x_g (y_g^{-1} x_g)^{-1} = t$, or $y_g^{-1} y_g^{-1} x_g t x_g^{-1}$.
for all $t \in T$. It is clear that the map $t \mapsto g(t)$ is a group automorphism of $T$. Furthermore, 
$g(ht) = (gh)t$ for all $g, h \in G_0$; in other words, the map that sends $g$ to left multiplication by 
g is a group homomorphism $\varphi : G_0 \rightarrow \text{Aut}(T)$. When there is such a group homomorphism, 
we will call $T$ a $G_0$-module. 

To give an example of a group extension, suppose that $T$ is a $G_0$-module. We thus have a 
group homomorphism $\varphi : G_0 \rightarrow \text{Aut}(T)$. We then can define the semidirect product $T \ltimes \varphi G_0$ 
to be, as a set, the direct product $T \times G_0$, but with the group operation defined by 

$$(s, g)(t, h) = (sg(t), gh).$$

An easy calculation shows that $T \times \varphi G_0$ is a group with this operation, that $T \cong \{(t, 1) : t \in T\}$, 
and that $G_0$ is isomorphic to the subgroup $\{(1, g) : g \in G_0\}$ of $T \ltimes \varphi G_0$. Furthermore, by 
identifying $T$ and $G_0$ with these isomorphic copies, we see that $(1, g)(t, 1)(1, g)^{-1} = (g(t), 1)$, 
so conjugation by $g$ induces the given action of $G_0$ on $T$. Furthermore, the homomorphisms 
t \mapsto (t, 1)$ and $\pi(t, g) = g$ yield a group extension $1 \rightarrow T \rightarrow T \ltimes \varphi G_0 \rightarrow G_0 \rightarrow 1$. 

Now assume that we have a group extension 

$$1 \rightarrow T \rightarrow G \xrightarrow{\pi} G_0 \rightarrow 1$$

that yields a given $G_0$-action on $T$. We will view $T$ as a subgroup of $G$. As before, we 
choose $x_g \in G$ with $\pi(x_g) = g$. While it is not necessary, we choose $x_1 = 1$; this will 
make the proof of Proposition A.1 easier. The function $g \mapsto x_g$ is not necessarily a group 
homomorphism. We can measure the failure of this function to be a homomorphism as 
follows. If $c(g, h) = x_g x_h x_g^{-1}$, then $g \mapsto x_g$ is a homomorphism if and only if $c(g, h) = 1$ 
for all $g, h \in G_0$. Note that $\pi(c(g, h)) = 1$, so $c(g, h) \in T$. This function is not arbitrary. 
Associativity in $G$ gives $(x_g x_h)x_k = x_g(x_h x_k)$. Using the formula $x_g x_h = c(g, h)x_{gh}$, we 
obtain 

$$(x_g x_h)x_k = c(g, h)x_{gh}x_k = c(g, h)c(gh, k)x_{ghk}$$

and 

$$x_g(x_h x_k) = x_g c(h, k)x_{hk} = x_g c(h, k)x_g^{-1}x_g x_{hk} = g(c(h, k))c(g, hk)x_{ghk}.$$ 

Therefore, we have the condition 

$$c(g, h)c(gh, k) = g(c(h, k))c(g, hk)$$

(A.1) 

for all $g, h, k \in G_0$. A function $c : G_0 \times G_0 \rightarrow T$ satisfying Equation (A.1) us called a 
2-cocycle. We point out that the choice $x_1 = 1$ also implies that $c(1, g) = c(g, 1) = 1$ for all 
g \in G_0; a cocycle satisfying this condition is said to be normalized. By defining pointwise 
multiplication of 2-cocycles, we obtain an Abelian group $Z^2(G_0, T)$ of all 2-cocycles from 
$G_0 \times G_0$ to $T$. If $1 \rightarrow T \rightarrow T \ltimes \varphi G_0 \rightarrow G_0 \rightarrow 1$ is the group extension corresponding to 
the semidirect product of $T$ by $G_0$, then we may choose $x_g = (1, g)$, and we find that the
cycloe class representing this extension is given by the cocycle \( c(g, h) = x_g x_h x_{gh}^{-1} = 1 \). In other words, the trivial cocycle arises from the semidirect product of \( T \) by \( G_0 \).

Looking at the construction above of a cocycle \( c \) from a group extension \( 1 \to T \to G \to G_0 \to 1 \), we note that \( c \) is not uniquely determined. If we make new choices \( y_g \) with \( \pi(y_g) = g \), this yields a different cocycle \( c' \) given by \( c'(g, h) = y_g y_h y_{gh}^{-1} \). To compare these cocycles, note that \( y_g = t_g x_g \) for some \( t_g \in T \) since \( \pi(y_g) = \pi(x_g) \). Therefore,

\[
c'(g, h) = y_g y_h y_{gh}^{-1} = (t_g x_g)(t_h x_h)(t_{gh} x_{gh})^{-1}
\]

\[
= t_g(x_g t_x g^{-1}) x_g x_h x_{gh}^{-1} t_{gh}^{-1} = t_g g(t_h) c(g, h) t_{gh}^{-1}
\]

\[
= (t_g g(t_h) t_{gh}^{-1}) c(g, h).
\]

The function \( b(g, h) = t_g g(t_h) t_{gh}^{-1} \) is then also a cocycle, being the element \( c' c^{-1} \in Z^2(G_0, T) \). Cocycles of this form are called 2-coboundaries. The set of 2-coboundaries from \( G_0 \times G_0 \to T \) is a subgroup of \( Z^2(G_0, T) \), which we denote by \( B^2(G_0, T) \). The quotient group

\[
H^2(G_0, T) = Z^2(G_0, T) / B^2(G_0, T)
\]

is called the second cohomology group of \( G_0 \) with coefficients in \( T \). By our procedure of obtaining a cocycle from a group extension \( 1 \to T \to G \to G_0 \to 1 \), we see that while the cocycle is not uniquely determined, its coset in \( H^2(G_0, T) \) is uniquely determined by the computation above.

We now describe how \( H^2(G_0, T) \) determines group extensions of \( T \) by \( G_0 \). We say that two group extensions are equivalent if there is a commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & T & \stackrel{i}{\longrightarrow} & G & \stackrel{\pi}{\longrightarrow} & G_0 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \varphi & & \downarrow & & \downarrow \text{id} \\
1 & \longrightarrow & T & \stackrel{i'}{\longrightarrow} & G' & \stackrel{\pi'}{\longrightarrow} & G_0 & \longrightarrow & 1 \\
\end{array}
\]

with \( \varphi : G \to G' \) a group isomorphism. The connection between group extensions and \( H^2(G_0, T) \) is given in the following proposition.

**Proposition A.1.** Let \( T \) be a \( G_0 \)-module. Then there is a 1–1 correspondence between the elements of \( H^2(G_0, T) \) and equivalence classes of group extensions of \( T \) by \( G_0 \) that induce the given \( G_0 \)-action on \( T \).

**Proof.** We have shown above that, given a group extension, there is a uniquely determined cocycle class in \( H^2(G_0, T) \). To go in the opposite direction, given a normalized cocycle \( c \), we produce a group extension \( 1 \to T \to G \to G_0 \to 1 \) whose cocycle class in \( H^2(G_0, T) \) is equal to the class of \( c \). Define \( G \) as a set by \( T \times G_0 \) and whose operation is given by

\[
(s, g)(t, h) = (sg(t)c(g, h), gh).
\]
A short calculation shows that $G$ is a group; associativity follows exactly from the 2-cocycle condition, and inverses are given by the formula $(s,g)^{-1} = ((g^{-1}(s)c(g^{-1},g))^{-1},g^{-1})$. Moreover, the map $T \to G$ with $t \mapsto (t,1)$ and the map $G \to G_0$ with $(t,g) \mapsto g$ are group homomorphisms, so we have a group extension $1 \to T \to G \to G_0 \to 1$. One consequence of the cocycle condition is that for the normalized cocycle $c$, we have $c(g,g^{-1}) = g(c(g^{-1},g))$; this follows by setting $h = g^{-1}$ and $k = g$ in Equation A.1. If we set $x_g = (1,g)$, then

$$x_g(t,1)x_g^{-1} = (1,g)(t,1)(c(g^{-1},g)^{-1},g^{-1}) = (g(t),g)(c(g^{-1},g)^{-1},g^{-1}) = (g(t)g(c(g^{-1},g))^{-1}c(g,g^{-1})) = (g(t),1).$$

Therefore, the $G_0$-action on $T$ is the same as that arising from this extension. Finally,

$$x_gx_hx_{gh}^{-1} = (1,g)(1,h)(c((gh)^{-1},gh)^{-1},(gh)^{-1}) = (c(g,h),gh)(c((gh)^{-1},gh)^{-1},(gh)^{-1}) = (c(g,h)gh(c((gh)^{-1},gh)^{-1}c(gh,(gh)^{-1})) = (c(g,h),1).$$

Therefore, the cocycle class for this extension is the same as the class of $c$.

Finally, we will finish the proof by showing that if two extensions have the same cocycle class, then they are equivalent. Suppose that

$$1 \to T \to G \to G_0 \to 1$$

and

$$1 \to T \to H \to G_0 \to 1$$

are two extensions giving rise to the same cocycle class. We may then assume that there are $x_g \in G$ and $y_g \in H$ with $x_gx_hx_{gh}^{-1} = y_gy_hy_{gh}^{-1}$ in $T$; note that we are viewing $T$ as a subgroup of both $G$ and $H$. Moreover, we may need to alter the $y_g$ by an element of $T$ to suppose that these group extensions give rise to the same cocycle, not just the same cocycle class. It is easy to see that $G = \{tx_g : t \in T, g \in G_0\}$ and $H = \{ty_g : t \in T, g \in G_0\}$. We then define a map $\varphi : G \to H$ by $\varphi(tx_g) = ty_g$. A short calculation shows that $\varphi$ is well-defined, and that $\varphi$ is a group isomorphism with $\varphi|_T = \text{id}$ and the induced map $G_0 \to G_0$ is also the identity. Thus, these extensions are equivalent.

**Example A.2.** To give some examples of group extensions, consider the case $T = \mathbb{Z}$ and $G_0 = \mathbb{Z}_2$. Since $\text{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$, there are two possible actions of $\mathbb{Z}_2$ on $\mathbb{Z}$. One, the *trivial action*, corresponds to the trivial homomorphism $\mathbb{Z}_2 \to \text{Aut}(\mathbb{Z})$, and so the action satisfies $gt = t$ for all $t \in \mathbb{Z}$ and $g \in \mathbb{Z}_2$. The other action arises from the nontrivial homomorphism $\mathbb{Z}_2 \to \text{Aut}(\mathbb{Z})$, and so the nonidentity element of $\mathbb{Z}_2$ acts as $-1$ on $\mathbb{Z}$. With respect to the trivial action the following two sequences are group extensions of $\mathbb{Z}$ by $\mathbb{Z}_2$:

$$0 \to \mathbb{Z} \to \frac{1}{2}\mathbb{Z} \to \mathbb{Z}_2 \to 0$$

and

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0,$$
These extensions are not equivalent since the middle groups are not even isomorphic. In
fact, one can show by the calculations of Section A.4 below that $H^2(\mathbb{Z}_2, \mathbb{Z}) \cong \mathbb{Z}_2$ for this
trivial action. The two group extensions above are exactly the two inequivalent extensions
of $\mathbb{Z}_2$ by $\mathbb{Z}$ with this action.

**Example A.3.** If we consider group extensions of $\mathbb{Z}$ by $\mathbb{Z}_2$ with the nontrivial action of $\mathbb{Z}_2$
on $\mathbb{Z}$, and if $G$ is the semidirect product of $\mathbb{Z}$ and $\mathbb{Z}_2$, then

$$0 \to \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0$$

is a group extension. We can see that this group extension is not equivalent to the previous
two because $G$ is not Abelian, so it is not isomorphic to either of the middle groups in those
sequences. However, another way to see this is to note that if two group extensions of $T$ by
$G_0$ are equivalent, then the action of $G_0$ on $T$ is the same for both group extensions. We do
not give a proof of this fact. Instead, Corollary 3.8 will be sufficient for our needs in terms
of comparing point groups acting in different ways on $T$.

The notion of equivalence of group extension is more subtle than that of isomorphism of
the middle terms of the sequence. Consider $T = \mathbb{Z}_p$ and $G_0 = \mathbb{Z}_p$ for $p$ an odd prime. Given
a group extension

$$0 \to \mathbb{Z}_p \to G \to \mathbb{Z}_p \to 0,$$

the group $G$ has order $p^2$, so is Abelian. This forces the action of $\mathbb{Z}_p$ on $\mathbb{Z}_p$ to be trivial.
One can also see this from the fact that $\text{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$, so there is no nontrivial group
homomorphism $\mathbb{Z}_p \to \text{Aut}(\mathbb{Z}_p)$. There are two isomorphism classes of groups of order $p^2$, the
cyclic group of order $p^2$, and the direct product $\mathbb{Z}_p \times \mathbb{Z}_p$. However, there are $p$ equivalence
classes of extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$. One can see this by showing that $H^2(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$.
However, to be more explicit, the direct product $\mathbb{Z}_p \times \mathbb{Z}_p$ corresponds to the trivial cocycle.
On the other hand, if $G = \langle a \rangle$ is cyclic of order $p^2$, then for $1 \leq i < p$, we get a group
extension

$$0 \to \langle a^p \rangle \to \langle a \rangle \xrightarrow{\pi_i} \mathbb{Z}_p \to 0$$
of $\mathbb{Z}_p$ by $\mathbb{Z}_p$ by defining $\pi_i(a) = i(\text{mod } p)$. A short calculation shows that, for $i \neq j$, the
 corresponding group extensions are not equivalent. Nor are any of these extensions trivial
 since $G$ is not isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. These yield $p - 1$ inequivalent group extensions, all
 with the middle group isomorphic to $\mathbb{Z}_{p^2}$.

**A.2 Group Extensions of $T$ by $G_0$**

Let $G$ be a wallpaper group with translation lattice $T$ and point group $G_0$. In Section 3.1,
we proved that $G_0 \cong G/T$, that $G_0$ is isomorphic to a cyclic group $C_n$ or a Dihedral group
$D_n$ with $n \in \{1, 2, 3, 4, 6\}$, and that, by considering the action of $G_0$ on $T$, we produced
thirteen subgroups of $\text{Gl}_2(\mathbb{Z})$, none of which are conjugate in $\text{Gl}_2(\mathbb{Z})$, as candidates for $G_0$. 
By Corollary 3.8, if two wallpaper groups are isomorphic, by representing their point groups as subgroups of $\text{Gl}_2(\mathbb{Z})$, these two groups must be the same from this list of thirteen groups. Furthermore, from $G$ we get a group extension $1 \to T \to G \to G_0 \to 1$, which then yields an element of the cohomology group $H^2(G_0, T)$.

We can use group extensions to limit the number of possibilities of nonisomorphic wallpaper groups. We first point out that any group extension of $T$ by $G_0$ corresponds to some wallpaper group; we prove this in the following lemma. To prove the lemma we need two results from group cohomology. One is simple. If $S \subseteq T$ are $G_0$-modules, then there is a natural group homomorphism $H^2(G_0, S) \to H^2(G_0, T)$ induced by viewing a 2-cocycle $f \in Z^2(G_0, S)$ as a 2-cocycle $G_0 \times G_0 \to T$. The other is the Lyndon-Hochschild-Serre spectral sequence, which we describe in Section A.4.

**Lemma A.4.** Let $1 \to T \to G \to G_0 \to 1$ be a group extension. Then, up to equivalence of group extension, $G$ can be taken to be a subgroup of $\text{Isom}(\mathbb{R}^2)$, and this subgroup is a wallpaper group.

**Proof.** By a calculation using the Lyndon-Hochschild-Serre spectral sequence, given in Lemma A.7 below, we have $H^2(G_0, \mathbb{R}^2) = 0$. If $c$ is the cocycle class of the given group extension, then $c$ goes to 0 under the natural map $H^2(G_0, T) \to H^2(G_0, \mathbb{R}^2)$. Therefore, there are $t_g \in \mathbb{R}^2$ with

$$c(g, h) = t_g + gt_h - t_{gh}$$

for all $g, h \in G_0$. We define $G'$ by

$$G' = \{(g, t + t_g) : g \in G_0, t \in T\}.$$

An easy calculation shows that $G'$ is a subgroup of $\text{Isom}(\mathbb{R}^2)$, and that the maps $t \mapsto (\text{id}, t)$ and $(g, t) \mapsto g$ from $T$ to $G$ and from $G$ to $G_0$, respectively, yield a group extension

$$1 \to T \to G' \to G_0 \to 1.$$

Furthermore the cocycle class representing this extension is $c$; this latter fact can be seen by choosing $x_g = (g, t_g)$, so $x_g x_h x_{gh}^{-1} = (I, t_g + gt_h - t_{gh}) = (I, c(g, h))$. Thus, this extension is equivalent to the original extension. Finally, we note that $G'$ is indeed a wallpaper group since $G'$ contains the two-dimensional lattice $T$. \qed

For a given point group $G_0$ and an action of $G_0$ on $T$, two nonisomorphic wallpaper groups $G$ and $G'$, both with translation lattice $T$ and point group $G_0$, correspond to inequivalent group extensions. Therefore, the number of nonisomorphic wallpaper groups is at most the number of inequivalent group extensions of $T$ by $G_0$. Since $H^2(G_0, T)$ classifies the group extensions of $T$ by $G_0$ for a given action of $G_0$ on $T$, the number $|H^2(G_0, T)|$ is an upper bound for the number of nonisomorphic wallpaper groups with point group $G_0$ with the given action. We determine these cohomology groups in all possible cases in the next section. Before we calculate them, we summarize the calculations in the following chart.
We will see in Section 4.1 that determining all nonisomorphic wallpaper groups is almost the same as determining all group extensions of $T$ by $G_0$ for the various $G_0$. In fact, in only one occasion will two inequivalent group extensions give rise to isomorphic wallpaper groups; thus, from 18 group extensions we will get 17 nonisomorphic wallpaper groups.

### A.3 Higher Cohomology Groups

The second cohomology group $H^2(G_0, T)$ describes the equivalence classes of extensions of $T$ by $G_0$; we need to calculate this cohomology group in order to determine the possible wallpaper groups. However, to do this we need to be able to calculate higher cohomology groups. We give an extremely brief description of what are these groups in this section. For a more complete description, see any book on homological algebra. A systematic description of cohomology groups would best involve derived functors. However, because we are being brief, we will give a more ad-hoc description. In fact, this description is essentially that given in the paper of Hochschild and Serre [2].

Let $G_0$ be a group and let $T$ be a $G_0$-module. If $n$ is a nonnegative integer, let $C^n(G_0, T)$ be the set of all functions from the Cartesian product $G_0^n = \prod_{i=1}^n G_0$ to $T$. If $n = 0$, we interpret $G_0^0$ as a single point and then identify $C^0(G_0, T)$ with $T$. Let $d^n : C^n(G_0, T) \rightarrow T$ be the map that sends each function $f : G_0^n \rightarrow T$ to the sum of its values.
Appendix A. Group Cohomology

Let \( C^{n+1}(G_0, T) \) be the map defined by

\[
d^n(f)(g_1, \ldots, g_{n+1}) = g_1 f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n).
\]

If \( n = 0 \), and if \( t \in C^0(G_0, T) = T \), then we define \( d^0(t)(g) = gt - t \). A tedious calculation will show that \( d^{n+1} \circ d^n = 0 \). That is, the collection \( \{C^n(G_0, T)\}_{n \geq 0} \) together with the sequence \( \{d^n\} \) of maps forms a chain complex. We obtain the cohomology groups \( H^n(G_0, T) \) as the homology of this complex. In other words,

\[
H^n(G_0, T) = \ker(d^n)/\text{im}(d^{n-1})
\]

if \( n > 0 \), and \( H^0(G_0, T) = \ker(d^0) = T_{G_0} \). Moreover, a quick check will show that \( f : G_0 \times G_0 \to T \) is a 2-cocycle if and only if \( f \in \ker(d^2) \), and \( f \) is a 2-coboundary if and only if \( f \in \text{im}(d^1) \). This definition of \( H^2(G_0, T) \) is then the same as that given in Section A.1.

For those familiar with derived functors, we mention the connection between group cohomology and derived functors. Let \( T^G_0 = \{t \in T : gt = g \text{ for all } g \in G_0\} \). Then \( H^0(G_0, T) = T^G_0 \), and in fact the functor \( H^0(G_0, -) \) is naturally equivalent to the fixed point functor \((-)^{G_0}\). Therefore, an alternative description of \( H^n(G_0, T) \) is that if \( F = (-)^{G_0} \), and if \( R^n(F) \) is the \( n \)-th right derived functor of \( F \), then \( H^n(G_0, T) = R^n(F)(T) \). Furthermore, \( H^n(G_0, T) = \text{Ext}^n_{ZG_0}(\mathbb{Z}, G_0) \). The complex \( \{C^n(G_0, T)\} \) arises by taking the free resolution \( \{P_n\} \) of \( \mathbb{Z} \) as a trivial \( G_0 \)-module, where \( P_n \) is the free \( \mathbb{Z}G_0 \)-module on the set \( G_0^n \). The differential \( d : P_n \to P_{n-1} \) is given by the formula \( d = \sum_{i=0}^{n} (-1)^i d_i \), where

\[
\begin{align*}
d_0(g_1, \ldots, g_n) &= g_1(g_2, \ldots, g_{n-1}), \\
d_i(g_1, \ldots, g_n) &= (g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_n), \quad 1 < i < n \\
d_n(g_1, \ldots, g_n) &= (g_1, \ldots, g_{n-1})
\end{align*}
\]

Applying the contravariant functor \( \text{hom}_{\mathbb{Z}G_0}(\mathbb{Z}, T) \) to the resolution \( \{P_n\} \), using the isomorphism \( \text{hom}_{\mathbb{Z}G_0}(P_n, T) \cong C^n(G_0, T) \), which holds since homomorphisms are determined by their action on a basis, we see that the chain complex \( \{\text{hom}_{\mathbb{Z}G_0}(P_n, T)\} \) is the complex \( \{C^n(G_0, T)\} \).

### A.4 Calculation of the Groups \( H^2(G_0, T) \)

In this section we calculate the cohomology groups \( H^2(G_0, T) \) for the various point groups \( G_0 \). In the next chapter we will see how to determine all wallpaper groups from these calculations. The calculations of these groups for \( G_0 = C_n \) are very easy. If the point group is a Dihedral group, then we have to work harder. If you do not wish to bother with spectral sequences, you should skip those calculations. However, if you are familiar with spectral
sequences or if you wish to see an example of their use, these calculations are a nice and easy illustration of the power of spectral sequences.

There are various facts about cohomology we will need for our calculations. Because seven of the thirteen $G_0$ are cyclic groups, we start with facts about the cohomology of cyclic groups. If $C = \langle c \rangle$ is a cyclic group of order $n$ and $M$ is a $C$-module, the norm map $N_C$ on $M$ is defined as $N_C(x) = \sum_{i=0}^{n-1} c^i x$. The set $M^C = \{ m \in M : cm = m \}$ is the subgroup of $M$ fixed by all elements of $C$. The cohomology groups of $C$ with coefficients in $M$ are then

\[
H^0(C, M) \cong M^C,
H^{2n}(C, M) \cong M^C/N_C(M),
H^{2n+1}(C, M) \cong \ker(N_C)/\text{im}(1 - c),
\]

for all positive integers $n$. This result can be found in [7, Chapter 6]. We give the description of $H^q(C, M)$ for all $q$ because we will need to know $H^q(C_n, T)$ for all $q$ to calculate $H^2(D_n, T)$ via the Lyndon-Hochschild-Serre spectral sequence.

$G_0 = C_n$.

Suppose that our point group $G_0$ is generated by a rotation. Then $G_0$ acts on $T$ without any nonzero fixed points; that is, $T^{G_0} = 0$. However, since

\[
H^2(G_0, T) \cong T^{G_0}/N_{G_0}(T),
\]

we obtain $H^2(G_0, T) = 0$.

$G_0 = D_{1,c}$.

In this case $G_0$ is generated by a reflection $f$ that interchanges the two basis vectors $t_1, t_2$ of $T$. Since $G_0$ is cyclic, we have

\[
H^2(G_0, T) \cong T^{G_0}/\text{im}(N_{G_0}).
\]

Furthermore, $T^{G_0} = \mathbb{Z}(t_1 + t_2)$, and $t_1 + t_2 = N_{G_0}(t_1)$ since $t_1$ is sent to $t_2$ by $f$. Therefore, $T^{G_0} = \text{im}(N_{G_0})$, so $H^2(G_0, T) = 0$.

$G_0 = D_{1,p}$.

Here we have $G_0 = \{f\}$, where $f(t_1) = t_1$ and $f(t_2) = -t_2$. As before, $H^2(G_0, T) \cong T^{G_0}/\text{im}(N_{G_0})$. In this case $T^{G_0} = \mathbb{Z} t_1$ and $\text{im}(N_{G_0}) = \mathbb{Z}(2t_1)$, since

\[
N_{G_0}(at_1 + bt_2) = (at_1 + bt_2) + f(at_1 + bt_2)
= 2at_1.
\]
Therefore, $H^2(D_{1,p}, T) \cong \mathbb{Z}/2\mathbb{Z}$. We thus have two inequivalent group extensions of $T$ by $D_{1,p}$.

We now consider the non-cyclic point groups. For $D_{2,c}$ and $D_6$ we need more machinery. If $G$ is a group with normal subgroup $N$, if $M$ is a $G$-module, and if $H^1(N, M) = 0$, then there is a five term exact sequence [4, p. 307] of low degree terms

$$0 \rightarrow H^2(G/N, M^N) \rightarrow H^2(G, M) \rightarrow H^2(N, M)^{G/N}$$
$$\rightarrow H^3(G/N, M^N) \rightarrow H^3(G, M).$$

This exact sequence arises from the Lyndon-Hochschild-Serre spectral sequence associated to the normal subgroup $N$ of $G$; an introduction to spectral sequences and the description of the Lyndon-Hochschild-Serre spectral sequence is given below. We will use this exact sequence for $G = D_6$ and $D_{2,c}$.

$G_0 = D_6$.

The group $D_6$ contains the normal subgroup $C_6$. Moreover,

$$C_6 = \langle \left( \begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array} \right) \rangle,$$

the matrix representing the $60^\circ$ rotation $r$. From the description above of the cohomology of a cyclic group,

$$H^1(C_6, T) \cong \ker(N_{C_6})/\text{im}(1 - r),$$

where $N_{C_6}$ is the norm map as defined above. It is easy to see that $N_{C_6} = 0$ since the image of this map is contained in $T_{C_6}$, which is zero. Therefore, $\ker(N_{C_6}) = T$. However, $1 - r$ is represented by the matrix

$$\left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - \left( \begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array} \right),$$

which is invertible in $\text{Gl}_2(\mathbb{Z})$. Therefore, $\text{im}(1 - r) = T$, so $H^1(C_6, T) = 0$. The first part of the five term exact sequence mentioned above is

$$0 \rightarrow H^2(D_6/C_6, T^{C_6}) \rightarrow H^2(D_6, T) \rightarrow H^2(C_6, T)^{D_6/C_6}.$$

The first term is zero since $T^{C_6} = 0$. The third term is zero since $H^2(C_6, T) = 0$, as we saw above. Therefore, $H^2(D_6, T) = 0$.

$G_0 = D_{2,c}$.

In this case we have $G_0 = \langle r, f \rangle$ with $r$ the $180^\circ$ rotation and $f$ the reflection given by $f(t_1) = t_2$ and $f(t_2) = t_1$. Note that $r(t) = -t$ for all $t \in T$. Our argument will be
A.4. Calculation of the Groups $H^2(G_0, T)$

slightly different from the previous case because $H^1(C_2, T) \neq 0$. Therefore, to apply the five term sequence, we need a different normal subgroup of $G_0$. Let $M = \langle f \rangle$. Then $H^1(M, T) = \ker(N_M)/\im(1 - f)$. We have $N_M(t) = t + f(t)$, so it is easy to see that \( \ker(N_M) = \mathbb{Z}(t_1 - t_2) \). Also, $$(1 - f)(at_1 + bt_2) = a(t_1 - t_2) + b(t_2 - t_1),$$ so $\im(1 - f) = \ker(N_M)$. Therefore, $H^1(M, T) = 0$. The first part of the five term exact sequence arising from the normal subgroup $N$ of $G_0$ is

$$0 \to H^2(G_0/M, T^M) \to H^2(G_0, T) \to H^2(M, T)^{G_0/M}.$$

The first term is a quotient of $(T^M)^{G_0/M} = T^{G_0} = 0$; we see that $T^{G_0} = 0$ since $r(t) = -t$ for all $t$. For the third term, we have $T^M = \mathbb{Z}(t_1 + t_2)$ and $\im(N_M) = \mathbb{Z}(t_1 + t_2)$ as $N_M(at_1 + bt_2) = (a + b)(t_1 + t_2)$. Therefore, the third term is zero, so $H^2(G_0, T) = 0$.

To finish the remaining cases we need to use the theory of spectral sequences. We give the definition of a spectral sequence and leave details to the books of Weibel [7] and Tamme [6]. A cohomological spectral sequence is the following collection of data: Abelian groups

$$\{ E^{p,q}_r : p, q, r \in \mathbb{Z}, r \geq 2 \};$$

homomorphisms

$$d_r^{p,q} : E^{p,q}_r \to E^{p+r,q-r+1}_r$$

with $d_r^{p,q} \circ d_r^{p-r,q+r-1} = 0$ for all $p, q, r$; isomorphisms

$$E^{p,q}_{r+1} \cong \ker(d_r^{p,q})/\im(d_r^{p-r,q+r-1});$$

filtered Abelian groups

$$\{ E^n : n \in \mathbb{Z} \}$$

where the filtration $\cdots \supseteq F^p(E^n) \supseteq F^{p+1}(E^n) \supseteq \cdots$ satisfies $F^p(E^n) = E^n$ for $p \ll 0$ and $F^p(E^n) = 0$ for $p \gg 0$; limit terms

$$\{ E^{p,q}_\infty : p, q \in \mathbb{Z} \};$$

and isomorphisms

$$E^{p,q}_\infty \cong F^p(E^{p+q})/F^{p+1}(E^{p+q}).$$

We will only consider the case where $E^{p,q}_r = 0$ if $p < 0$ or $q < 0$. In this case we can define $E^{p,q}_\infty$. If $r > \max \{ p, q + 1 \}$, consider the sequence

$$E^{p-r,q+r-1}_r \xrightarrow{d_r^{p-r,q+r-1}} E^{p,q}_r \xrightarrow{d_r^{p,q}} E^{p+r,q-r+1}_r.$$

The assumption on $r$ gives $E^{p-r,q+r-1}_r = 0 = E^{p+r,q-r+1}_r$. Therefore, \( \ker(d_r^{p,q}) = E^{p,q}_r \) and \( \im(d_r^{p-r,q+r-1}) = 0 \). Therefore, $E^{p,q}_{r+1} = E^{p,q}_r$. This implies that the sequence $E^{p,q}_r$ becomes constant, up to isomorphism, once $r$ is large enough. We denote $E^{p,q}_\infty$ to be this constant value. This large amount of data is typically denoted $E^{p,q}_2 \Rightarrow E^{p+q}$ for short.

We note the following immediate consequences of the definition.
1. If $E^p_q = 0$, then $E^p_{s,q} = 0$ for all $s \geq r$, and $E^p_{\infty} = 0$.

2. For any $n \geq 0$, we have $F^0(E^n) = E^n$. This follows from the first fact because our assumption that $E^p_{r,q} = 0$ if $p < 0$ or $q < 0$ yields $E^p_{\infty} = 0$.

3. For any $n \geq 0$, we have $F^{n+1}(E^n) = 0$. This follows for similar reasons as in the previous fact.

In many situations, including ours, the $E^n$ are $n$-th cohomology groups. Having a spectral sequence allows one to get information about $E^n$ since determining the limit terms $E^p_{\infty}$ determines the factors $F^p(E^{p+q})/F^{p+1}(E^{p+q})$ of the filtration of $E^{p+q}$. In several situations, including ours, the spectral sequence is simple enough to completely determine $E^2$ from the limit terms.

We give two general but simple examples that we will use below.

**Example A.5.** Suppose that $E^p_{r,q} = 0$ for all $p, q \geq 0$. Then $E^p_{\infty} = 0$ for all $p, q$. Therefore, $F^p(E^2)/F^{p+1}(E^2) = 0$ for all $p \geq 0$. This means $F^p(E^2) = F^{p+1}(E^2)$ for all $p \geq 0$. Since $F^0(E^2) = E^2$ and $F^3(E^2) = 0$, as noted in the second and third facts above, this yields $E^2 = 0$.

**Example A.6.** Suppose that $E^0_{\infty} = E^2_{\infty} = 0$ and $E^1_{\infty} = A$ for some group $A$. Then the relation to the filtration for $E^2$ gives

\[
0 = F^0(E^2)/F^1(E^2) = E^2/F^1(E^2),
\]

\[
A = F^1(E^2)/F^2(E^2),
\]

\[
0 = F^2(E^2)/F^3(E^3) = F^2(E^2).
\]

Again we are using the second and third facts above. These three equations tell us that $E^2 = F^1(E^2)$ and $F^1(E^2) = A$. Therefore, $E^2 = A$.

To determine $H^2(D_n, T)$ for $n \geq 2$, we will use the *Lyndon-Hochschild-Serre* spectral sequence. If $N$ is a normal subgroup of a group $G$, and if $A$ is a $G$-module, this is the spectral sequence

\[
E^p_{2,q} = H^p(G/N, H^q(N, A)) \implies H^{p+q}(G, A).
\]

A proof of the existence of this sequence can be found in [7, Section 6.8]. We will apply this sequence with $G = D_n$ and $N = C_n$ for $A = T$. As a first use of this spectral sequence, we prove the result quoted above in Lemma A.4.

**Lemma A.7.** If $G_0$ is a finite subgroup of $O_2(\mathbb{R})$, then $H^2(G_0, \mathbb{R}^2) = 0$.

**Proof.** By Proposition 2.27, $G_0$ is isomorphic to $C_n$ or $D_n$ for some integer $n$. If $G_0 = C_n$, then $G_0$ is generated by a rotation $r$. Since a rotation has no nonzero fixed point, we see that $(\mathbb{R}^2)^{G_0} = \{0\}$. Therefore, $H^2(G_0, \mathbb{R}^2) \cong (\mathbb{R}^2)^{G_0}/\text{im}(N_{G_0}) = 0$. On the other hand, if
G_0 = D_n, let N = G_0 \cap SO_2(\mathbb{R}), the normal subgroup of rotations in G_0. We consider the spectral sequence
\[ H^p(G_0/N, H^q(N, \mathbb{R}^2)) \implies H^{p+q}(G_0, \mathbb{R}^2). \]
For q = 2n even, we have \( H^q(N, \mathbb{R}^2) = H^2(N, \mathbb{R}^2) = 0 \) by the argument just given. For q = 2n + 1 odd, \( H^q(N, \mathbb{R}^2) \cong \ker(N_{G_0})/\im(1 - r) \), where the subgroup N is generated by r. Since r has no nonzero fixed points, 1 is not an eigenvalue of r. Therefore, 1 - r is invertible on \( \mathbb{R}^2 \). This forces \( \im(1 - r) = \ker(N_{G_0}) = \mathbb{R}^2 \), and so \( H^q(N, \mathbb{R}^2) = 0 \). Therefore, the \( E_2^{p,q} \) terms of the spectral sequence are \( H^p(G_0/N, 0) = 0 \). By Example A.5, we get \( H^n(G_0, \mathbb{R}^2) = 0 \) for all n. In particular, \( H^2(G_0, \mathbb{R}^2) = 0 \).

\[ G_0 = D_{3,l}, D_{3,s}. \]
In these two cases we do not need to keep track of the action of \( G_0 \) on T except for how the 120° rotation acts, so we consider \( G_0 = D_3 \) without worrying about the difference between the different actions we have. To determine \( H^2(D_3, T) \), we use the Lyndon-Hochschild-Serre spectral sequence arising from the normal subgroup \( C_3 \) of \( D_3 \), which is
\[ E_2^{p,q} = H^p(D_3/C_3, H^q(C_3, T)) \implies H^{p+q}(D_3, T) = E_2^{p+q}. \]
If \( C_3 = \langle r \rangle \), then the calculation of the cohomology of a cyclic group gives
\[ H^q(C_3, T) = \begin{cases} T^{C_3}/N_C(T) = 0 & \text{if } q \equiv 0 \pmod{2} \\ \ker(N)/\im(1 - r) & \text{if } q \equiv 1 \pmod{2} \end{cases}. \]
However, the norm map \( N : T \to T \) with respect to \( C_3 \) is \( N(t) = t + r(t) + r^2(t) \). Since \( 1 + r + r^2 = 0 \) as a linear transformation, \( \ker(N) = T \). The linear transformation \( 1 - r \) is represented by the matrix
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}. \]
Therefore, \( \im(1 - r) = \{(a + b, -a + 2b) : a, b \in \mathbb{Z}\} \), and it is easy to see that this is equal to \( \{(x, y) : x + y \equiv 0 \pmod{3}\} \). Therefore, \( H^q(C_3, T) \cong \mathbb{Z}_3 \) if \( q \) is odd. The nontrivial element of \( D_3/C_3 \cong \mathbb{Z}_2 \) either acts trivially on \( \mathbb{Z}_3 \) or it acts as \(-1\). In either of these cases, one of which occurs with \( D_{3,l} \) and the other with \( D_{3,s} \), we see that \( H^n(D_3/C_3, \mathbb{Z}_3) = 0 \) for all \( n \) by the calculation of the cohomology of a cyclic group. Therefore, putting all these pieces together, we see that \( E_2^{p,q} = 0 \) for all \( p, q \geq 0 \). By Example A.5, this yields \( H^2(D_3, T) = 0 \).

\[ G_0 = D_4. \]
Again, we use the Lyndon-Hochschild-Serre spectral sequence arising from the normal subgroup \( C_n \) of \( D_n \), which in this case is
\[ H^p(D_4/C_4, H^q(C_4, T)) \implies H^{p+q}(D_4, T). \]
From the case of $C_4 = \langle r \rangle$, we saw that $H^{2n}(C_4, T) = 0$ for all $n$. For odd integers, if $N : T \to T$ is the norm map $N(t) = t + r(t) + r^2(t) + r^3(t)$, we have

$$H^{2n+1}(C_4, T) \cong \ker(N)/\operatorname{im}(1 - r).$$

Since $r^2$ acts as $-1$ on $T$, we see that $N = 0$. Therefore, $\ker(N) = T$. Also,

$$(1 - r)(a, b) = (a, b) - (b, -a) = (a - b, a + b).$$

Therefore, it follows that $\operatorname{im}(1 - r) = \{(x, y) : x \equiv y \pmod{2}\}$. The two elements $(0, 0)$ and $(1, 0)$ then represent all cosets in $T/\operatorname{im}(N)$, so $H^{2n+1}(C_4, T) \cong \mathbb{Z}_2$. The group $D_4/C_4$ then acts trivially on the groups $H^q(C_4, T)$ since these groups are either trivial or have two elements. The norm map for $D_4/C_4$ on $H^q(C_4, T)$ is trivial in either case. Therefore, we obtain the formulas

$$E_2^{p,q} = \begin{cases} 0 & \text{if } q \equiv 0 \pmod{2} \\ \mathbb{Z}_2 & \text{if } q \equiv 1 \pmod{2} \end{cases}.$$ 

Since $E_2^{2,0} = 0 = E_2^{0,2}$, we have $E_2^{0,0} = 0 = E_\infty^{0,2}$. Also, from the relationship between $E_\infty^{p,q}$ and the $E_2^{p,q}$, we have $E_2^{1,1} = E_3^{1,1}$, and this is the homology of the complex $0 = E_2^{2,0} \to E_2^{1,1} \to E_2^{0,2} = 0$. Thus, $E_3^{1,1} = E_2^{1,1}$. Example A.6 then yields

$$H^2(D_4, T) = E_\infty^{1,1} = \mathbb{Z}_2.$$

$G_0 = D_{2,p}$. 

In this final case we continue to use the Lyndon-Hochschild-Serre spectral sequence, and we apply it to the normal subgroup $C_2$ of $D_2$, where we write $D_2$ for $D_{2,p}$. We have $D_2 = \langle r, f \rangle$, and the action on $T$ satisfies $rt = -t$ for all $t$, and there is a basis $\{t_1, t_2\}$ of $T$ with $ft_1 = t_1$ and $ft_2 = -t_2$. From our description of the cohomology of a cyclic group, it is easy to see that

$$H^q(C_2, T) = \begin{cases} 0 & \text{if } q \equiv 0 \pmod{2} \\ T/2T & \text{if } q \equiv 1 \pmod{2} \end{cases}.$$ 

as Abelian groups. Without giving any details, to see what is the $D_2/C_2$-action on $H^q(C_2, T)$, we point out that Example 6.7.10 of [7] can be modified to show that the isomorphism $H^{2n+1}(C_2, T) \cong T/2T$ sends the action of $f$ to $(-1)^n f$. However, $f$ acts as the identity on $T/2T$ by the description of $f$ and because $-1$ acts as the identity on a group of exponent 2. Therefore, $D_2/C_2$ acts trivially on $H^{2n+1}(C_2, T)$. Because $D_2/C_2$ is cyclic,

$$E_2^{p,q} = H^p(D_2/C_2, H^q(C_2, T)) = \begin{cases} 0 & \text{if } q \equiv 0 \pmod{2} \\ T/2T & \text{if } q \equiv 1 \pmod{2} \end{cases}.$$ 

Again, by Example A.6, we conclude that

$$H^2(D_2, T) = T/2T \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$
# List of Symbols

Listed in the following table are most of the symbols used in the text, along with the meaning and a page reference for each symbol.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}$</td>
<td>set of real numbers</td>
<td>3</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>space of n-tuples of real numbers</td>
<td>3</td>
</tr>
<tr>
<td>$\text{Isom}(\mathbb{R}^n)$</td>
<td>group of isometries of $\mathbb{R}^n$</td>
<td>3</td>
</tr>
<tr>
<td>Sym($W$)</td>
<td>symmetry group of $W$</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>zero vector</td>
<td>25</td>
</tr>
<tr>
<td>$T$</td>
<td>translation group</td>
<td>25</td>
</tr>
<tr>
<td>$\text{id}$</td>
<td>identity function</td>
<td>25</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>ring of integers</td>
<td>25</td>
</tr>
<tr>
<td>$|u|$</td>
<td>length of the vector $u$</td>
<td>17</td>
</tr>
<tr>
<td>$u \cdot v$</td>
<td>dot product of $u$ and $v$</td>
<td>17</td>
</tr>
<tr>
<td>$A^T$</td>
<td>transpose of $A$</td>
<td>18</td>
</tr>
<tr>
<td>$I_n$</td>
<td>$n \times n$ identity matrix</td>
<td>18</td>
</tr>
<tr>
<td>$O_n(\mathbb{R})$</td>
<td>orthogonal group</td>
<td>13</td>
</tr>
<tr>
<td>$\text{Aut}(T)$</td>
<td>automorphism group of $T$</td>
<td>20</td>
</tr>
<tr>
<td>$\text{det}(A)$</td>
<td>determinant of $A$</td>
<td>20</td>
</tr>
<tr>
<td>$\text{SO}_2(\mathbb{R})$</td>
<td>special orthogonal group</td>
<td>20</td>
</tr>
<tr>
<td>$D_n$</td>
<td>dihedral group of order $2n$</td>
<td>21</td>
</tr>
<tr>
<td>$T$</td>
<td>translation lattice of a wallpaper group</td>
<td>29</td>
</tr>
<tr>
<td>$(A, b)$</td>
<td>notation for the isometry $x \mapsto Ax + b$</td>
<td>29</td>
</tr>
<tr>
<td>$I$</td>
<td>identity transformation</td>
<td>29</td>
</tr>
<tr>
<td>$G_0$</td>
<td>point group of a wallpaper group $G$</td>
<td>29</td>
</tr>
<tr>
<td>$C_n$</td>
<td>cyclic group of order $n$</td>
<td>30</td>
</tr>
<tr>
<td>$\text{GL}_2(\mathbb{Z})$</td>
<td>general linear group</td>
<td>32</td>
</tr>
<tr>
<td>$D_{3,1}$</td>
<td>one representation of $D_3$</td>
<td>38</td>
</tr>
<tr>
<td>$D_{3,s}$</td>
<td>another representation of $D_3$</td>
<td>38</td>
</tr>
<tr>
<td>$D_{1,p}$</td>
<td>$D_1$ for rectangular lattice</td>
<td>42</td>
</tr>
<tr>
<td>$D_{1,c}$</td>
<td>$D_1$ for rhombic lattice</td>
<td>42</td>
</tr>
<tr>
<td>$D_{2,p}$</td>
<td>$D_2$ for rectangular lattice</td>
<td>42</td>
</tr>
<tr>
<td>Symbol</td>
<td>Definition</td>
<td>Page</td>
</tr>
<tr>
<td>----------</td>
<td>-----------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>$D_{2,c}$</td>
<td>$D_2$ for rhombic lattice</td>
<td>42</td>
</tr>
<tr>
<td>ker($f$)</td>
<td>kernel of $f$</td>
<td>101</td>
</tr>
<tr>
<td>$T \times \varphi G_0$</td>
<td>semidirect product of $T$ and $G_0$</td>
<td>102</td>
</tr>
<tr>
<td>$Z^2(G_0, T)$</td>
<td>group of 2-cocycles</td>
<td>103</td>
</tr>
<tr>
<td>$B^2(G_0, T)$</td>
<td>group of 2-coboundaries</td>
<td>103</td>
</tr>
<tr>
<td>$H^2(G_0, T)$</td>
<td>second cohomology group</td>
<td>103</td>
</tr>
<tr>
<td>$\mathbb{Z}_n$</td>
<td>integers modulo $n$</td>
<td>104</td>
</tr>
<tr>
<td>$\langle a \rangle$</td>
<td>cyclic group generated by $a$</td>
<td>105</td>
</tr>
<tr>
<td>$N_C$</td>
<td>norm map for a $C$-module</td>
<td>109</td>
</tr>
<tr>
<td>$M_C$</td>
<td>fixed points of $C$-module $M$</td>
<td>109</td>
</tr>
<tr>
<td>im($f$)</td>
<td>image of $f$</td>
<td>109</td>
</tr>
<tr>
<td>$E_r^{p,q}$</td>
<td>terms of a spectral sequence</td>
<td>111</td>
</tr>
<tr>
<td>$d_r^{p,q}$</td>
<td>differentials of a spectral sequence</td>
<td>111</td>
</tr>
<tr>
<td>$F^p(E)$</td>
<td>filtration of a spectral sequence</td>
<td>111</td>
</tr>
<tr>
<td>$E_\infty^{p,q}$</td>
<td>limit terms of a spectral sequence</td>
<td>111</td>
</tr>
</tbody>
</table>
Bibliography


Index

2-coboundary, 103
2-cocycle, 102
    normalized, 102
\text{Aut}(T), 20
Cauchy-Schwartz inequality, 17
cohomology group, 108
    higher, 108
    second, 103
dihedral group, 9, 21
dot product, 17
equivalent extension, 103
Escher, M. C., 1
five term exact sequence, 110
fundamental domain, 90
\text{G}_0\text{-module}, 102
glide reflection, 5
    non-trivial, 5
group extension, 101
hexagonal lattice, 37
integral basis, 26
isometry, 3
lattice, 25
    dimension of, 25
    hexagonal, 37
    parallelogram, 35
    square, 36
limit terms of a spectral sequence, 111
Lyndon-Hochschild-Serre spectral sequence, 112
    norm map, 109
    normalized cocycle, 102
Orthogonal Group, 13
orthogonal group
    special, 20
parallelogram lattice, 35
point group, 29
Reflection, 4
rotation, 4
second cohomology group, 103
semidirect product, 19, 102
Special Linear Group, 13
Special Orthogonal Group, 13
special orthogonal group, 20
spectral sequence, 111
    Lyndon-Hochschild-Serre, 112
split wallpaper group, 48
square lattice, 36
symmetry group, 3
translation, 3
translation lattice, 30
translation subgroup, 25
wallpaper group, 26
    split, 48
wallpaper pattern, 26