Exercises From Section 5.1

In this note we fill in some of the details from Chapter 5.1 of Weibel. Let $E$ be a double complex of modules. The first result we do is part 1 of Exercise 5.1.2, which gives a description of $E^2_{pq}$.

**Problem 1** (5.1.2). The module $E^2_{pq}$ is isomorphic to the quotient $U/V$, where

$$U = \{ (a, b) \in E^0_{p-1,q+1} \oplus E^0_{pq} : d^v(a) + d^h(b) = 0 = d^v(b) \}$$

and $V$ is generated by all elements of the form $(a, 0)$ with $d^v(a) = 0$; $(d^h(x), d^v(x))$ for $x \in E^0_{p,q+1}$; and $(0, d^v(c))$ with $c \in E^1_{p,q+1}$ and $d^v(c) = 0$.

**Proof.** Let $(a, b) + V \in U/V$. Then $b \in Z^1_{pq}$. The definition of $\tilde{d}^h$ is $\tilde{d}^h(b + B^1_{pq}) = d^h(b) + B^1_{p-1,q}$. However, $d^h(b) = d^v(-a) \in B^1_{p-1,q}$. Therefore, $b + B^1_{pq} \in \ker(\tilde{d}^h) = Z^2_{pq}$. Thus, we have a homomorphism $\varphi : U \to E^2_{pq}$ given by $\varphi(a, b) = (b + B^1_{pq}) + B^2_{pq}$. To cut down on notation, we will write $\overline{x}$ for $x + B^1_{pq}$ in $E^1_{pq}$. Thus, $\varphi(a, b) = \overline{b} + B^2_{pq}$. To show that $\varphi$ is surjective, take $\overline{t} + B^2_{pq} \in E^2_{pq}$. Then $\overline{t} \in Z^2_{pq}$, so $\overline{d}^h(\overline{t}) = 0$. Moreover, we can represent $\overline{t}$ by $t \in Z^1_{pq}$. Therefore, $d^v(t) = 0$. The condition $\overline{d}^h(\overline{t}) = 0$ means $d^h(t) \in B^1_{p-1,q}$; say $d^h(t) = d^v(-s)$ for some $s \in E^0_{p-1,q+1}$. The pair $(s, t)$ is then in $U$; it clearly maps to $\overline{t} + B^2_{pq}$. Therefore, $\varphi$ is surjective. Finally, we need to show that $\ker(\varphi) = V$. To do this, we first see that the inclusion $V \subseteq \ker(\varphi)$ is easy. For $\varphi(a, 0) = 0$ is clear. Next, $\varphi(d^h(x), d^v(x)) = \overline{d}^v(x) + B^2_{pq}$. However, $d^v(x) = d^v(x) + B^1_{pq}$, but $d^v(x) \in B^1_{pq}$ by definition. Finally, if $d^v(c) = 0$, then $\varphi(0, d^h(c)) = \overline{d}^h(c) + B^2_{pq}$. However, $c \in Z^1_{p,q+1}$, so $\overline{d}^h(c) = \overline{d}^h(c + B^1_{p-1,q}) \in B^2_{pq}$ by definition. Thus, $\varphi(0, d^h(c)) = 0$. For the reverse inclusion, let $(a, b) \in \ker(\varphi)$. Then $\overline{b} + B^2_{pq} = 0$. So, $\overline{b} = \overline{d}^h(\overline{c})$ for some $\overline{c} \in E^1_{p+1,q}$. We pick representatives $b \in Z^1_{pq}$ and $c \in Z^1_{p+1,q}$. We then have $b - d^h(c) = d^v(a')$ for some $a' \in E^0_{p,q+1}$. We can then write out pair $(a, b)$ as

$$(a, b) = (a - d^h(a'), 0) + (d^h(a'), d^v(a')) + (0, d^h(c)).$$

Each of these three terms are in $V$; the second and third are clearly in $V$, and the first is also in $V$ since $d^v(a - d^h(a')) = d^v(a) + d^h(a') = 0$ by the condition $d^v(a) + d^h(b) = 0$ and $d^h(b) = d^h(d^v(a')) = d^h(d^v(c)) = 0$. Therefore, $\ker(\varphi) = V$, and this proves our description of $E^2_{pq}$. $\square$
Problem 2 (5.1.3). There is an exact sequence of low degree terms

\[ H_2(T) \rightarrow E^{2}_{20} \rightarrow E^{2}_{01} \rightarrow H_1(T) \rightarrow E^{2}_{10} \rightarrow 0. \]

Proof. We have already defined the map \( E^{2}_{2,0} \rightarrow E^{2}_{0,1} \). We now define all the other maps.

- \( f : H_1(T) \rightarrow E^{2}_{10} \): Take \((a, b) \in Z_1(T) \subseteq E^{0}_{01} \oplus E^{0}_{10} \). Then \( d(a, b) = d^v(a) + d^h(b) \), so \( d^v(a) + d^h(b) = 0 \). Also, \( d^h(b) = 0 \) since \( E_{1,-1} = 0 \). Therefore, \((a, b)\) defines an element in \( E^{2}_{10} \). If \((a, b) \in B_1(T), then\)

\[
(a, b) = d(x, y, z) = (d^v(x) + d^h(y), d^v(y) + d^h(z)) = (d^v(x), 0) + (d^h(y), d^v(y)) + (0, d^h(z)),
\]

which shows \((a, b) = 0\) in \( E^{2}_{10} \). Therefore, we have a map \( f : H_1(T) \rightarrow E^{2}_{10} \), given by \( f ((a, b) + B_1(T)) = (a, b) \in E^{2}_{1,0} \).

- \( g : E^{2}_{01} \rightarrow H_1(T) \): We have that \( E^{2}_{01} \) is a subquotient of \( E^{0}_{-1,2} \oplus E^{0}_{01} = E^{0}_{01} \). Furthermore, \( b \in E^{2}_{01} \) yields the element \((b, 0) \in Z_1(T) \) since \( 0 = d^v(b) = d(b, 0) \). Therefore, the map \( g \) is given by \( g(b + V) = (b, 0) + B_1(T) \in H_1(T) \). To see this is well defined, we need \( V \rightarrow 0 \). An element in \( V \) is a sum \( d^v(x) + d^h(c) \) with \( x \in E^{0}_{02} \) and \( c \in Z^{1}_{11} \). But then 

\[
(d^v(x) + d^h(c), 0) = d(x, c, 0) \in B_1(T).
\]

- \( h : H_2(T) \rightarrow E^{2}_{20} \): let \((a, b, c) \in Z_2(T) \subseteq E^{0}_{02} \oplus E^{0}_{11} \oplus E^{0}_{20} \). Then \( d^v(a) + d^h(b) = 0 \) and \( d^v(b) + d^h(c) = 0 \). Note that \( d^v(c) = 0 \) since \( d^v_{20} = 0 \). Then \((b, c)\) represents an element of \( E^{2}_{20} \). We have

\[
B_2(T) = \{(d(x, y, z, w) : (x, y, z, w) \in C_2(T) \}
\]

\[
= \{(d^v(x) + d^h(y), d^v(y) + d^h(z), d^v(z) + d^h(w)) : (x, y, z, w) \in C_2(T) \}.
\]

Therefore

\[
(d^v(y) + d^h(z), d^v(z) + d^h(w)) = (d^v(y), 0) + (d^h(z), d^v(z)) + (0, d^h(w)).
\]

Furthermore, all three of these terms are zero in \( E^{0}_{20} \); the first is zero since \( d^v(d^v(y)) = 0 \) and the third is zero since \( d^v(w) = 0 \) (as \( d^w_{20} = 0 \)). Therefore, there is an induced map \( h : H_2(T) \rightarrow E^{2}_{20} \) given by \( h(a, b, c) + B_2(T) = (b, c) \in E^{2}_{20} \).

We now show that the sequence above is exact.

**Exactness at \( E^{2}_{10} \):** If \((a, b) \in E^{2}_{10} \), then \( d^v(a) + d^h(b) = 0 \). Therefore, \((a, b)\) represents a pair in \( Z_1(T) \), and \((a, b) \in H_1(T) \) maps onto \((a, b) \in E^{2}_{10} \).

**Exactness at \( H_1(T) \):** Let \( b \in E^{2}_{01} \). Then \( d^v(b) = 0 \). The element \( b \) is sent to \((0, b) \in H_1(T) \), which is sent to \((b, 0) \in E^{2}_{10} \). However, this element is 0 in \( E^{2}_{10} \) since \( d^v(b) = 0 \). Therefore,
the sequence $E^2_{01} \to H_1(T) \to E^2_{10}$ is a zero sequence. Conversely, take $(a, b) \in H_1(T)$ with $f(a, b) = 0$ in $E^2_{10}$. Then $(a, b) = 0$ in $E^2_{10}$. Therefore, in $E^0_{01} \oplus E^0_{10}$, we have

$$(a, b) = (x, 0) + (d^h(y), d^v(y)) + (0, d^h(z))$$

with $x \in Z^1_{01}$, $y \in E^0_{11}$, and $z \in E^0_{20} = Z^1_{20}$. We see that $(d^h(y), d^v(y)) = d(0, y, 0)$ and $(0, d^h(z)) = d(0, 0, z)$. Therefore, $(a, b) + B_1(T) = (x, 0) + B_1(T)$. However, $(x, 0) + B_1(T) = g(x)$, so $\ker(f) = \text{im}(g)$.

**Exactness at $E^2_{01}$:** If $(a, b) \in E^2_{01}$, then it gets mapped to $d(a, b) = d^h(a) \in E^2_{01}$, then to $(d^h(a), 0) + B_1(T) \to H_1(T)$. However, since $d^v(a) + d^h(b) = 0$, $(d^h(a), 0) = d(0, a, b) \in B_1(T)$. Therefore, the sequence $E^2_{20} \to E^2_{01} \to H_1(T)$ is a zero sequence. Conversely, if $b \in E^2_{01}$ with $(b, 0) = 0$ in $H_1(T)$, then

$$(b, 0) = d(x, y, z) = (d^v(x) + d^h(y), d^v(y) + d^h(z))$$

for some $(x, y, z) \in C_2(T)$. This means $b = d^v(x) + d^h(y)$ and $0 = d^v(y) + d^h(z))$. We can then view $(y, z) \in E^2_{20}$. Writing $E^2_{01} = U/V$, we have $d^v(x) \in V$ by our description of $E^2_{01}$. So, $b + V = d^h(y) + V$. Therefore, $d(y, z) = d^h(y) = b \in E^2_{01}$. This shows that the sequence is exact at $E^2_{01}$.

**Exactness at $E^2_{20}$:** If $(a, b, c) \in Z_2(T)$, then $h(a, b, c) = (b, c) \in E^2_{20}$. Then $d(b, c) = d^h(b) \in E^2_{01}$. However, $d^v(a) + d^h(b) = 0$, so $d^h(b) = d^v(-a) \in B^2_{01}$. Therefore, $d^h(b) = 0$ in $E^2_{01}$. The sequence $H_2(T) \to E^2_{20} \to E^2_{01}$ is thus a zero sequence. Conversely, if $(b, c) \in E^2_{20}$ with $d^h(b) = 0$ in $E^2_{01}$, then as an element of $E^0_{01}$,

$$d^h(b) = d^v(u) + d^h(w)$$

for some $u \in E^0_{02}$ and $w \in Z^1_{11}$. In particular, $d^v(w) = 0$. Therefore, $(w, 0) = 0$ in $E^2_{20}$, so $(b, c) = (b - w, c)$. Let $b' = b - w$. We have $d^v(b') + d^h(c) = 0$, so $d^v(b') + d^h(c) = 0$. Moreover, $d^h(b') = d^v(u)$. Therefore, $(-u, b', c) \in Z_2(T)$. This element maps to $(b', c) = (b, c) \in E^2_{20}$.

Therefore, the sequence is exact at $E^2_{20}$.

To finish this note, we point out that there is a natural finite filtration on $H_n(T)$ for $T$ a first quadrant double complex. First, $C_n(T) = E^0_{0n} \oplus E^0_{1,n-1} \oplus \cdots \oplus E^0_{n0}$. Then $C_n(T)$ has a filtration of length (at most) $n + 1$

$$0 \subseteq E^0_{n0} \subseteq E^0_{n-1,1} \oplus E^0_{n0} \subseteq \cdots \subseteq C_n(T).$$

This restricts to a filtration of length $n + 1$ on $Z_n(T)$. If this is $0 \subseteq F^0(Z_n(T)) \subseteq \cdots \subseteq F^n(Z_n(T)) = Z_n(T)$, then we obtain a filtration on $H_n(T)$ as

$$0 \subseteq \frac{F^0(Z_n(T)) + B_n(T)}{B_n(T)} \subseteq \frac{F^1(Z_n(T)) + B_n(T)}{B_n(T)} \subseteq \cdots \subseteq \frac{F^n(Z_n(T)) + B_n(T)}{B_n(T)} = H_n(T).$$