Root Space Decomposition of the Classical Lie Algebras

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September 18, 1998

1 Introduction

Let L be a semisimple Lie algebra. We assume that $\mathrm{char}(F) \neq 2$. If T is a maximal toral subalgebra of L, we may write

$$L = T \oplus \bigoplus_{\alpha \in \Phi} L_\alpha,$$

where L_α is the generalized Eigenspace of the root α. Suppose that $L_\alpha = Fx_\alpha$. Let κ be the Killing form of L, defined by $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$. Then we get an element $t_\alpha \in T$ by $[x_\alpha, x_{-\alpha}] = \kappa(x_\alpha, x_{-\alpha})t_\alpha$. We also define $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$, and the bracket $\langle\ ,\ \rangle$ by

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}.$$

Recall that if α and β are simple roots, then the Dynkin diagram of L connects α and β with $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ edges. Moreover, if $\|\alpha\| < \|\beta\|$, then we draw an arrow on the edge(s) toward α. If they have the same length, then we have an undirected edge.

To simplify some of the calculations, we note that in all four cases of the classical Lie algebras, the form $\mathrm{tr}(xy)$ is an associative bilinear form. Thus, there is a scalar a with $\kappa(x, y) = a\,\mathrm{tr}(xy)$. For a root α, if we set $s_\alpha = at_\alpha$, then we will have $[x_\alpha, x_{-\alpha}] = a\,\mathrm{tr}(x_\alpha x_{-\alpha})t_\alpha = \mathrm{tr}(x_\alpha x_{-\alpha})s_\alpha$. So, we can determine s_α from this formula. Moreover, $(\alpha, \beta) = \kappa(t_\alpha, t_\beta) = a^{-2}\kappa(s_\alpha, s_\beta) = a^{-1}\,\mathrm{tr}(s_\alpha s_\beta)$ and $(\beta, \beta) = a^{-1}\,\mathrm{tr}(s_\beta^2)$. Thus, we can calculate $\langle \alpha, \beta \rangle$ by the formula

$$\langle \alpha, \beta \rangle = \frac{2\,\mathrm{tr}(s_\alpha s_\beta)}{\mathrm{tr}(s_\beta^2)}.$$

Because of this we do not need to calculate the Killing form κ.

Throughout we will denote by e_{ij} the $n \times n$ matrix with a 1 in the ij position and 0 elsewhere. In the section B_n, we write e_i for the i-th standard basis vector of F^n.

We now investigate the classical algebras, describe their roots, simple roots, and Dynkin diagrams.

2 A_n

Let $L = \mathrm{sl}(n+1, F)$. We first describe a toral subalgebra of L. Let $d(n+1, F)$ be the set of diagonal matrices, and define $T = d(n+1, F) \cap L$. Note that

$$T = \{\mathrm{diag}(a_1, \dots, a_n, -a_1 - \cdots - a_n) : a_i \in F\}.$$

We claim that T is a maximal toral subalgebra of L. To prove this, note that it is clear that T is a toral subalgebra; every element of T is diagonal, hence semisimple. To show maximality , if $T \subseteq T'$ with T' toral, then for $a \in T'$, we have $[a, b] = 0$ for all $b \in T$. Thus, $ab = ba$ for all $b \in T$. If $b = \mathrm{diag}(b_1, \dots, b_{n+1}) \in L$ and $a = (a_{ij})$, then $b_i a_{ij} = b_j a_{ij}$ for all i, j. Since b is arbitrary (but $\mathrm{tr}(b) = 0$), we can pick a b with $b_i \neq b_j$. So, $a_{ij} = 0$ if $i \neq j$. Thus, a is diagonal. So, $a \in T$, and thus $T' = T$ as desired.

We now describe the roots of L. For $i \neq j$ with $i, j \leq n$, define $\varphi_{ij} : T \to F$ by $\varphi_{ij}(\sum b_i e_{ii}) = b_i - b_j$. Then $\varphi_{ij} \in T^* = \hom_F(T, F)$. Moreover, a short calculation shows that

$$[t, e_{ij}] = \varphi_{ij}(t) e_{ij}$$

for all $t \in T$. Thus, $\varphi_{ij} \in \Phi$. Since we have $\dim_F(L) = (n+1)^2 - 1$ and $\dim_F(T) = n$, we must have $(n-1)^2 - 1 - n = n(n+1)$ roots. We have exhibited that many with the φ_{ij}, so $\Phi = \left\{\varphi_{ij} : i \neq j\right\}$. To determine the simple roots, note that if $i < j$, then $\varphi_{ij} = \sum_{k=i}^{j-1} \varphi_{k,k+1}$. Also, $\varphi_{ji} = -\varphi_{ij}$, so the $\left\{\varphi_{i,i+1} : 1 \leq i \leq n\right\}$ span Φ, and since they are clearly independent, they form a basis for T^*. Thus, the set of simple roots of L is

$$\left\{\varphi_{12}, \varphi_{23}, \dots, \varphi_{n,n+1}\right\}.$$

For simplicity, we will write $\alpha_i = \varphi_{i,i+1}$ and $s_i = s_{\alpha_i}$. To determine the numbers $\langle \alpha_i, \alpha_j \rangle$, we first need to determine s_i. We have, since $L_{\varphi_{ij}} = F e_{ij}$ for each i, j, and $-\varphi_{ij} = \varphi_{ji}$, that

$$s_i = \frac{[e_{i,i+1}, e_{i+1,i}]}{\mathrm{tr}(e_{i,i+1}, e_{i+1,i})} = \frac{1}{2}(e_{ii} - e_{i+1,i+1})$$

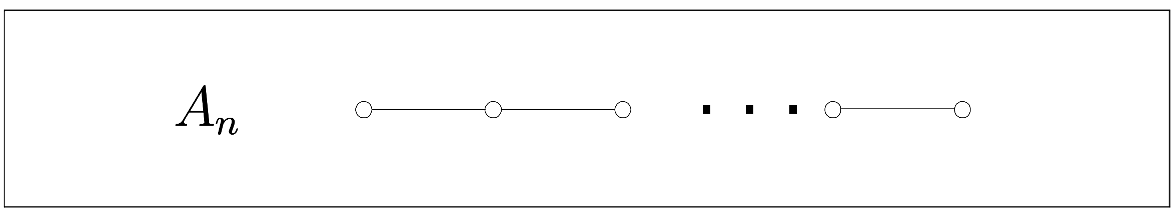
From this we get $\operatorname{tr}(s_i^2) = 1/2$, and so

$$\begin{aligned}\langle \alpha_i, \alpha_j \rangle = \frac{2\operatorname{tr}(s_i s_j)}{\operatorname{tr}(s_j^2)} &= \operatorname{tr}(e_{ii} - e_{i+1,i+1})(e_{jj} - e_{j+1,j+1}) \\ &= \begin{cases} 0 & \text{if } |i-j| \neq 1 \\ -1 & \text{if } |i-j| = 1 \end{cases}\end{aligned}$$

as $(e_{ii} - e_{i+1,i+1})(e_{jj} - e_{j+1,j+1}) = -e_{jj}$ if $i+1 = j$, and the product is 0 otherwise. Furthermore, this tells us that α_i and α_j are not connected if $|i-j| \neq 1$, and that α_i and α_{i+1} are connected with $(-1)(-1) = 1$ edge for all i. The length's of the α_i are all equal, since we may think of $\alpha_i = (0, \dots, 1, -1, 0, \dots, 0) \in \mathbb{R}^n$. So, the Cartan matrix for L is

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

and the Dynkin diagram for $L = \operatorname{sl}(n+1, F)$ is



3 B_n

Let $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$, a $(2n+1) \times (2n+1)$ matrix. The classical Lie algebra of type B_n is $L = o(2n+1, F) = \{x \in \operatorname{gl}(2n+1, F) : x^t M + Mx = 0\}$. Thus,

$$L = \left\{ \begin{pmatrix} 0 & (a_1, \dots, a_n) & (-b_1, \dots, -b_n) \\ (b_1, \dots, b_n)^t & C & D \\ (-a_1, \dots, -a_n)^t & E & -C^t \end{pmatrix} : D^t = D, E^t = E, a_i, b_i \in F \right\}.$$

To see why this is true, suppose that

$$x = \begin{pmatrix} a & (a_1, \dots, a_n) & (a_{n+1}, \dots, a_{2n}) \\ (b_1, \dots, b_n)^t & C & D \\ (b_{n+1}, \dots, b_{2n})^t & E & F \end{pmatrix}.$$

Then we have

$$x^t M + Mx = \begin{pmatrix} 2a & (\{a_i + b_{n+i}\}) & (\{a_{n+i} + b_i\}) \\ (\{a_i + b_{n+i}\})^t & E^t + E & C^t + F \\ (\{a_{n+i} + b_i\})^t & F^t + C & D^t + D \end{pmatrix}.$$

From this we obtain the description of L. From this we see that $\dim_F(L) = n^2 + n(n-1) + 2n = 2n^2 + n$. Let $d(2n+1, F)$ be the set of diagonal matrices, and let $T = d(2n+1, F) \cap L$. Note that the description above of L gives

$$T = \{\operatorname{diag}(0, a_1, \dots, a_n, -a_1, \dots, -a_n) : a_i \in F\}.$$

We claim that T is a maximal toral subalgebra of L. It is clear that T is a toral subalgebra. To prove maximality, suppose that T' is a toral subalgebra of L that contains T. Let $a \in T'$. Then $[a, b] = 0$ for all $b \in T$, so $ab = ba$ for all $b \in T$. If $a = (a_{ij})$, then $a_{ij}b_i = b_j a_{ij}$ for all i, j, where $b = \operatorname{diag}(0, \{b_i\})$. Since b is arbitrary, this forces the first row and first column of a to be 0, and $a_{ij} = 0$ for $i \neq j$ otherwise. Thus, $a \in T$. Therefore, T is a maximal toral subalgebra of L. Note that $\dim_F(T) = n$, so there will be $2n^2$ roots for L.

We determine the roots for L. Define, maps $\sigma_{ij}, \tau_{ij} : T \to F$ for $i < j$ and $\rho_i : T \to F$ for all i by

$$\begin{aligned} \sigma_{ij}(\operatorname{diag}(0, a_1, \dots, a_n, -a_1, \dots, -a_n)) &= a_i + a_j, \\ \tau_{ij}(\operatorname{diag}(0, a_1, \dots, a_n, -a_1, \dots, -a_n)) &= a_i - a_j, \\ \rho_i(\operatorname{diag}(0, a_1, \dots, a_n, -a_1, \dots, -a_n)) &= a_i. \end{aligned}$$

Then short calculations show that, for all $t \in T$,

$$\left[t, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{ij}+e_{ji} \\ 0 & 0 & 0 \end{pmatrix}\right] = \sigma_{ij}(t) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_{ij}+e_{ji} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left[t, \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ij} & 0 \\ 0 & 0 & -e_{ji} \end{pmatrix}\right] = \tau_{ij}(t) \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ij} & 0 \\ 0 & 0 & -e_{ji} \end{pmatrix},$$

$$\left[t, \begin{pmatrix} 0 & 0 & e_i \\ -e_i^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \rho_i(t) \begin{pmatrix} 0 & 0 & e_i \\ -e_i^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left[t, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_{ij}+e_{ji} & 0 \end{pmatrix}\right] = -\sigma_{ij}(t) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_{ij}+e_{ji} & 0 \end{pmatrix},$$

$$\left[t, \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ji} & 0 \\ 0 & 0 & -e_{ij} \end{pmatrix}\right] = -\tau_{ij}(t) \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ji} & 0 \\ 0 & 0 & -e_{ij} \end{pmatrix},$$

$$\left[t, \begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & 0 \\ -e_i^t & 0 & 0 \end{pmatrix}\right] = -\rho_i(t) \begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & 0 \\ -e_i^t & 0 & 0 \end{pmatrix},$$

Therefore, $\{\pm\sigma_{ij}, \pm\tau_{ij}, \pm\rho_i\}$ are all roots. Since there are $2n(n-1)+2n = 2n^2$ of them, this is the set of roots of L. We claim that a set of simple roots is

$$\{\tau_{12}, \dots, \tau_{n-1,n}, \rho_n\}.$$

To verify this, we have $\tau_{ij} = \sum_{k=i}^{j-1} \tau_{k,k+1}$. Next,

$$\begin{aligned} \sigma_{in} &= \tau_{i,i+1} + \cdots + \tau_{n-2,n-1} + \sigma_{n-1,n}, \\ \sigma_{ij} &= \tau_{ir} + \sigma_{jr}, \quad i < j < r, \\ \rho_i &= \tau_{in} + \rho_n. \end{aligned}$$

This shows that our set spans the set of roots, and since there is the correct number, they form a set of simple roots. For simplicity of notation, we define $\alpha_i = \tau_{i,i+1}$ if $i < n$, and $\alpha_n = \rho_n$, and write $s_i = s_{\alpha_i}$.

We now determine the s_i. Then by our eigenvectors above, we have, for $i < n$,

$$s_i = \frac{\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{i,i+1} & 0 \\ 0 & 0 & -e_{i+1,i} \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{i+1,i} & 0 \\ 0 & 0 & -e_{i,i+1} \end{pmatrix}\right]}{\operatorname{tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{i,i+1} & 0 \\ 0 & 0 & -e_{i+1,i} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{i+1,i} & 0 \\ 0 & 0 & -e_{i,i+1} \end{pmatrix}}$$
$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ii} - e_{i+1,i+1} & 0 \\ 0 & 0 & e_{i+1,i+1} - e_{ii} \end{pmatrix}.$$

For $i = n$, we have

$$s_n = \frac{\left[\begin{pmatrix} 0 & 0 & e_n \\ -e_n^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e_n & 0 \\ 0 & 0 & 0 \\ -e_n^t & 0 & 0 \end{pmatrix}\right]}{\operatorname{tr}\begin{pmatrix} 0 & 0 & e_n \\ -e_n^t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & e_n & 0 \\ 0 & 0 & 0 \\ -e_n^t & 0 & 0 \end{pmatrix}}$$
$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -e_n & 0 \\ 0 & 0 & e_n \end{pmatrix}.$$

We can now determine the numbers $\langle \alpha_i, \alpha_j \rangle$. We first see that $\operatorname{tr}(s_i^2) = 1$ and $\operatorname{tr}(s_n^2) = 1/2$. Next, if $i, j < n$, then

$$\operatorname{tr}(s_i s_j) = \frac{1}{4}\operatorname{tr}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ii} - e_{i+1,i+1} & 0 \\ 0 & 0 & e_{ii} - e_{i+1,i+1} \end{pmatrix}\right)\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{jj} - e_{j+1,j+1} & 0 \\ 0 & 0 & e_{j+1,j+1} - e_{jj} \end{pmatrix}\right),$$

so $\operatorname{tr}(s_i s_j) = 0$ if $|j - i| \neq 1$. If $j = i + 1$, then $\operatorname{tr}(s_i s_{i+1}) = -1/2$. Next,

$$\operatorname{tr}(s_i s_n) = \frac{1}{4}\operatorname{tr}\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_{ii} - e_{i+1,i+1} & 0 \\ 0 & 0 & e_{i+1,i+1} - e_{ii} \end{pmatrix}\right)\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & -e_n & 0 \\ 0 & 0 & e_n \end{pmatrix}\right),$$

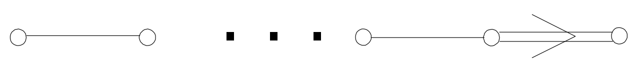
so $\operatorname{tr}(s_i s_n) = 0$ if $i < n - 1$. If $i = n - 1$, then $\operatorname{tr}(s_{n-1} s_n) = -1/2$. Therefore, we have

$\langle \alpha_i, \alpha_j \rangle = 0$ if $i, j < n$ and $|j - i| \neq 1$, and $\langle \alpha_i, \alpha_{i+1} \rangle = -1$. Also, $\langle \alpha_i, \alpha_n \rangle = 0$ if $i < n - 1$, and $\langle \alpha_{n-1}, \alpha_n \rangle = -2$, and $\langle \alpha_n, \alpha_{n-1} \rangle = -1$. Therefore, α_i and α_{i+1} are connected with $(-1)(-1) = 1$ edge, α_{n-1} and α_n are connected with $(-2)(-1) = 2$ edges, and no other vertices are connected. Furthermore, $\|\alpha_i\| = \sqrt{2}$ if $i < n$, and $\|\alpha_n\| = 1$. This comes from viewing $\alpha_i = (0, \ldots, 1, -1, 0, \ldots, 0)$ and $\alpha_n = (0, \ldots, 1)$ in $\mathbb{R}^n$. Thus, the double edge connecting α_{n-1} and α_n points toward α_n. Therefore, the Cartan matrix for L is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & \ddots & \ddots & -1 & 2 & -2 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}$$

and the Dynkin diagram for L is

B_n



4 C_n

Let $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then the Lie algebra of type C_n is

$$L = sp(2n, F) = \left\{ x \in \mathrm{gl}(2n, F) : x^t M + Mx = 0 \right\}.$$

Thus,

$$L = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} : B^t = B, C^t = C \right\}.$$

Therefore, $\dim_F(L) = n^2 + n(n+1) = 2n^2 + n$. We claim that a maximal toral subalgebra of L is

$$T = \{\mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n) : a_i \in F\}.$$

We first claim that if $d(2n, F)$ is the space of diagonal matrices, then $T = d(2n, F) \cap L$. The inclusion $\subseteq$ is clear. For the reverse, if A, B are diagonal matrices with $\operatorname{diag}(A, B) \in L$, then by the description above, we see that $B = -A$, forcing $\operatorname{diag}(A, B) \in T$ as desired. It is clear that T is a toral subalgebra of L. To show maximality, suppose that T' is a toral subalgebra of L that contains T. If $a \in T'$, then $[a, b] = 0$ for all $b \in T$ since T' is Abelian. But then $ab = ba$, which forces a to be a diagonal matrix by the argument in the A_n case. So, T is a maximal toral subalgebra. Since $\dim_F(T) = n$, we must have $2n^2 + n - n = 2n^2$ roots for L. For $i < j$, define σ_{ij} and τ_{ij} on T by

$$\begin{aligned}
\sigma_{ij}(\operatorname{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)) &= a_i + a_j, \\
\tau_{ij}(\operatorname{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)) &= a_i - a_j,
\end{aligned}$$

and define, for all i, the map ρ_i by

$$\rho_i(\operatorname{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n)) = 2a_i.$$

We see that the collection $\{\pm\sigma_{ij}, \pm\tau_{ij}, \pm\rho_i\}$ are all roots by checking that for all $t \in T$,

$$\begin{aligned}
\left[t, \begin{pmatrix} 0 & e_{ij} + e_{ji} \\ 0 & 0 \end{pmatrix}\right] &= \sigma_{ij}(t) \begin{pmatrix} 0 & e_{ij} + e_{ji} \\ 0 & 0 \end{pmatrix}, \\
\left[t, \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}\right] &= \tau_{ij}(t) \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}, \\
\left[t, \begin{pmatrix} 0 & e_{ii} \\ 0 & 0 \end{pmatrix}\right] &= \rho_{ij}(t) \begin{pmatrix} 0 & e_{ii} \\ 0 & 0 \end{pmatrix}, \\
\left[t, \begin{pmatrix} 0 & 0 \\ e_{ij} + e_{ji} & 0 \end{pmatrix}\right] &= -\sigma_{ij}(t) \begin{pmatrix} 0 & 0 \\ e_{ij} + e_{ji} & 0 \end{pmatrix}, \\
\left[t, \begin{pmatrix} e_{ji} & 0 \\ 0 & -e_{ij} \end{pmatrix}\right] &= -\tau_{ij}(t) \begin{pmatrix} e_{ji} & 0 \\ 0 & -e_{ij} \end{pmatrix}, \\
\left[t, \begin{pmatrix} 0 & 0 \\ e_{ii} & 0 \end{pmatrix}\right] &= -\rho_{ij}(t) \begin{pmatrix} 0 & 0 \\ e_{ii} & 0 \end{pmatrix}.
\end{aligned}$$

Thus, we have shown that the given elements are roots, and produced appropriate eigenvectors. Note that there are $2n^2$ roots produced, so we have produced all of the roots for L.

We now determine the simple roots. We claim that

$$\{\tau_{12}, \dots, \tau_{n-1,n}, \rho_n\}$$

is a set of simple roots. To verify this, first note that $\tau_{ij} = \sum_{k=i}^{j-1} \tau_{k,k+1}$, so the τ_{ij} are spanned by these elements. Moreover, we have

$$\begin{aligned}
\sigma_{in} &= \tau_{i,i+1} + \cdots + \tau_{n-2,n-1} + \sigma_{n-1,n}, \\
\sigma_{ij} &= \tau_{ir} + \sigma_{jr}, \quad i < j < r.
\end{aligned}$$

From these equations, we see by induction that each σ_{ij} spanned by this set of roots. Finally, $\rho_i = \sigma_{ij} + \tau_{ij}$ for any j. Thus, this set spans the set of roots. Since the number of them is equal to $\dim_F(T)$, they do indeed form a set of simple roots. For ease of notation, we write $\alpha_i = \tau_{i,i+1}$ for $i < n$ and $\alpha_n = \rho_n$, and $s_i = s_{\alpha_i}$.

We next find the s_i. For $i < n$, we have

$$\begin{aligned}
s_i &= \frac{\left[\begin{pmatrix} e_{i,i+1} & 0 \\ 0 & -e_{i+1,i} \end{pmatrix}, \begin{pmatrix} e_{i+1,i} & 0 \\ 0 & -e_{i,i+1} \end{pmatrix}\right]}{\operatorname{tr}\left(\begin{pmatrix} e_{i,i+1} & 0 \\ 0 & -e_{i+1,i} \end{pmatrix} \begin{pmatrix} e_{i+1,i} & 0 \\ 0 & -e_{i,i+1} \end{pmatrix}\right)} \\
&= \frac{1}{2} \begin{pmatrix} e_{ii} - e_{i+1,i+1} & 0 \\ 0 & e_{i+1,i+1} - e_{ii} \end{pmatrix}.
\end{aligned}$$

For $i = n$, we have

$$\begin{aligned}
s_n &= \frac{\left[\begin{pmatrix} 0 & e_{ii} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e_{ii} & 0 \end{pmatrix}\right]}{\operatorname{tr}\begin{pmatrix} 0 & e_{ii} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ e_{ii} & 0 \end{pmatrix}} \\
&= \begin{pmatrix} e_{nn} & 0 \\ 0 & -e_{nn} \end{pmatrix}.
\end{aligned}$$

We now determine the numbers $\langle \alpha_i, \alpha_j \rangle$. We have $\operatorname{tr}(s_i^2) = 1$ if $i < n$ and $\operatorname{tr}(s_n^2) = 2$. If $j < i < n$, then

$$\operatorname{tr}(s_i s_j) = \frac{1}{4} \operatorname{tr} \begin{pmatrix} e_{i+1,i+1} - e_{ii} & 0 \\ 0 & e_{ii} - e_{i+1,i+1} \end{pmatrix} \begin{pmatrix} e_{j+1,j+1} - e_{jj} & 0 \\ 0 & e_{jj} - e_{j+1,j+1} \end{pmatrix},$$

so $\operatorname{tr}(s_i s_j) = 0$ if $|i - j| \neq 1$. If $|i - j| = 1$, then $j + 1 = i$, so $\operatorname{tr}(s_i s_{i+1}) = -1/2$. Thus,

$\langle \alpha_i, \alpha_j \rangle = 0$ if $|j - i| \neq 1$, and $\langle \alpha_i, \alpha_{i+1} \rangle = -1$. Next, we consider $\langle \alpha_i, \alpha_n \rangle$. We have

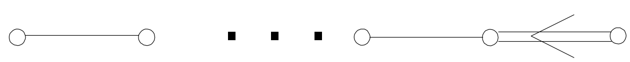
$$\operatorname{tr}(s_i s_n) = \frac{1}{2} \operatorname{tr} \begin{pmatrix} e_{ii}e{-}_{i+1,i+1} & 0 \\ 0 & e_{i+1,i+1} - e_{ii} \end{pmatrix} \begin{pmatrix} e_{nn} & 0 \\ 0 & -e_{nn} \end{pmatrix}.$$

So, this is 0 if $i < n-1$, and for $i = n-1$ we get $\operatorname{tr}(s_{n-1}s_n) = -1$. Thus, $\langle \alpha_{n-1}, \alpha_n \rangle = -1$ and $\langle \alpha_n, \alpha_{n-1} \rangle = -2$. From this information, we see that α_i and α_{i+1} are connected with one edge for $i < n-1$, and α_{n-1} and α_n are connected with two edges. Moreover, $\|\alpha_i\| = \sqrt{2}$ if $i < n$, and $\|\alpha_n\| = 2$ for all $i < n$, since we may view $\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$ and $\alpha_n = (0, \dots, 0, 2)$ in $\mathbb{R}^n$. So, the double edge connecting α_{n-1} and α_n points toward α_{n-1}. Thus, the Cartan matrix for L is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & \ddots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -2 & 2 \end{pmatrix}$$

and the Dynkin diagram for L is

C_n



5 D_n

We describe the Lie algebra of type D_n in the following way. Let $M = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, and define $L = o(2n, F) = \{x \in \operatorname{gl}(2n, F) : x^t M + Mx = 0\}$. Thus,

$$L = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} : B^t = -B, C^t = -C \right\}.$$

We first describe a maximal toral subalgebra of L. Let $d(2n, F)$ be the set of diagonal matrices, and let

$$T = \{\mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n) : a_i \in F\}.$$

We first claim that $T = d(2n, F) \cap L$. The inclusion $\subseteq$ is clear. For the reverse, let A and B be $n \times n$ diagonal matrices. If

$$\begin{aligned} 0 &= \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^t \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \\ &= \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & A+B \\ A+B & 0 \end{pmatrix}, \end{aligned}$$

we see that $B = -A$, showing that the inclusion $\supseteq$ is also true.

Next we show that T is a maximal toral subalgebra of L. It is clear that T is a toral subalgebra, since every element of T is a diagonal matrix, hence semisimple. If T' is a toral subalgebra of L that contains T, then take $a \in T'$. Since T' is Abelian, $[a, b] = 0$ for all $b \in T$. Thus, $ab = ba$ for all $b \in T$. A computation similar to the A_n case shows that $a \in T$, as desired. Thus, T is a maximal toral subalgebra.

We now determine the root system. Note that $\dim_F(L) = n(2n-1)$ and $\dim_F(T) = n$, so there are $2n^2 - 2n$ roots. For $i < j$, we define σ_{ij} and τ_{ij} by

$$\begin{aligned} \sigma_{ij}\left(\mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)\right) &= a_i + a_j, \\ \tau_{ij}\left(\mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)\right) &= a_i - a_j. \end{aligned}$$

Then it is clear that σ_{ij} and $\tau_{ij} \in T^* = \hom_F(T, F)$ for all $i < j$. A short computation shows that, for all $t \in T$,

$$\begin{aligned} \left[t, \begin{pmatrix} 0 & e_{ij} - e_{ji} \\ 0 & 0 \end{pmatrix}\right] &= \sigma_{ij}(t) \begin{pmatrix} 0 & e_{ij} - e_{ji} \\ 0 & 0 \end{pmatrix}, \\ \left[t, \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}\right] &= \tau_{ij}(t) \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}, \\ \left[t, \begin{pmatrix} 0 & 0 \\ e_{ij} - e_{ji} & 0 \end{pmatrix}\right] &= -\sigma_{ij}(t) \begin{pmatrix} 0 & 0 \\ e_{ij} - e_{ji} & 0 \end{pmatrix}, \\ \left[t, \begin{pmatrix} e_{ji} & 0 \\ 0 & -e_{ij} \end{pmatrix}\right] &= -\tau_{ij}(t) \begin{pmatrix} e_{ji} & 0 \\ 0 & -e_{ij} \end{pmatrix}. \end{aligned}$$

Therefore, the elements $\pm\sigma_{ij}, \pm\tau_{ij}$ are roots, and we have found corresponding eigenvectors. Moreover, these four classes of roots total $4\binom{n}{2} = 2n(n-1)$ elements, the number of total roots. So, $\Phi = \{\pm\sigma_{ij}, \pm\tau_{ij} : i < j\}$.

We now determine the simple roots. We claim that they are

$$\{\tau_{12}, \tau_{23}, \dots, \tau_{n-1,n}, \sigma_{n-1,n}\}.$$

To verify this, we see that $\tau_{ij} = \sum_{k=i}^{j-1} \tau_{k,k+1}$, so the τ_{ij} are spanned by these elements. Moreover, we have

$$\begin{aligned} \sigma_{in} &= \tau_{i,i+1} + \dots + \tau_{n-2,n-1} + \sigma_{n-1,n}, \\ \sigma_{ij} &= \tau_{ir} + \sigma_{jr}, \quad i < j < r. \end{aligned}$$

From these equations, we see by induction that our set does indeed span the set of roots. Since the number of them is equal to $\dim_F(T)$, they do indeed form a set of simple roots. For ease of notation, we write $\alpha_i = \tau_{i,i+1}$ for $i < n$ and $\alpha_n = \sigma_{n-1,n}$. We also write s_i for s_{α_i}.

We now determine the s_{α_i}. Since the Eigenspace L_{α_i} is spanned by $\begin{pmatrix} e_{i,i+1} & 0 \\ 0 & -e_{i+1,i} \end{pmatrix}$ for $i < n$, and $L_{-\alpha_i}$ is spanned by $\begin{pmatrix} e_{i+1,i} & 0 \\ 0 & -e_{i,i+1} \end{pmatrix}$, we have

$$\begin{aligned} s_i &= \frac{\left[\begin{pmatrix} e_{i,i+1} & 0 \\ 0 & -e_{i+1,i} \end{pmatrix}, \begin{pmatrix} e_{i+1,i} & 0 \\ 0 & -e_{i,i+1} \end{pmatrix} \right]}{\operatorname{tr} \begin{pmatrix} e_{i,i+1} & 0 \\ 0 & -e_{i+1,i} \end{pmatrix} \begin{pmatrix} e_{i+1,i} & 0 \\ 0 & -e_{i,i+1} \end{pmatrix}} \\ &= \frac{1}{2} \begin{pmatrix} e_{ii} - e_{i+1,i+1} & 0 \\ 0 & e_{i+1,i+1} - e_{ii} \end{pmatrix}. \end{aligned}$$

For $i = n$, the Eigenspace L_{α_n} is spanned by $\begin{pmatrix} 0 & e_{n-1,n} - e_{n,n-1} \\ 0 & 0 \end{pmatrix}$ and the space $L_{-\alpha_n}$ is spanned by $\begin{pmatrix} 0 & 0 \\ e_{n-1,n} - e_{n,n-1} & 0 \end{pmatrix}$, so

$$s_n = \frac{\left[\begin{pmatrix} 0 & e_{n-1,n} - e_{n,n-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ e_{n-1,n} - e_{n,n-1} & 0 \end{pmatrix} \right]}{\operatorname{tr} \begin{pmatrix} 0 & e_{n-1,n} - e_{n,n-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ e_{n-1,n} - e_{n,n-1} & 0 \end{pmatrix}}$$
$$= \frac{-1}{2} \begin{pmatrix} e_{nn} + e_{n-1,n-1} & 0 \\ 0 & -e_{nn} - e_{n-1,n-1} \end{pmatrix}.$$

We are now in a position to determine the integers $\langle \alpha_i, \alpha_j \rangle = 2 \operatorname{tr}(s_i, s_j) / \operatorname{tr}(s_j, s_j)$. Moreover, if $i, j < n$, then

$$\operatorname{tr}(s_i, s_j) = \frac{1}{4} \operatorname{tr} \begin{pmatrix} e_{ii} - e_{i+1,i+1} & 0 \\ 0 & e_{i+1,i+1} - e_{ii} \end{pmatrix} \begin{pmatrix} e_{jj} - e_{j+1,j+1} & 0 \\ 0 & e_{j+1,j+1} - e_{jj} \end{pmatrix}.$$

If $|i - j| \neq 1$, then the product inside the trace is 0, so $\operatorname{tr}(s_i, s_j) = 0$. If $i + 1 = j$, then the product is $\begin{pmatrix} -e_{jj} & 0 \\ 0 & -e_{jj} \end{pmatrix}$, and so $\operatorname{tr}(s_i, s_j) = -1/2$. Also,

$$\operatorname{tr}(s_j, s_j) = \frac{1}{4} \operatorname{tr} \begin{pmatrix} e_{jj} - e_{j+1,j+1} & 0 \\ 0 & e_{j+1,j+1} - e_{jj} \end{pmatrix} \begin{pmatrix} e_{jj} e-_{j+1,j+1} & 0 \\ 0 & e_{j+1,j+1} - e_{jj} \end{pmatrix}$$
$$= \frac{1}{4} \operatorname{tr} \begin{pmatrix} e_{jj} + e_{j+1,j+1} & 0 \\ 0 & e_{j+1,j+1} + e_{jj} \end{pmatrix} = 1.$$

This tells us for $i, j < n$, that $\langle \alpha_i, \alpha_j \rangle = -1$ if $|i - j| = 1$ and $\langle \alpha_i, \alpha_j \rangle = 0$ otherwise. We now consider the case of $\langle \alpha_i, \alpha_n \rangle$. A short calculation shows us that $\operatorname{tr}(s_n, s_n) = 1$. Also,

$$\operatorname{tr}(s_i, s_n) = \frac{-1}{4} \operatorname{tr} \begin{pmatrix} e_{ii} - e_{i+1,i+1} & 0 \\ 0 & e_{i+1,i+1} - e_{ii} \end{pmatrix} \begin{pmatrix} e_{nn} + e_{n-1,n-1} & 0 \\ 0 & -e_{nn} - e_{n-1,n-1} \end{pmatrix}.$$

So, $\operatorname{tr}(s_i, s_n) = 0$ if $i < n - 2$. For $i = n - 2$, we have

$$\operatorname{tr}(s_{n-2}, s_n) = -\frac{1}{4} \operatorname{tr} \begin{pmatrix} e_{n-1,n-1} & 0 \\ 0 & e_{n-1,n-1} \end{pmatrix} = -\frac{1}{2}.$$

So, $\langle \alpha_{n-2}, \alpha_n \rangle = -1$. For $i = n-1$, we have

$$\begin{aligned}
\operatorname{tr}(s_{n-1}, s_n) &= \frac{-1}{4} \operatorname{tr} \begin{pmatrix} e_{n-1,n-1} - e_{n,n} & 0 \\ 0 & e_{nn} - e_{n-1,n-1} \end{pmatrix} \begin{pmatrix} e_{nn} + e_{n-1,n-1} & 0 \\ 0 & -e_{nn} - e_{n-1,n-1} \end{pmatrix} \\
&= \frac{-1}{4} \operatorname{tr} \begin{pmatrix} e_{n-1,n-1} - e_{nn} & 0 \\ 0 & e_{n-1,n-1} - e_{nn} \end{pmatrix} = 0,
\end{aligned}$$

so $\langle \alpha_{n-1}, \alpha_n \rangle = 0$. This tells us that the roots α_i and α_{i+1} are connected with $(-1)(-1) = 1$ edge, that α_i and α_j are not connected if $i, j < n$ and $|i-j| > 1$, and that α_{n-2} and α_n are connected with one edge. Moreover, each root has the same length, since we may view $\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$ and $\alpha_n = (0, \dots, 0, 1, 1)$ in $\mathbb{R}^n$. Therefore, the Cartan matrix for L is

$$\begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & \ddots & \ddots & \vdots \\
0 & -1 & 2 & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & -1 & -1 \\
\vdots & \ddots & 0 & -1 & 2 & 0 \\
0 & \cdots & 0 & -1 & 0 & 2
\end{pmatrix}$$

and the Dynkin diagram for L is