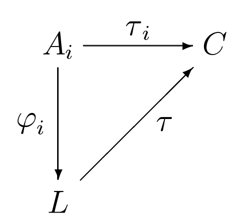
Direct Limits

In this note we define direct limits and prove their basic properties. This notion is important in various places in algebra. In particular, in algebraic geometry and complex analysis, the fundamental notion of a stalk of a sheaf uses direct limits.

We recall the definition of a direct limit. Let I be a set with a partial order $\leq$ satisfying the property that for any $i, j \in I$, there is a $k \in I$ with $i \leq k$ and $j \leq k$. Such a set is called a *directed set*. Suppose we have the following data: an Abelian group A_i for each i, and for each pair $i \leq j$ a map $\varphi_{ij} : A_i \to A_j$ with $\varphi_{ii} = \mathrm{id}_{A_i}$ for each i, and such that whenever $i \leq j \leq k$, we have $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$. Then $\{A_i, \varphi_{ij}\}$ is called a *directed system* of groups. The *direct limit* $\varinjlim A_i$ is the unique up to isomorphism group L satisfying the following universal mapping property: there are maps $\varphi_i : A_i \to L$ such that $\varphi_i = \varphi_j \circ \varphi_{ij}$ for every pair $i \leq j$, and if there is an Abelian group C together with maps $\tau_i : A_i \to C$ such that $\tau_i = \tau_j \circ \varphi_{ij}$ for each $i \leq j$, then there is a unique group homomorphism $\tau : L \to C$ with $\sigma = \tau \circ \varphi_i$.



A routine exercise involving universal mapping properties shows that the direct limit of a group, if it exists, is unique up to isomorphism. Direct limits of Abelian groups do exist; here is a construction: Let M be the direct sum of the A_i, and let N be the subgroup generated by all elements of the form $a - \varphi_{ij}(a)$ for all $i \leq j$ and all $a \in A_i$. Then M/N, together with φ_i the compositions of the natural maps $A_i \to M \to M/N$, satisfy the mapping property for the direct limit.

In order to prove the basic properties of direct limits in the lemma below, we give an alternative description of them. This description is often how you may see direct limits used in complex analysis and algebraic geometry. Let $\{A_i, \varphi_{ij}\}$ be a directed system of groups. Consider pairs (A_i, a_i) with $a_i \in A_i$. Define a relation $\sim$ on such pairs by $(A_i, a_i) \sim (A_j, a_j)$ if there is a $k \geq i, j$ with $\varphi_{ik}(a_i) = \varphi_{jk}(a_j)$. A short calculation shows that $\sim$ is an equivalence relation. We will write $[A_i, a_i]$ for the equivalence class of a pair (A_i, a_i). If G is the set of equivalence classes, then we can define an operation on G by

$$[A_i, a_i] + [A_j, a_j] = [A_k, \varphi_{ik}(a_i) + \varphi_{jk}(a_j)],$$

where k is any index with $k \geq i, j$. Another short calculation shows that this operation is well defined, and that G is an Abelian group under this operation. The map $\sigma_i : A_i \to G$ given by $\sigma_i(a) = [A_i, a]$ is a group homomorphism. Furthermore, $\sigma_i = \sigma_j \circ \varphi_{ij}$ for any pair $i \leq j$ since $\sigma_j(\varphi_{ij}(a)) = [A_j, \varphi_{ij}(a)] = [A_i, a]$ by the definition of the equivalence relation. We will prove that $G \cong \varinjlim A_i$ by proving that G has the same mapping property as does the direct limit. To do this, suppose that B is an Abelian group and that for each i there is a homomorphism $\tau_i : A_i \to B$ with $\tau_i = \tau_j \circ \varphi_{ij}$ for each $i \leq j$. Define $\tau : G \to B$ by $\tau([A_i, a]) = \tau_i(a)$. This is well defined since if $[A_i, a_i] = [A_j, a_j]$, then there is a k with $i, j \leq k$ and $\varphi_{ik}(a_i) = \varphi_{jk}(a_j)$. Therefore,

$$\tau_i(a_i) = \tau_k(\varphi_{ik}(a_i)) = \tau_k(\varphi_{jk}(a_j)) = \tau_j(a_j).$$

The map τ is clearly a group homomorphism. Furthermore, $\tau_i = \tau \circ \sigma_i$ is clear from the definition of τ. Finally, if $\tau' : G \to B$ satisfies $\tau_i = \tau' \circ \sigma_i$ for each i, then $\tau'([A_i, a_i]) = \tau'(\sigma_i(a_i)) = \tau_i(a_i) = \tau([A_i, a_i])$. Thus, $\tau' = \tau$. This shows that G satisfies the mapping property of direct limits (along with the maps σ_i). A standard application of mapping properties will then show $G \cong \varinjlim A_i$, as we now give. The maps σ_i induce a unique homomorphism $\sigma : \varinjlim A_i \to G$ with $\sigma_i = \sigma \circ \varphi_i$ for each i. Similarly, the mapping property applied to G yields a map $\tau : G \to \varinjlim A_i$ satisfying $\varphi_i = \tau \circ \sigma_i$. Therefore, $\sigma \circ \tau : G \to G$ satisfies $\sigma_i = (\sigma \circ \tau) \circ \varphi_i$. However, $\mathrm{id}_G : G \to G$ also satisfies $\sigma_i = \mathrm{id}_G \circ \varphi_i$. By the uniqueness part of the mapping property, we conclude that $\sigma \circ \tau = \mathrm{id}_G$. Similarly, $\tau \circ \sigma = \mathrm{id}_{\varinjlim A_i}$, so σ (and τ) is an isomorphism.

We now prove the two most basic computational properties of direct limits.

Lemma 1. *Let $\varinjlim A_i$ be the direct limit of a directed system of groups. (1) Every element of $\varinjlim A_i$ can be written in the form $\varphi_i(a)$ for some $a \in A_i$. (2) If $a \in A_i$ satisfies $\varphi_i(a) = 0$, then there is a $j \geq i$ with $\varphi_{i,j}(a) = 0$.*

Proof. These properties are easy to see from the definition of the group G defined above, which is isomorphic to $\varinjlim A_i$. Every element of G is of the form $[A_i, a_i] = \sigma_i(a_i)$. Also, if $[A_i, a_i] = 0$, then $(A_i, a_i) \sim (A_i, 0)$, so by definition of the relation, there is a $j \geq i$ with $\varphi_{ij}(a_i) = \varphi_{ij}(0) = 0$. Finally, following these properties for G by the isomorphism $\tau : G \to \varinjlim A_i$ yields the corresponding properties for $\varinjlim A_i$. □

Example 2. Let A be an Abelian group. Let $\{A_i\}_{i \in I}$ be the set of finitely generated subgroups of A. Then, by ordering I by $i \leq j$ if $A_i \subseteq A_j$, the set I is a directed set, since for any pair i, j, the group $A_i + A_j$ is both finitely generated and contains A_i and A_j. If we let $\varphi_{i,j} : A_i \to A_j$ be the inclusion map whenever $i \leq j$, we have a directed system $\{A_i, \varphi_{i,j}\}$. Thus, the direct limit $\varinjlim A_i$ exists. We claim that $A = \varinjlim A_i$. We write $H = \varinjlim A_i$ for convenience. To prove this, we have inclusion maps $\sigma_i : A_i \to A$. We also have the canonical maps $\varphi_i : A_i \to H$ for each i. Since $\sigma_i = \sigma_j \circ \varphi_{i,j}$, as both sides are the inclusion maps $A_i \to A$, the universal mapping property gives a unique homomorphism $\sigma : H \to A$ with $\sigma \circ \varphi_i = \sigma_i$ for each i. The map σ is surjective, since if $g \in A$, then $g \in A_i$ for some i; the cyclic group $\langle g \rangle$ is a finitely generated subgroup of A, so it is equal to A_i for some i. Thus, $g = \sigma_i(g) = \sigma(\varphi_i(g))$, proving that σ is surjective. Finally, if $h \in \ker(\sigma)$, then let $h = \varphi_i(g)$ for some $g \in A_i$. Then $0 = \sigma(h) = \sigma(\varphi_i(g)) = \sigma_i(g) = g$ since σ_i is the inclusion map. Thus, $g = 0$, so $h = \varphi_i(g) = 0$. We have thus proven that σ is bijective, so $A \cong H$.

Example 3. Let I be a directed set that has a maximum element k. That is, $i \leq k$ for every $i \in I$. We claim that $\varinjlim A_i = A_k$ for any directed system. Write $A = \varinjlim A_i$. To prove this claim, for each i we have the canonical maps $\varphi_{i,k} : A_i \to A_k$, and since $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ for any j with $i \leq j$, the universal mapping property gives a uniquely determined map $\sigma : A \to A_k$ with $\sigma \circ \varphi_i = \varphi_{i,k}$ for every i. In particular, for $i = k$, we have $\sigma \circ \varphi_k = \varphi_{k,k} = \mathrm{id}_{A_k}$. Thus, σ is surjective. For injectivity, take $g \in A$ with $\sigma(g) = 0$. Write $g = \varphi_i(g_i)$ for some $g_i \in A_i$. Then $0 = \sigma(g) = \sigma(\varphi_i(g_i)) = \varphi_{i,k}(g_i)$. By definition of directed systems, we then have $0 = \varphi_k(\varphi_{i,k}(g_i)) = \varphi_i(g_i) = g$. Therefore, σ is also injective, so $A \cong A_k$.

Example 4. Let $\mathcal{F}$ be a sheaf on a topological space X. Then the stalk $\mathcal{F}_x$ is defined as $\mathcal{F}_x = \varinjlim \mathcal{F}(U)$, where the direct limit is over all open neighborhoods of x. This set is a directed set by ordering it with reverse inclusion: if U and V are neighborhoods of x, then $U \cap V$ is a neighborhood of x contained in both U and V. If $V \subseteq U$, then the canonical map $\mathcal{F}(U) \to \mathcal{F}(V)$ is the restriction map $\mathrm{res}_{U,V}$ that comes along with the sheaf $\mathcal{F}$. Since $\mathrm{res}_{V,W} \circ \mathrm{res}_{U,V} = \mathrm{res}_{U,W}$ whenever $W \subseteq V \subseteq U$, and $\mathrm{res}_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$, these maps satisfy the axioms to have a directed system of groups. Therefore, the direct limit $\varinjlim \mathcal{F}(U)$ does exist. We write f_x for the image of $f \in \mathcal{F}(U)$ in $\mathcal{F}_x$ for $x \in U$. The properties of the lemma translate to the following two: (i) if $\alpha \in \mathcal{F}_x$, then $\alpha = f_x$ for some open neighborhood U of x and some $f \in \mathcal{F}(U)$, and (ii) if $f \in \mathcal{F}(U)$ with $f_x = 0$, then there is some $V \subseteq U$ with $\mathrm{res}_{U,V}(f) := f|_V = 0$.

Example 5. Here is another example that arises in the theory of sheaves. Let $f : X \to Y$ be a continuous map of topological spaces. If $\mathcal{G}$ is a sheaf on Y, then we can define the inverse image presheaf $\mathcal{F}$ by $\mathcal{F}(U) = \varinjlim \mathcal{G}(V)$, as the limit runs over all open sets V of Y with $f(U) \subseteq V$. This set of open sets is a directed set by ordering it with reverse inclusion, since if V and W are open and containing $f(U)$, then so is $V \cap W$. Example 3 shows that if $U = f^{-1}(V)$, then this set of open sets has $f^{-1}(V)$ as a maximum element, so$\mathcal{F}(f^{-1}(V)) = \mathcal{G}(V)$.

Example 6. To continue the previous example further, suppose that X is a topological space and $x \in X$. The unique map $f : X \to \{x\}$ is continuous. An Abelian group A gives rise to a sheaf on $\{x\}$ which we will also denote by A, since the only nonempty open set of $\{x\}$ is $\{x\}$ itself. We then have the inverse image presheaf $\mathcal{F}$ on X, defined by $\mathcal{F}(U) = \varinjlim A(V)$, as V runs over open sets of $\{x\}$ containing $f(U)$. However, the only choice for V is $\{x\}$. Therefore, $\mathcal{F}(U) = A(\{x\}) = A$. Therefore, $\mathcal{F}$ is the "constant" presheaf that sends every open set to the same Abelian group.

Example 7. To give another version of the inverse image presheaf example, let X be a topological space and let $x \in X$. If $j : \{x\} \to X$ is the inclusion map, then j is continuous.

Let $\mathcal{G}$ be a sheaf on X. Then the inverse image presheaf $\mathcal{F}$ is a presheaf on $\{x\}$. Since $\{x\}$ is a one point space, this sheaf is nothing more than the Abelian group $\mathcal{F}(\{x\}) = \varinjlim \mathcal{G}(V)$, as the limit ranges over open sets V of X containing $j(x) = x$. Thus, this group is just the stalk $\mathcal{G}_x$.

Example 8. For a final example, we look at pushouts. Given the diagram of Abelian groups

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \\ C & & \end{array}$$

the pushout is a group P with maps α and β such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow[\beta]{} & P \end{array}$$

and such that for any Abelian group D and maps $\alpha' : C \to D$ and $\beta' : B \to D$, there is a unique map $\theta : L \to D$ with $\alpha' = \theta \circ \alpha$ and $\beta' = \theta \circ \beta$. We see that L exists and is just a direct limit. Let I be the directed set $\{1, 2, 3\}$ ordered by divisibility. That is, $1 \leq 2$ and $1 \leq 3$, but $2 \nleqq 3$. Set $A = A_1$, $B = A_2$, and $C = A_3$. Then with the maps f and g, we have a directed set. Thus, set $L = \varinjlim A_i$. Then the universal mapping property for direct limits is exactly that described above, where we set the map $A \to L$ to be $\beta \circ f = \alpha \circ g$.

In the remainder of this note we look at direct limits from a categorical point of view. Let I be a directed set. From I we have a category $\mathcal{I}$ whose objects are the elements of I, and whose morphisms are arrows $i \to j$ for each pair $i \leq j$ in I. In other words, $\hom_{\mathcal{I}}(i, j)$ contains one map $i \to j$ if $i \leq j$, and is empty otherwise. This mimics the category $\mathrm{Top}(X)$ of open sets of a topological space X. A directed system of Abelian groups is then nothing but a functor from $\mathcal{I}$ to Ab. Thus, the functor category $Ab^{\mathcal{I}}$ is then the category of directed systems of Abelian groups on I. We claim that the direct limit gives a functor from $Ab^{\mathcal{I}}$ to Ab. We already know how it acts on objects, it sends a directed system $\{A_i\}$ to the direct limit $\varinjlim A_i$. To see how it acts on morphisms, let $f : \{A_i, \varphi_{ij}\} \to \{B_i, \phi_{ij}\}$ be a morphism of directed systems. Recall from the study of functor categories that f is a natural transformation of functors. In other words, for each i there is a group homomorphism $f_i : A_i \to B_i$, and if $i \leq j$, then the following diagram commutes.

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ {\scriptstyle \varphi_{ij}}\downarrow & & \downarrow{\scriptstyle \phi_{ij}} \\ A_j & \xrightarrow[f_j]{} & B_j \end{array}$$

If $\varphi_i : A_i \to \varinjlim A_i$ and $\phi_i : B_i \to \varinjlim B_i$ are the canonical maps, then $\phi_i \circ f_i : A_i \to \varinjlim B_i$ is a homomorphism for each i that has the correct properties to yield a unique group homomorphism $f' : \varinjlim A_i \to \varinjlim B_i$ with $\varphi_i \circ f' = f_i$ for all i. The association $f \mapsto f'$ is then how the direct limit functor operates on maps. It is an easy exercise to show that composition of morphisms is preserved, and that $\mathrm{id}'_{\{A_i\}} = \mathrm{id}_{\varinjlim A_i}$. In other words, direct limit does give a functor $Ab^{\mathcal{I}} \to Ab$. We have a simple functor C in the opposite direction. For B an Abelian group, let $C(B) = \{B\}$. This is the direct system with $B_i = B$ for all i, and the map $B_i \to B_j$ for $i \le j$ is the identity map. It is easy to see that C is an exact functor. Furthermore, the universal mapping property for direct limits yields immediately that

$$\hom(\varinjlim A_i, B) \cong \hom_{Ab^{\mathcal{I}}}(\{A_i\}, C(B)).$$

In other words, $\varinjlim$ is a left adjoint to C. Therefore, by Proposition 2.6.1 of [1], the direct limit functor $\varinjlim$ is right exact. In fact, a straightforward calculation shows that $\varinjlim$ is actually an exact functor.

One can dualize definitions to define inverse systems of Abelian groups: such a system $\{A_i\}$ has, for each $i \le j$, a homomorphism $\varphi_{ij} : A_j \to A_i$. One can define the *inverse limit* $\varprojlim A_i$, together with homomorphisms $\varprojlim A_i \to A_i$, via the following universal mapping property: if B is an Abelian group and if $f_i : B \to A_i$ are homomorphisms with $f_j = \varphi_{ij} \circ f_i$ for each $i \le j$, then there is a unique homomorphism $f : B \to \varprojlim A_i$ with $f_i = \varphi_i \circ f$ for all i. As we did above, we obtain a category of inverse systems over I, and $\varprojlim$ is a functor from this category to Ab. The "constant" functor C that sends B to $\{B\}_{i\in I}$ is a functor in the opposite direction, and we have $\hom(B, \varprojlim A_i) = \hom(C(B), \{A_i\})$ from the universal mapping property of inverse limits. This means that $\varprojlim$ is a right adjoint to C. Since C is clearly exact, $\varprojlim$ is then left exact. Unlike direct limits, however, $\varprojlim$ is not exact. Therefore, one can study the derived functors of $\varprojlim$, whereas the derived functors of the direct limit $\varinjlim$ are trivial since this functor is exact.

References

[1] Charles A. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.