Dualities in Lattice Theory

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In this note we prove several duality theorems in lattice theory. We also discuss the connection between spectral spaces and Priestley spaces, and interpret Priestley duality in terms of spectral spaces.

The organization of this note is as follows. In the first section we collect appropriate definitions and basic results common to many of the various topics. The next four sections consider Birkhoff duality between finite distributive lattices and finite posets, Stone's duality between Boolean algebras and Stone spaces, Priestley duality between bounded distributive lattices and Priestley spaces, and Heyting between Heyting algebras and Heyting spaces. Finally, in Section 6 we consider the relation between the Stone topology and the spectral topology on the set of prime filters of a distributive lattice. The spectral topology more closely resembles the topology on the spectrum of a commutative ring, while the Stone topology yields a compact Hausdorff space. We will then interpret Priestley duality in terms of spectral spaces. As we will see, Priestley duality generalizes both Birkhoff and Stone duality, and Heyting duality considers special cases of bounded distributive lattices and Priestley spaces.

1 Preliminaries

A tuple $(L, \vee, \wedge)$ is a *lattice* if L is a nonempty set with binary operations $\vee$ and $\wedge$ containing elements $0, 1$ such that the following identities hold:

$$\begin{array}{ll} a \vee b = b \vee a, & a \wedge b = b \wedge a, \\ a \vee (b \vee c) = (a \vee b) \vee c), & a \wedge (b \wedge c) = (a \wedge b) \wedge c, \\ a \vee a = a, & a \wedge a = a, \\ a \vee (a \wedge b) = a, & a \wedge (a \vee b) = a. \end{array}$$

If, in addition, L contains elements $0, 1$ satisfying

$$a \vee 0 = a, \quad a \wedge 1 = a,$$

then L is said to be a bounded lattice. We define a partial order $\leq$ on a lattice L by $a \leq b$ if $a \vee b = b$, or, equivalently, $a \wedge b = a$. An alternative way to define a lattice is as a poset $(L, \leq)$

in which for any pair $a, b \in L$, the least upper bound $\mathrm{lub}(a, b)$ and the greatest lower bound $\mathrm{glb}(a, b)$ exist. We then define $a \vee b = \mathrm{lub}(a, b)$ and $a \wedge b = \mathrm{glb}(a, b)$. A straightforward but somewhat long exercise proves the equivalence of these two notions. If a lattice L satisfies the identities

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

then we call L a *distributive lattice*. In fact, these identities are equivalent in a lattice.

Example 1.1. We recall that a *Boolean algebra* is a bounded distributive lattice B with a unary operation $-$ satisfying

$$a \vee -a = 1,$$
$$a \wedge -a = 0$$

for all $a \in B$. For specific examples, let X be a set, and let B be the power set of X. By setting $\vee$ to be union and $\wedge$ to be intersection, then B is a Boolean algebra with $1 = X$ and $0 = \varnothing$. Moreover, $-A = X \backslash A$ for any $A \in B$.

Example 1.2. Let $(C, \leq)$ be a totally ordered poset. Then C is lattice, since $\mathrm{lub}(a, b) = \max\{a, b\}$ and $\mathrm{glb}(a, b) = \min\{a, b\}$ for any $a, b \in C$. If C has a top and a bottom element, then C is a bounded lattice.

Throughout this note L will be a bounded lattice. A *filter* of a lattice L is a nonempty subset F such that (i) if $a \in F$ and $a \leq b$, then $b \in F$, and (ii) if $a, b \in F$, then $a \wedge b \in F$. Sets satisfying (i) are called upsets. The dual notation is that of an ideal; an *ideal* of a lattice L is a nonempty subset I such that (i) if $a \in F$ and $b \leq a$, then $a \in I$, and (ii) if $a, b \in I$, then $a \vee b \in I$. Sets satisfying Condition (i) of an ideal are called downsets. We may phrase duality in terms of ideals or filters; we choose to work with filters. For example, if $a \in L$, then we define $\uparrow a = \{b \in L : a \leq b\}$ and $\downarrow a = \{b \in L : b \leq a\}$, the principal filter and ideal generated by a, respectively. It is easy to see that $\uparrow a$ is a filter of F and $\downarrow a$ is an ideal of L. A *prime filter* P is a proper filter on L such that if $a \vee b \in P$, then $a \in P$ or $b \in P$. A proper filter maximal with respect to inclusion is called a maximal filter or, more commonly, an *ultrafilter*.

Definition 1.3. *The set of all prime filters on a bounded lattice L is denoted $\mathcal{PF}(L)$. For $a \in L$, we set $\varphi(a) = \{P \in \mathcal{PF}(L) : a \in P\}$.*

There are two natural topologies on $\mathcal{PF}(L)$ we will study. One, the spectral topology, we will investigate later in Section 6. This topology is a direct generalization of the Zariski topology on the spectrum of a commutative ring. However, we first consider the topology defined by Stone, to which we refer as the *Stone topology* on $\mathcal{PF}(L)$. The relation between these topologies will be analyzed in Section 6. We define the Stone topology τ on $\mathcal{PF}(L)$ by letting $\mathcal{B} = \{\varphi(a) : a \in L\} \cup \{\varphi(a)^c : a \in L\}$ be a subbasis of open sets for τ. To analyze this topology, we first note that $\varphi(0) = \varnothing$ and $\varphi(1) = \mathcal{PF}(L)$. Next, we show that $\varphi(a \vee b) = \varphi(a) \cup \varphi(b)$ and $\varphi(a \wedge b) = \varphi(a) \cap \varphi(b)$. To prove these, the inclusions $\varphi(a) \cup \varphi(b) \subseteq \varphi(a \vee b)$ and $\varphi(a \wedge b) \subseteq \varphi(a) \cap \varphi(b)$ are clear since filters are upsets. The inclusion $\varphi(a) \cap \varphi(b) \subseteq \varphi(a \wedge b)$ is true since filters are closed under finite meets. Finally, if $P \in \varphi(a \vee b)$, then $a \vee b \in P$, so either $a \in P$ or $b \in P$, since P is prime. Thus, $P \in \varphi(a) \cup \varphi(b)$. These relations imply that the set $\{\varphi(a) \cap \varphi(b)^c : a, b \in L\}$ is a basis for the topology on $\mathcal{PF}(L)$. The sets in $\mathcal{B}$ are, by construction, clopen subsets of $\mathcal{PF}(L)$.

Lemma 1.4. *Let F be a filter and I an ideal of a bounded distributive lattice L. If $F \cap I = \varnothing$, then there is a prime filter P of L such that $F \subseteq P$ and $P \cap I = \varnothing$.*

Proof. Let $\mathcal{F}$ be the set of all filters of L containing F and disjoint from I. Then $\mathcal{F}$ is nonempty since it contains F. Let $\mathcal{C} = \{P_\alpha : \alpha \in I\}$ be a chain in $\mathcal{F}$. Then $\bigcup_\alpha P_\alpha$ is easily seen to be a filter, and $(\bigcup_\alpha P_\alpha) \cap I = \varnothing$. Therefore, by Zorn's lemma, $\mathcal{F}$ contains a maximal element P. We prove that P is prime. Suppose that $a \vee b \in P$. Consider the filters F_1 and F_2 generated by $P \cup \{a\}$ and $P \cup \{b\}$, respectively. Suppose that $a, b \notin P$. Then P is properly contained in both F_1 and F_2; therefore, F_1 and F_2 are not elements of $\mathcal{F}$. Thus, $F_i \cap I$ is nonempty for each i; let $x_i \in F_i \cap I$. By construction of F_i, there is $p_i \in P$ with $p_1 \wedge a \leq x_1$ and $p_2 \wedge b \leq x_2$. Therefore,

$$x_1 \vee x_2 \geq (p_1 \wedge a) \vee (p_2 \wedge b) = (p_1 \vee p_2) \wedge (p_1 \vee a) \wedge (p_2 \vee b) \wedge (a \vee b).$$

All four terms are in P, so their meet is in P. This implies that $x_1 \vee x_2 \in P \cap I$, a contradiction. Thus, either $a \in P$ or $b \in P$, so P is a prime filter. □

Corollary 1.5. *If L is a bounded distributive lattice, then $\varphi(a) = \varphi(b)$ if and only if $a = b$.*

Proof. If $a \neq b$, then either $a \not\leq b$ or $b \not\leq a$. Suppose that $a \not\leq b$. If $F = {\uparrow}a$ is the filter generated by a and $I = {\downarrow}b$ the ideal generated by b, then the condition $a \not\leq b$ implies that $F \cap I = \varnothing$, so Lemma 1.4 yields a prime filter P with $F \subseteq P$ and $P \cap I = \varnothing$. Thus, $P \in \varphi(a)$ and $P \notin \varphi(b)$, showing that $\varphi(a) \neq \varphi(b)$. □

2 Birkhoff Duality

In this section we prove the first of our duality results, that the category **FDL** of finite distributive lattices is dual to the category **FPos** of finite posets. Let L be a bounded distributive lattice. Then $p \in L$ is said to be a *join prime* if $p \leq a \vee b$ implies that $p \leq a$ or $p \leq b$. Let $\mathcal{JP}(L)$ be the set of join prime elements of L. We order $\mathcal{JP}(L)$ by pRq if $q \leq p$. Then $(\mathcal{JP}(L), R)$ is a poset. Note that pRq if and only if ${\uparrow}p \subseteq {\uparrow}q$.

For a poset $(X, \leq)$, we denote by $\mathcal{U}(X, \leq)$ the set of all upsets of $(X, \leq)$. With respect to union and intersection, $\mathcal{U}(X, \leq)$ is a bounded distributive lattice, with top and bottom X and $\varnothing$, respectively.

Lemma 2.1. *Let L be a finite distributive lattice. If P is a filter of L, thenP is prime if and only if $P = \uparrow p$ for some $p \in \mathcal{JP}(L)$.*

Proof. Since L is finite, $\bigwedge P \in L$; set $p = \bigwedge P$. It is then clear that $P = \uparrow p$. Suppose that P is prime. To see $p \in \mathcal{JP}(L)$, suppose that $p \le a \vee b$. Then $a \vee b \in P$, so $a \in P$ or $b \in P$. Thus, $p \le a$ or $p \le b$. Thus, $p \in \mathcal{JP}(L)$. Conversely, if $p \in \mathcal{JP}(L)$, let $a, b \in L$ with $a \vee b \in P$. Then $p \le a \vee b$. Since $p \in \mathcal{JP}(L)$, we have $p \le a$ or $p \le b$. Thus, $a \in P$ or $b \in P$. Thus, P is prime. $\square$

More generally, if L is a lattice and $P = \uparrow p$ is a principal filter, then P is a prime filter if and only if $p \in \mathcal{JP}(L)$. However, prime filters need not be principal in general.

Proposition 2.2. *Let L be a finite distributive lattice. Then $F_L : L \to \mathcal{U}(\mathcal{JP}(L), R)$, defined by $F_L(a) = \{p \in \mathcal{JP}(L) : p \le a\}$, is a lattice isomorphism*

Proof. Lemma 1.4 shows that $\uparrow a = \bigcap \varphi(a)$. If $\varphi(a) = \{P_1, \ldots, P_r\}$, and if $P_i = \uparrow p_i$ with $p_i \in \mathcal{JP}(L)$, then

$$\uparrow a = P_1 \cap \cdots \cap P_r = \uparrow p_1 \cap \cdots \cap \uparrow p_r = \uparrow p_1 \vee \cdots \vee p_r \, .$$

Consequently, $a = p_1 \vee \cdots \vee p_r$. The definition of F_L shows that $F_L(a)$ is an upset, since if $p \le a$ and pRq, then $q \le p$, so $q \le a$. We have $a = \bigvee F_L(a)$, so F_L is an injection. It is a surjection, since if U is an upset of $\mathcal{JP}(L)$, let $a = \bigvee U$. If $U = \{p_1, \ldots, p_r\}$, then $a = p_1 \vee \cdots \vee p_r$, so each $p_i \in F_L(a)$; this shows $U \subseteq f(a)$. If $q \in F_L(a)$, then $q \le a$. Since $q \in \mathcal{JP}(L)$, we get $q \le p_i$ for some i, so $p_i R q$. Therefore, $q \in U$ since U is an upset. Therefore, $U = F_L(a)$, so F_L is surjective. It is clear that F_L is a lattice homomorphism. Thus, F_L is an isomorphism. $\square$

We now consider the situation categorically. We have $\mathcal{JP} : \mathbf{FDL} \to \mathbf{FPos}$ defined on objects. For maps, if $f : L \to M$ is a lattice homomorphism, define $\mathcal{JP}(f) : \mathcal{JP}(M) \to \mathcal{JP}(L)$ by $\mathcal{JP}(f)(q) = \bigwedge f^{-1}(\uparrow q)$. We show that $\mathcal{JP}(f)(q) \in \mathcal{JP}(L)$. Before doing so, note that if $q \in \mathcal{JP}(M)$, then $\uparrow q \in \mathcal{PF}(M)$, and so $f^{-1}(\uparrow q) \in \mathcal{PF}(L)$; thus, $f^{-1}(\uparrow q) = \uparrow p$ for some $p \in \mathcal{JP}(L)$. We recover p as $p = \bigwedge \uparrow p = \bigwedge f^{-1}(\uparrow q)$; this motivates the definition of $\mathcal{JP}(f)$, and proves that $\mathcal{JP}(f)(q) \in \mathcal{JP}(L)$. It is easy to see that $\mathcal{JP}$ is indeed a functor.

To obtain a functor in the opposite direction, we have defined $\mathcal{U} : \mathbf{FPos} \to \mathbf{FDL}$ on objects. On maps, if $g : (X, \le) \to (Y, \le)$ is a map of posets, then we define $\mathcal{U}(g) : \mathcal{U}(Y, \le) \to \mathcal{U}(X, \le)$ by $\mathcal{U}(g) = g^{-1}$. In other words, $\mathcal{U}(g)(V) = g^{-1}(V)$ for any upset V of Y. The set $g^{-1}(V)$ is indeed an upset of X, for if $x \in g^{-1}(V)$ and $x \le x'$, then $g(x) \in V$, and since $g(x) \le g(x')$, we have $g(x') \in V$, so $x' \in g^{-1}(V)$. This map is clearly a lattice homomorphism.

Proposition 2.3. *Let $(X, \le)$ be a finite poset. Then $G_X : (X, \le) \to (\mathcal{JP}(\mathcal{U}(X, \le)), R)$ given by $G_X(x) = \uparrow x$, is a poset isomorphism.*

Proof. First of all, it is clear that $\uparrow x$ is a join prime element of $\mathcal{U}(X, \leq)$ for any $x \in X$, so G_X is well-defined. Moreover, if $U \in \mathcal{JP}(\mathcal{U}(X \leq))$, then as U is the (finite) union of the $\uparrow x$ for $x \in U$, we see that $U = \uparrow x$ for some $x \in U$ since U is a join prime. Thus, G_X is a bijection. Next, we see that $\uparrow x \subseteq \uparrow y$ if and only if $y \leq x$. Thus, by the definition of the order R, we have $y \leq x$ in X if and only if $\uparrow y \, R \uparrow x$ in $\mathcal{JP}(\mathcal{U}(X \leq))$. Thus, G_X and G_X^{-1} are order preserving. This proves that G_X is a poset isomorphism. □

Theorem 2.4. *The functors $\mathcal{JP}$ and $\mathcal{U}$ give a co-equivalence between the categories* **FDL** *and* **FPos**.

Proof. Define a natural transformation $F : \mathrm{id}_{\mathbf{FDL}} \to \mathcal{U} \circ \mathcal{JP}$ by $F_L : L \to \mathcal{U}(\mathcal{JP}(L))$ sends a to $F_L(a) = \{p \in \mathcal{JP}(L) : p \leq a\}$. Then F_L is an isomorphism by Proposition 2.2. We see that F is indeed a natural transformation; if $f : L \to M$ is a morphism of finite distributive lattices, then if we consider the diagram

$$\begin{tikzcd} L \arrow[r, "F_L"] \arrow[d, "f"'] & \mathcal{U}(\mathcal{JP}(L)) \arrow[d, "\mathcal{U}(\mathcal{JP}(f))"] \\ M \arrow[r, "F_M"] & \mathcal{U}(\mathcal{JP}(M)) \end{tikzcd}$$

we see that it is commutative; if $a \in L$, then $F_L(a) = \{p \in \mathcal{JP}(L) : p \leq a\}$, and then this is sent to $\{q \in \mathcal{JP}(M) : \mathcal{JP}(f)(q) \leq a\}$ in $\mathcal{U}(\mathcal{JP}(M))$. On the other hand, going around the diagram the other direction, we have $a \mapsto f(a) \mapsto \{q \in \mathcal{JP}(M) : q \leq f(a)\}$. We note that $q \leq f(a)$ iff $f(a) \in \uparrow q$ iff $a \in f^{-1}(\uparrow q)$, and since this is the principal upset $\uparrow \mathcal{JP}(f)(q)$, we see that $a \in f^{-1}(\uparrow q)$ iff $\mathcal{JP}(f)(q) \leq a$. Thus, our diagram commutes.

We next define a natural transformation $G : \mathrm{id}_{\mathbf{FPos}} : \mathcal{JP} \circ \mathcal{U}$ by $G_X : (X, \leq) \to \mathcal{JP}(\mathcal{U}(X))$ by $G_X(x) = \uparrow x$, the isomorphism defined in Proposition 2.3. Let $g : (X, \leq) \to (Y, \leq)$ be a poset morphism. To see that

$$\begin{tikzcd} (X, \leq) \arrow[r, "G_X"] \arrow[d, "g"'] & \mathcal{JP}(\mathcal{U}(X, \leq)) \arrow[d, "\mathcal{JP}(\mathcal{U}(g))"] \\ (Y, \leq) \arrow[r, "G_Y"] & \mathcal{JP}(\mathcal{U}(Y, \leq)) \end{tikzcd}$$

commutes, let $x \in X$. Then going around the top right of the diagram, we get the element

$$\begin{aligned} \mathcal{JP}(\mathcal{U}(G_X(x)) &= \bigcap \left\{ \mathcal{U}(g) \left(^{-1} \{ U \in \mathcal{U}(X, \leq) : \uparrow x \subseteq U \} \right) \right\} \\ &= \bigcap \left\{ V \in \mathcal{U}(Y) : x \in g^{-1}(V) \right\} = \bigcap \left\{ V \in \mathcal{U}(Y) : g(x) \in V \right\} \\ &= G_Y(g(x). \end{aligned}$$

Thus, the diagram commutes. This finishes the proof that **FDL** and **FPos** are dual. □

Let B be a finite Boolean algebra. Then join prime elements of B are atoms; consequently, the order relation R is the equality relation; thus, $(\mathcal{JP}(B), R)$ is identified just with the set

$\mathcal{JP}(B)$. Conversely, let X be a finite set, and define $\leq$ to be the equality relation. Then $\leq$ is the trivial partial order. Then $\mathcal{U}(X \leq) = \mathcal{P}(X)$, and so $\mathcal{P}(X)$ is a Boolean algebra. The propositions above then yield isomorphisms $B \cong \mathcal{P}(\mathcal{JP}(B))$ as Boolean algebras and $\mathcal{JP}(\mathcal{P}(X)) \cong X$ as sets. Moreover, our duality restricts to a duality $\mathbf{FBA} \cong \mathbf{FSet}$ between the category of finite Boolean algebras and the category of finite sets. This is then a special case of Stone duality, where a Stone topology on a finite set is necessarily the discrete topology.

Let L be a finite distributive lattice. The lemma above proves $\mathcal{PF}(L) = \{\uparrow p : p \in \mathcal{JP}(L)\}$. Since $\uparrow p \subseteq \uparrow q$ if and only if $q \leq p$, we see that pRq if and only if $\uparrow p \subseteq \uparrow q$, and so $(\mathcal{PF}(L), \subseteq) \cong (\mathcal{JP}(L), R)$. Looking ahead, we see that Birkhoff duality is a special case of Priestley duality; in general we do not recover a lattice from the set $\mathcal{JP}(L)$ and so must work with $\mathcal{PF}(L)$. This is because prime filters need not be principal.

3 Stone Duality

We now consider Stone duality for Boolean algebras. We recall that a topological space X is said to be a *Stone space* if X is compact, Hausdorff, and 0-dimensional. Thus, X has a basis of clopen sets. Let $\mathbf{BA}$ denote the category of Boolean algebras. We also denote by $\mathbf{Stone}$ the category of Stone spaces. Maps in $\mathbf{Stone}$ are simply continuous maps.

Lemma 3.1. *Let L be a bounded distributive lattice. Then $\mathcal{PF}(L)$ is a Stone space.*

Proof. We first note that $\mathcal{PF}(L)$ is 0-dimensional, since our standard basis consists of clopen sets. Next, to see that $\mathcal{PF}(L)$ is Hausdorff, let P and Q be distinct prime filters on L. Either $P \not\subseteq Q$ or $Q \not\subseteq P$; suppose the former happens. Take $a \in P$ with $a \notin Q$. Therefore, $P \in \varphi(a)$ and $Q \in \varphi(a)^c$. We have thus separated P and Q by disjoint open sets, so $\mathcal{PF}(L)$ is Hausdorff. Finally, we show that $\mathcal{PF}(L)$ is compact. To do this, it is enough to prove that if we have a cover of $\mathcal{PF}(L)$ of basic open sets, then the cover has a finite subcover. So, suppose that $\mathcal{PF}(L) = \bigcup_i \varphi(a_i) \cup \bigcup_j \varphi(b_j)^c$ for some $a_i, b_j \in L$. This implies that $\bigcap_j \varphi(b_j) \subseteq \bigcup_i \varphi(a_i)$. Let I be the ideal generated by the a_i and let F be the filter generated by the b_j. If $F \cap I = \varnothing$, then Lemma 1.4 shows that there is a prime filter P with $F \subseteq P$ and $P \cap I = \varnothing$. However, we see that $P \in \bigcap_j \varphi(b_j)$ since $F \subseteq P$. Therefore, $P \in \varphi(a_i)$ for some i. However, this means $a_i \in P$, so $P \cap I$ is nonempty. This contraction shows that $F \cap I \neq \varnothing$. By definition, we have an inequality of the form $b_1 \wedge \cdots \wedge b_n \leq a_1 \vee \cdots \vee a_m$. This yields $\varphi(b_1) \cap \cdots \cap \varphi(b_n) \subseteq \varphi(a_1) \cup \cdots \cup \varphi(a_m)$. Therefore, $\mathcal{PF}(L)$ is covered by the finite collection $\varphi(a_1), \ldots \varphi(a_m), \varphi(b_1)^c, \ldots, \varphi(b_n)^c$. Thus, $\mathcal{PF}(L)$ is compact. $\square$

Let B be a Boolean algebra. We point out that prime filters are ultrafilters; to see this if P is a prime filter, then $a \notin P$ implies that $-a \in P$ since $a \vee -a = 1 \in P$. Then the filter generated by $P \cup \{a\}$ is not proper, so P is an ultrafilter.

We now define contravariant functors from $\mathbf{BA}$ to $\mathbf{Stone}$ and vice-versa. For $\mathcal{PF}$: $\mathbf{BA} \to \mathbf{Stone}$, we define it on objects by sending B to $\mathcal{PF}(B)$. For maps, given a Boolean homomorphism $f : B \to C$, we define $\mathcal{PF}(f) : \mathcal{PF}(C) \to \mathcal{PF}(B)$ by $Q \mapsto f^{-1}(Q)$. It is easy to see that this is a well-defined function, since the pullback of a prime filter under a Boolean map is a prime filter. Moreover,

$$\begin{aligned}\mathcal{PF}(f)^{-1}(\varphi(b)) &= \{Q \in \mathcal{PF}(C) : b \in f^{-1}(Q)\} \\ &= \{Q \in \mathcal{PF}(C) : f(b) \in Q\} = \varphi(f(b)).\end{aligned}$$

Thus, $\mathcal{PF}(f)$ is continuous. It is elementary to see that $\mathcal{PF}$ does define a functor.

Let X be a Stone space. We denote by $\mathcal{C}(X)$ the set of clopen sets in X. It is clear that $\mathcal{C}(X)$ is a Boolean algebra under ordinary set union, intersection, and compliment. Define $\mathcal{C} : \mathbf{Stone} \to \mathbf{BA}$ on objects by $X \mapsto \mathcal{C}(X)$, and for maps, if $f : X \to Y$ is a continuous map between Stone spaces, then define $\mathcal{C}(f) : \mathcal{C}(Y) \to \mathcal{C}(X)$ by $U \mapsto f^{-1}(U)$. It is clear that a continuous map pulls back clopen sets to clopen sets, and because the pullback map is a Boolean homomorphism from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$, its restriction to $\mathcal{C}(Y)$ is also a Boolean map. It is then elementary to see that $\mathcal{C}$ is indeed a functor. Our goal in this section is to prove that $\mathcal{PF}$ and $\mathcal{C}$ provide a duality between **BA** and **Stone**. We start by seeing that we recover B as $\mathcal{C}(\mathcal{PF}(B))$.

Lemma 3.2. *Let C be a clopen subset of $\mathcal{PF}(B)$. Then $C = \varphi(a)$ for some $a \in B$.*

Proof. We have already noted that $\varphi(a)$ is clopen for every $a \in B$; we are claiming the converse in the lemma. Let C be a clopen subset of $\mathcal{PF}(B)$. Then $C = \varphi(A)$ for some subset A of B since C is closed. Because $\varphi(a)$ is open for each a, and since $\varphi(a) \subseteq C$ if and only if $a \in A$, we have an open cover $\{\varphi(a) : a \in A\}$ of C. Since C is compact, $C = \varphi(a_1) \cup \cdots \cup \varphi(a_n)$ for some $a_i \in C$. However, this yields $C = \varphi(a_1 \vee \cdots \vee a_n)$, as desired. □

Proposition 3.3. *Let B be a Boolean algebra, and let $\mathcal{PF}(B)$ be the space of prime filters on B. Then the map $F_B : B \to \mathcal{C}(\mathcal{PF}(B))$, given by $F_B(b) = \varphi(b)$, is an isomorphism of Boolean algebras.*

Proof. We first point out that $\mathcal{C}(\mathcal{PF}(B))$ is a Boolean algebra. It is clear that intersections and unions of two clopens is again clopen, that the compliment of a clopen is clopen, and that $\mathcal{PF}(B)$ and $\varnothing$ are clopen. Thus, $\mathcal{C}(\mathcal{PF}(B))$ is a Boolean subalgebra of the power set of $\mathcal{PF}(B)$. The map F_B is well defined, and is surjective by the lemma. Moreover, it is a Boolean homomorphism since $\varphi(a) \cup \varphi(b) = \varphi(a \vee b)$ and $\varphi(a) \cap \varphi(b) = \varphi(a \wedge b)$, along with $\varphi(a)^c = \varphi(-a)$; this last equality is clear since no filter contains both a and $-a$, but every maximal filter contains one or the other. Finally, we prove that F_B is injective. Suppose that $a \neq b$. We may assume that $a \nleq b$, so $a \wedge -b \neq 0$. By Lemma 1.4, there is a prime filter P containing $a \wedge -b$. Therefore, $a, -b \in P$. It thus does not contain b, and so $P \in \varphi(a)$ but $P \notin \varphi(b)$. Therefore, $\varphi(a) \neq \varphi(b)$, finishing the proof. □

Proposition 3.4. *Let X be a Stone space. Then the map $G_X : X \to \mathcal{PF}(\mathcal{C}(X))$, defined by $G_X(x) = \{U \in \mathcal{C}(X) : x \in U\}$, is a homeomorphism.*

Proof. We first show that $G_X(x)$ is a prime filter of $\mathcal{C}(X)$ to see that G_X is well-defined. It is clear that $G_X(x)$ is an upset in $\mathcal{C}(X)$. If $U, V \in G_X(x)$, then $x \in U$ and $y \in V$, so $x \in U \cap V$; thus, $U \cap V \in G_X(x)$. Next, to see that $G_X(x)$ is prime, suppose that $U \cup V \in G_X(x)$. Then $x \in U \cup V$, so either $x \in U$ or $x \in V$. Thus, $U \in G_X(x)$ or $V \in G_X(x)$. Thus, $G_X(x)$ is a prime filter. Next, we show that G_X is continuous. Let $U \in \mathcal{C}(X)$, and consider the basic clopen set $\varphi(U) = \{P \in \mathcal{PF}(\mathcal{C}(X)) : U \in P\}$. Then

$$\begin{aligned} G_X^{-1}(\varphi(U)) &= \{x \in X : g(x) \in \varphi(U)\} = \{x \in X : U \in g(x)\} \\ &= \{x \in X : x \in U\} = U. \end{aligned}$$

Thus, G_X is continuous. We note that $\{z\} = \bigcap G_X(z)$ for any $z \in X$; this follows because X is 0-dimensional, so every point of X can be separated from z by a clopen set. Therefore, if $G_X(x) = G_X(y)$, then $x = y$. The map G_X is then 1-1. It is also surjective; if P is a prime filter in $\mathcal{C}(X)$, then consider $\bigcap P$. This is nonempty since P, as a collection of closed subsets of X, has the finite intersection property. If $\bigcap P$ contains distinct points x, y, then there is a clopen set U with $x \in U$ and $y \in X \backslash U$. Moreover, either $U \in P$ or $X \backslash U \in P$. If $U \in P$, then $y \notin \bigcap P$; a similar contradiction happens if $X \backslash U \in P$. Thus, $\bigcap P = \{x\}$ for some $x \in X$. Thus, $P \subseteq G_X(x)$. However, P and $G_X(x)$ are both prime, hence maximal, filters on $\mathcal{C}(X)$. Thus, $P = G_X(x)$. Finally, we point out that since G_X is a bijective continuous map between Stone spaces, it is a homeomorphism. □

Theorem 3.5. *The functors $\mathcal{PF}$ and $\mathcal{C}$ give a co-equivalence between* **BA** *and* **Stone**.

Proof. To show that $\mathcal{PF}$ and $\mathcal{C}$ yield an equivalence of categories, we define a natural isomorphism $F : \mathrm{id}_{\mathbf{BA}} \to \mathcal{C} \circ \mathcal{PF}$. For B a Boolean algebra, define $F_B : B \to \mathcal{C}(\mathcal{PF}(B))$ by $F_B(b) = \varphi(b)$. As we saw in Proposition 3.3, F is an isomorphism of Boolean algebras. It is easy to see that F is a natural transformation, since if $f : A \to B$ is a Boolean homomorphism the diagram

$$\begin{array}{ccc} A & \xrightarrow{F_A} & \mathcal{C}(\mathcal{PF}(A)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(\mathcal{PF}(f))} \\ B & \xrightarrow{F_B} & \mathcal{C}(\mathcal{PF}(B)) \end{array}$$

clearly commutes, since if $a \in A$, then

$$\begin{aligned} \mathcal{C}(\mathcal{PF}(f))(F_A(a)) &= \mathcal{C}(\mathcal{PF}(f))\,(\varphi(b)) = \mathcal{PF}(f)^{-1}(\varphi(a)) \\ &= \left\{Q \in \mathcal{PF}(B) : f^{-1}(Q) \in \varphi(b)\right\} = \left\{Q \in \mathcal{PF}(B) : a \in f^{-1}(Q)\right\} \\ &= \{Q : f(a) \in Q\} = G_B(f(a)). \end{aligned}$$

Next, define $G : \mathrm{id}_{\mathbf{Stone}} \to s \circ t$ by $G_X : X \to \mathcal{PF}(\mathcal{C}(X))$, for a Stone space X, as $G_X(x) = \{U \in \mathcal{C}(X) : x \in U\}$. We say in Proposition 3.4 that G_X is an isomorphism. Moreover, the diagram

$$\begin{array}{ccc} X & \xrightarrow{G_X} & \mathcal{PF}(\mathcal{C}(X)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathcal{PF}(\mathcal{C}(g))} \\ Y & \xrightarrow{G_Y} & \mathcal{PF}(\mathcal{C}(Y)) \end{array}$$

is commutative, since if $x \in X$, then $G_Y(f(x)) = \{V \in \mathcal{C}(Y) : f(x) \in V\}$ and

$$\begin{aligned}\mathcal{PF}(\mathcal{C}(f))(G_X(x)) &= \mathcal{PF}(\mathcal{C}(f))\left(\{U \in \mathcal{C}(X) : x \in U\}\right) \\ &= \mathcal{C}(f)^{-1}\left(\{U \in \mathcal{C}(X) : x \in U\}\right) = \left\{V \in \mathcal{C}(Y) : x \in f^{-1}(V)\right\} \\ &= \{V \in \mathcal{C}(Y) : f(x) \in V\} = G_Y(f(x)).\end{aligned}$$

This finishes the proof that we have a co-equivalence of categories. $\square$

4 Priestley Duality

In this section we extend Stone duality to the case of bounded distributive lattices to obtain Priestley duality. If L is a bounded distributive lattice L, then we do not necessarily recover L as $\mathcal{C}(\mathcal{PF}(L))$, since this construction produces a Boolean algebra. Thus, we need to determine how to recover L from $\mathcal{PF}(L)$. We point out the simple fact that if $a \in L$ and if $P \in \varphi(a)$, then $Q \in \varphi(a)$ for any prime filter Q with $P \subseteq Q$. Keeping this in mind, inclusion is a partial order on $\mathcal{PF}(L)$. In other words, if P and Q are prime filters on L, then $P \leq Q$ if $P \subseteq Q$. Then $\varphi(a)$ is a clopen upset of $\mathcal{PF}(L)$ for each $a \in L$. We now discuss the general situation.

Definition 4.1. *Suppose that $(X, \leq)$ is a Stone space with a partial order. We say that $(X, \leq)$ satisfies the* Priestley separation axiom *if for all $x, y \in X$ with $x \not\leq y$, there is a clopen upset U with $x \in U$ and $y \notin U$.*

We consider the category **PS** of ordered Stone spaces, where the maps are continuous and order preserving maps. We call objects of this category Priestley spaces. Let L be a bounded distributive lattice. The space $\mathcal{PF}(L)$ then is a Stone space by Lemma 3.1. We start by showing that $(X, \leq)$ satisfies the Priestley separation axiom: if $x \not\leq y$, then there is a clopen upset U with $x \in U$ and $y \notin U$.

Lemma 4.2. *If L is a bounded distributive lattice, then $(\mathcal{PF}(L), \subseteq)$ is a Priestley space.*

Proof. Let P and Q be prime filters on L with $P \not\subseteq Q$. Then there is an $a \in P$ with $a \notin Q$. Thus, $P \in \varphi(a)$ but $Q \notin \varphi(a)$. Therefore, $\varphi(a)$ is a clopen upset containing P but not Q. $\square$

We now see how to recover L from $\mathcal{PF}(L)$ using the partial order.

Lemma 4.3. *The clopen upsets of $\mathcal{PF}(L)$ are precisely the sets $\varphi(a)$ for $a \in L$.*

Proof. We have already noted that each $\varphi(a)$ is a clopen upset. Conversely, let U be a clopen upset of $\mathcal{PF}(L)$. For each $P \in U$ and $Q \in \mathcal{PF}(L)\backslash U$, we have $P \not\subseteq Q$ since U is an upset. Thus, there is $a_{PQ} \in L$ with $a_{PQ} \in P$ and $a_{PQ} \notin Q$. Therefore, $P \in \varphi(a_{PQ})$ and $Q \in \varphi(a_{PQ})^c$. Thus, $\mathcal{PF}(L)\backslash U$ is covered by the various $\varphi(a_{PQ})^c$; by compactness, we have

$$\mathcal{PF}(L)\backslash U \subseteq \bigcup_{i=1}^{n} \varphi(a_{PQ_i})^c = \varphi(a_P)^c,$$

where $a_P = a_{PQ_1} \wedge \cdots \wedge a_{PQ_n}$. Consequently, $P \in \varphi(a_P) \subseteq U$. Because U is the union of the various $\varphi(a_P)$, compactness implies that U is a finite union $\bigcup_{i=1}^{m} \varphi(a_{P_i}) = \varphi(a)$, where $a = a_{P_1} \vee \cdots \vee a_{P_m}$. □

If $(X, \leq)$ is a Priestley space, we denote by $\mathcal{CU}(X \leq)$ the clopen upsets of $(X, \leq)$.

Proposition 4.4. *If L is a distributive lattice, then the map $F_L : L \to \mathcal{CU}(\mathcal{PF}(L), \subseteq)$, defined by $F_L(l) = \varphi(l)$, is a lattice isomorphism.*

Proof. We have the injective lattice homomorphism $\varphi = F_L : L \to \mathcal{CU}(\mathcal{PF}(L), \subseteq)$. It is surjective by the previous lemma classifying the clopen upsets of $\mathcal{PF}(L)$; thus, it is a lattice isomorphism. □

Proposition 4.5. *If $(X, \leq)$ is a Priestley space, then $G_X : (X \leq) \to \mathcal{PF}(\mathcal{CU}\,(X, \leq)\,, \subseteq)$ is an isomorphism of Priestley spaces.*

Proof. Define $G_X : X \to \mathcal{PF}(\mathcal{CU}\,(X, \leq)\,, \subseteq)$ by $G_X(x) = \{U \in \mathcal{CU}\,(X, \leq) : x \in U\}$. It is easy to see that $G_X(x)$ is a prime filter; the argument is the same as in the proof of Proposition 3.4. Furthermore, if $x \leq y$ and $U \in G_X(x)$, then $x \in U$, so $y \in U$. Thus, $U \in G_X(y)$. This proves that $G_X(x) \subseteq G_X(y)$, so G_X is order preserving. Next, let U be a clopen upset in X, and consider the basic clopen set $\varphi(U) = \{P \in \mathcal{PF}(\mathcal{CU}\,(X, \leq) : U \in P\}$. Then

$$\begin{aligned} G_X^{-1}(\varphi(U)) &= \{x \in X : g(x) \in \varphi(U)\} = \{x \in X : U \in g(x)\} \\ &= \{x \in X : x \in U\} = U, \end{aligned}$$

so G_X is continuous. The Priestley separation axiom shows that, if $z \in X$, then any point in X not above z can be separated from z by a clopen upset. Thus, $\uparrow z = \bigcap G_X(z)$. From this it is clear that if $G_X(x) = G_X(y)$, then $x = y$, and if $G_X(x) \subseteq G_X(y)$, then $x \leq y$. Moreover, to see that G_X is surjective, we note that $G_X(X)$ is closed in $\mathcal{PF}(\mathcal{CU}\,(X, \leq)\,, \subseteq)$ since X is compact and the target is Hausdorff. If G_X is not surjective, then there is a prime filter P of $\mathcal{CU}\,(X, \leq)$ not contained in $G_X(X)$. Therefore, there is a clopen set V of $\mathcal{CU}\,(X, \leq)$ missing $G_X(X)$ and containing P. We may assume that $V = \varphi(U_1) \cap \varphi(U_2)^c$ for some $U_1, U_2 \in \mathcal{CU}\,(X, \leq)$ since V is a finite union of such sets. Now, $\varnothing = G_X^{-1}(V) = G_X^{-1}(\varphi(U_1)) \cap G_X^{-1}(\varphi(U_2)^c$. However, $G_X^{-1}(\varphi(U)) = U$, as we say earlier. Therefore, $\varnothing = U_1 \cap U_2^c$, implying that $U_1 \subseteq U_2$. But then $\varphi(U_1) \cap \varphi(U_2)^c = \varnothing$. This contradiction shows that G_X is surjective.

Therefore, G_X is a bijection, and is an isomorphism of posets. Furthermore, since it is a continuous bijection between Stone spaces, it is a homeomorphism. Thus, $(X, \leq)$ and $\mathcal{PF}(\mathcal{CU}(X, \leq), \subseteq)$ are isomorphic as Priestley spaces.

We now work categorically. Let **BDL** be the category of bounded distributive lattices and lattice homomorphisms, and let **PS** be the category of Priestley spaces and continuous order-preserving maps. We have defined $\mathcal{PF} : \mathbf{BDL} \to \mathbf{PS}$ on objects. On maps, let $f : L \to M$ be a lattice homomorphism. Define $\mathcal{PF}(f) : \mathcal{PF}(M) \to \mathcal{PF}(L)$ by $\mathcal{PF}(f)(Q) = f^{-1}(Q)$, the same way we defined it for maps between Boolean algebras. It is easy to see that $\mathcal{PF}$ is then a functor. Next, we have defined $\mathcal{CU} : \mathbf{PS} \to \mathbf{BDL}$ on objects. On maps, if $g : X \to Y$ is a Priestley morphism, define $\mathcal{CU}(g) : \mathcal{CU}(Y) \to \mathcal{CU}(X)$ by $\mathcal{CU}(g)(V) = g^{-1}(V)$. It is easy to see that this is well-defined, and that $\mathcal{CU}$ is a functor. □

Theorem 4.6. *The functors $\mathcal{CU}$ and $\mathcal{PF}$ give co-equivalence of categories between* **BDL** *and* **PS**.

Proof. Define natural transformations $F : \mathrm{id}_{\mathbf{BDL}} \to \mathcal{CU} \circ \mathcal{PF}$ by $F_L : L \to \mathcal{CU}(\mathcal{PF}(L), \subseteq)$ as in Proposition 4.4. Then F_L is an isomorphism. To see that the diagram

$$\begin{tikzcd} L \arrow[r, "F_L"] \arrow[d, "f"'] & \mathcal{CU}(\mathcal{PF}(L)) \arrow[d, "\mathcal{CU}(\mathcal{PF}(f))"] \\ M \arrow[r, "F_M"] & \mathcal{CU}(\mathcal{PF}(M)) \end{tikzcd}$$

commutes, let $l \in L$. Then

$$\begin{aligned} \mathcal{CU}(\mathcal{PF}(f))(\varphi(l)) &= \mathcal{PF}(f)^{-1}(\varphi(l)) = \left\{Q \in \mathcal{PF}(M) : f^{-1}(Q) \in \varphi(l)\right\} \\ &= \left\{Q \in \mathcal{PF}(M) : l \in f^{-1}(Q)\right\} = \{Q \in \mathcal{PF}(M) : f(l) \in Q\} \\ &= F_M(f(l)). \end{aligned}$$

Next, define $G : \mathrm{id}_{\mathbf{PS}} \to \mathcal{PF} \circ \mathcal{CU}$ as in Proposition 4.5. Let $g : X \to Y$ be a morphism of Priestley spaces. To see that the diagram

$$\begin{tikzcd} X \arrow[r, "G_X"] \arrow[d, "g"'] & \mathcal{PF}(\mathcal{CU}(X)) \arrow[d, "\mathcal{PF}(\mathcal{CU}(g))"] \\ Y \arrow[r, "G_Y"] & \mathcal{PF}(\mathcal{CU}(Y)) \end{tikzcd}$$

commutes, let $x \in X$. Then

$$\begin{aligned} \mathcal{PF}(\mathcal{CU}(g))(G_X(x)) &= \left\{V \in \mathcal{CU}(Y) : g^{-1}(V) \in G_X(x)\right\} = \left\{V \in \mathcal{CU}(Y) : x \in g^{-1}(V)\right\} \\ &= \{V \in \mathcal{CU}(Y) : g(x) \in V\} = G_Y(g(x)). \end{aligned}$$

Therefore, F and G yield a co-equivalence between **BDL** and **PS**. □

5 Heyting Duality

In this section we extend Priestley duality to the category of Heyting algebras. Recall that a bounded distributive lattice H is said to be a *Heyting algebra* if H has a binary operation $\to$, called implication, such that $x \leq a \to b$ if and only if $a \wedge x \leq b$. Symbolically, implication is given by

$$a \to b = \bigvee \{x \in H : a \wedge x \leq b\}.$$

The complication is that arbitrary joins of elements need not exist in a lattice, so the existence of an implication is not automatic. While we do not need the following fact in this note, if H is a lattice in which joins of arbitrary subsets exist, then H is a Heyting algebra if and only if H satisfies the infinite distributive law

$$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$$

for all subsets B of H.

Let **HA** be the category whose objects are Heyting algebras and whose morphisms are lattice homomorphisms which preserve implication; these maps are called *Heyting morphisms*. We wish to restrict Priestley duality to the category **HA**; we thus need to determine which Priestley spaces are duals of Heyting algebras, and which morphisms of such spaces are dual to Heyting morphisms. Let $g : (X, \leq) \to (Y, \leq)$ be a morphism of posets. We say that g is a *p-morphism* if for every $x \in X$ and $z \in Y$ with $g(x) \leq z$, there is an $x' \in X$ with $x \leq x'$ and $g(x') = z$. A *Heyting space* is a Priestley space $(X, \leq)$ such that if U is clopen in X, then $\downarrow U$ is clopen. Alternatively, $(X, \leq)$ is a Heyting space if for every open set U, then downset $\downarrow U$ is open. The equivalence of these conditions follows from Lemma 5.1 below. A morphism of Heyting spaces is a continuous p-morphism. The category **HS** consists of all Heyting spaces and morphisms of Heyting spaces. In this section we will see that Heyting spaces are exactly those Priestley spaces which are dual to Heyting algebras, and that Priestley duality restricts to a duality between **HA** and **HS**. We start with some preliminary lemmas.

Lemma 5.1. *Let $(X, \leq)$ be a Priestley space.*

1. *The set $\leq$ is a closed subset of $X \times X$.*

2. *If C is closed in X, then $\uparrow C$ and $\downarrow C$ are closed in X.*

Proof. Let $(x, y) \in (X \times X) \setminus \leq$. Then $x \not\leq y$. Therefore, there is a clopen upset U with $x \in U$ and $y \in X \backslash U$. Since U is an upset and $X \backslash U$ a downset, we see that $U \times (X \backslash U) \cap \leq = \varnothing$. Therefore, $U \times (X \backslash U)$ is an open neighborhood of (x, y) missing $\leq$. Therefore, $\leq$ is closed in $X \times X$.

To prove the second statement, we note that $\uparrow C = \pi_2 (\wedge^{-1}(C) \cap \leq)$. Since X is compact, the projection maps are closed. Thus, $\uparrow C$ is closed. Similarly, as $\downarrow C = \pi_1 (\vee^{-1}(C) \cap \leq)$, we see that $\downarrow C$ is closed. □

Lemma 5.2. *Let H be a Heyting algebra. If $a, b \in H$, then $\downarrow \varphi(a) \cap \varphi(b)^c = \varphi(a \to b)^c$.*

Proof. Let $a, b \in H$. Since $a \wedge (a \to b) \le b$, we have $\varphi(a) \cap \varphi(a \to b) \subseteq \varphi(b)$, so $\varphi(a) \cap \varphi(b)^c \subseteq \varphi(a \to b)^c$. Because $\varphi(a \to b)^c$ is a downset, $\downarrow \varphi(a) \cap \varphi(b)^c \subseteq \varphi(a \to b)^c$. For the reverse inclusion, let $P \in \varphi(a \to b)^c$. Then P is a prime filter with $a \to b \notin P$. We wish to find a prime filter Q with $P \cup \{a\} \subseteq Q$ and $b \notin Q$. Note that if $a, a \to b \in Q$, then $b \in Q$. Therefore, it is enough to make sure that $a \to b \notin Q$. Such a prime filter Q exists if the filter F generated by $P \cup \{a\}$ does not contain $a \to b$, by Lemma 1.4. If F contains $a \to b$, then there is $x \in P$ with $a \wedge x \le a \to b$. Then $(a \wedge x) \wedge a \le b$, by the definition of $a \to b$. But then $a \wedge x \le b$, forcing $x \le a \to b$. This is a contradiction since $x \in P$ and $a \to b \notin P$. Thus, we have a prime filter Q with $P \subseteq Q$, $a \in Q$, and $b \notin Q$. Therefore, $P \in \downarrow \varphi(a) \cap \varphi(b)^c$. This finishes the proof. □

Lemma 5.3. *Let $f : X \to Y$ be a poset morphism. Then the following conditions are equivalent.*

1. *f is a p-morphism.*
2. *$f^{-1}(\downarrow A) = \downarrow f^{-1}(A)$ for every subset $A \subseteq Y$.*
3. *$f^{-1}(\downarrow y) = \downarrow f^{-1}(\{y\})$ for every $y \in Y$.*

Proof. Suppose that f is a p-morphism and $A \subseteq Y$. Since $f^{-1}(A) \subseteq f^{-1}(\downarrow A)$, and the latter is a downset, we have $\downarrow f^{-1}(A) \subseteq f^{-1}(\downarrow A)$. For the reverse inclusion, let $x \in f^{-1}(\downarrow A)$. Then $f(x) \le a$ for some $a \in A$. Since f is a p-morphism, $a = f(x')$ for some $x' \in X$ with $x \le x'$. Then $x' \in f^{-1}(A)$, so $x \in \downarrow f^{-1}(A)$. Therefore, $f^{-1}(\downarrow A) = \downarrow f^{-1}(A)$ for every $A \subseteq Y$. The implication (2) $\Rightarrow$ (3) is trivial. Finally, assume that (3) holds. Suppose that $x \in X$ and $y \in Y$ with $f(x) \le y$. Then $f(x) \in \downarrow y$, so $x \in f^{-1}(\downarrow y) = \downarrow f^{-1}(y)$. Therefore, $x \le z$ for some $z \in f^{-1}(y)$. Therefore, $y = f(z)$. This proves that f is a p-morphism. □

We now consider the functor $\mathcal{PF} : \mathbf{BDL} \to \mathbf{PS}$, but restricted to $\mathbf{HA}$.

Lemma 5.4. *If H is a Heyting algebra, then $(\mathcal{PF}(H), \subseteq)$ is a Heyting space.*

Proof. Let U be a clopen set in $\mathcal{PF}(H)$. Then $U = \bigcup_{i=1}^n \varphi(a_i) \cap \varphi(b_i)^c$ for some $a_i, b_i \in H$. By Lemma 5.2, we have $\downarrow U = \bigcup \downarrow \varphi(a_i) \cap \varphi(b_i)^c = \bigcup \varphi(a_i \to b_i)^c$, a clopen set. Therefore, $(\mathcal{PF}(H), \subseteq)$ is a Heyting space. □

Lemma 5.5. *Let $f : H \to H'$ be a Heyting morphism. Then $\mathcal{PF}(f) : \mathcal{PF}(H') \to \mathcal{PF}(H)$ is a p-morphism.*

Proof. Let $Q \in \mathcal{PF}(H')$ and $P \in \mathcal{PF}(H)$ with $f^{-1}(Q) \subseteq P$. For notational convenience, we will denote $\mathcal{PF}(f) = g$. Let C be a clopen set in $\mathcal{PF}(H)$ containing P. Then C is a finite union of sets of the form $\varphi(a) \cap \varphi(b)^c$ with $a \in P$ and $b \notin P$. We have

$$\begin{aligned} g^{-1}\left(\downarrow \varphi(a) \cap \varphi(b)^c\right) &= g^{-1}\left(\varphi(a \to b)^c\right) = g^{-1}\left(\varphi(a \to b)^c\right. \\ &= \varphi(f(a \to b))^c = \varphi(f(a) \to f(b))^c \\ &= \downarrow(\varphi(f(a)) \cap \varphi(f(b))^c)\,. \end{aligned}$$

By considering finite unions, we then see that $g^{-1}(\downarrow C) = \downarrow g^{-1}(C)$ for any clopen set. Since $g(Q) \subseteq P$, we see that $Q \in g^{-1}(\downarrow C) = \downarrow g^{-1}(C)$ for any clopen C containing P. Thus, $\uparrow Q \cap g^{-1}(C) \neq \varnothing$. Since the set of clopens containing P are closed under finite intersections, compactness implies that $\bigcap (\uparrow Q \cap g^{-1}(C)) \neq \varnothing$, where the intersection is over all clopens C containing P. This yields $\uparrow Q \cap \bigcap g^{-1}(C) \neq \varnothing$, so $\uparrow Q \cap g^{-1}(\{P\}) \neq \varnothing$. Therefore, there is Q' with $Q \subseteq Q'$ and $g(Q') = P$. This proves that $g = \mathcal{PF}(f)$ is a p-morphism. $\square$

The previous two lemmas show that $\mathcal{PF}$ is a functor from **HA** to **HS**. We now consider the restriction of the functor $\mathcal{CU} : \mathbf{PS} \to \mathbf{BDL}$ restricted to **HS**.

Lemma 5.6. *Let $(X, \leq)$ be a Heyting space. Then $\mathcal{CU}(X, \leq)$ is a Heyting algebra, where implication is defined by $U \to V = (\downarrow U \cap V^c)^c$.*

Proof. Let U, V be clopen upsets. Then $U \cap V^c$ is clopen, so $\downarrow U \cap V^c$ is clopen, since $(X, \leq)$ is a Heyting space. Thus, $(\downarrow U \cap V^{c\,c})$ is a clopen upset. We then define $U \to V = (\downarrow U \cap V^c)^c$. To see that $\mathcal{CU}(X, \leq)$ is a Heyting algebra, we only need to check that for any clopen upset W, we have $U \cap W \subseteq V$ if and only if $W \subseteq U \to V$. One direction is easy; since $U \to V \subseteq (U \cap V^c)^c$, we have $U \cap (U \to V) \subseteq U \cap (U \cap V^c)^c = U \cap (U^c \cup V) \subseteq V$. For the converse, suppose that $U \cap W \subseteq V$. Then $U \cap V^c \subseteq W^c$. Since W^c is a downset, we get $\downarrow U \cap V^c \subseteq W^c$. Thus, $W \subseteq (\downarrow U \cap V^c)^c = U \to V$. $\square$

Lemma 5.7. *Let $g : (X, \leq) \to (Y, \leq)$ be a morphism of Heyting spaces. Then $g^{-1} : \mathcal{CU}(Y, \leq) \to \mathcal{CU}(X, \leq)$ is a Heyting morphism.*

Proof. We know that g^{-1} is a lattice homomorphism, so we only need to show that g^{-1} preserves implication. Let U, V be clopen upsets of Y. Since $g^{-1}(U) \cap g^{-1}(U \to V) = g^{-1}(U \cap (U \to V)) \subseteq g^{-1}(V)$, we see that $g^{-1}(U \to V) \subseteq g^{-1}(U) \to g^{-1}(V)$. For the reverse inclusion, suppose that $x \notin g^{-1}(U \to V)$. Since $U \to V = (\downarrow U \cap V^c)^c$, we have $x \in g^{-1}(\downarrow U \cap V^c)$, so $g(x) \in \downarrow U \cap V^c$. Therefore, there is $y \in U \cap V^c$ with $g(x) \leq y$. Since g is a p-morphism, there is $z \in X$ with $x \leq z$ and $y = g(z)$. Then $z \in g^{-1}(U \cap V^c) = g^{-1}(U) \cap g^{-1}(V)^c$. Thus, $x \in \downarrow g^{-1}(U) \cap g^{-1}(V)^c$, and so $x \notin g^{-1}(U) \to g^{-1}(V)$. This proves the reverse inclusion. Thus, $g^{-1}(U \to V) = g^{-1}(U) \to g^{-1}(V)$, so g^{-1} is a Heyting morphism. $\square$

Thus, $\mathcal{CU}$ is a functor from **HS** to **HA**. We now see that these categories are dual to each other. Much of the work involved we did verifying Priestley duality.

Proposition 5.8. *Let H be a Heyting algebra. Then there is a Heyting isomorphism $F_H : H \to \mathcal{CU}(\mathcal{PF}(H))$, given by $F_H(h) = \varphi(h) = \{P \in \mathcal{PF}(H) : h \in P\}$.*

Proof. We have seen in Proposition 4.4 that F_H is an isomorphism of distributive lattices. Thus, we only need to check that F_H preserves implication. Let $a, b \in H$. Then $F_H(a \to b) = \varphi(a \to b)$. Now, $\varphi(a \to b) = \left(\downarrow \varphi(a) \cap \varphi(b)^c\right)^c = \varphi(a) \to \varphi(b)$, by Lemma 5.6. Thus, F_H is an isomorphism of Heyting algebras. $\square$

Proposition 5.9. *Let $(X, \leq)$ be a Heyting space. Then there is an isomorphism of Heyting spaces $G_X : (X, \leq) \to \mathcal{PF}(\mathcal{CU}(X, \leq), \subseteq)$, given by $G_X(x) = \{U \in \mathcal{CU}(X, \leq) : x \in U\}$.*

Proof. We saw in Proposition 4.5 that G_X is an isomorphism of Priestley spaces. Since G_X and G_X^{-1} are then, in particular, poset isomorphisms, they are both clearly p-morphisms. Thus, G_X is an isomorphism of Heyting spaces. □

Theorem 5.10. *The functors $\mathcal{CU}$ and $\mathcal{PF}$ give a co-equivalence of categories between* **HA** *and* **HS**.

Proof. We have natural transformations $F : \mathrm{id}_{\mathbf{HA}} \to \mathcal{CU} \circ \mathcal{PF}$ and $G : \mathrm{id}_{\mathbf{HS}} : \mathcal{PF} \circ \mathcal{CU}$, defined in the previous two propositions; that they are natural transformations follows from the arguments of Theorem 4.6 along with the lemmas above. □

6 Priestley and Spec

In this section we prove that the categories of spectral spaces and Priestley spaces are isomorphic. Moreover, we interpret Priestley duality in terms of spectral spaces. To start, we make a note of the choice of topology on the spectrum of a distributive lattice. Recall that if R is a commutative ring, then the Zariski topology on $\mathrm{Spec}(R)$ is defined by a set is closed if and only if it is of the form $Z(I) = \{P \in \mathrm{Spec}(R) : I \subseteq P\}$. To mimic this for a distributive lattice, let $X = \mathcal{PF}(L)$, and let Y be the set of prime ideals of L. Then complementation is an order-reversing bijection from X to Y (and vice-versa). If we define the analogue of the Zariski topology on Y, it pulls back to a topology on X. Moreover, if $a \in L$, then $Z(\downarrow a)^c = \{Q \in Y : a \notin Q\}$ is pulled back to $\{P \in X : a \in P\} = \varphi(a)$. Thus, the corresponding topology has the sets $\{\varphi(a) : a \in L\}$ as basic open sets, but not necessarily clopen. As we will see, defining a topology on $\mathcal{PF}(L)$ with basis $\{\varphi(a) : a \in L\}$ will give a spectral topology on $\mathcal{PF}(L)$. We will see that the Stone topology on $\mathcal{PF}(L)$ is obtained from this spectral topology by means of the patch topology

The category **Spec** of spectral spaces has as its objects *spectral spaces*. These are topological spaces (X, τ) which are compact and T_0, the collection of compact open subsets forms a basis for the topology and is closed under finite intersections, and (X, τ) is *sober*; meaning that each closed irreducible subset of X is the closure of a point. Morphisms in **Spec** are continuous maps such that the preimage of a compact open set is compact open; these maps are called *spectral maps*. We recall that if (X, τ) is a topological space, then the specialization order $\leq_\tau$ is defined by $x \leq_\tau y$ if $x \in \overline{\{y\}}$. This is a quasi-order in general, and is a partial order if and only if (X, τ) is T_0. For some notation, we denote by $\mathcal{CO}(X, \tau)$ the set of compact open subsets of (X, τ).

We define functors $\mathcal{P} : \mathbf{Spec} \to \mathbf{PS}$ and $\mathcal{S} : \mathbf{PS} \to \mathbf{Spec}$ as follows. If (X, τ) is a spectral space, then $\mathcal{P}(X, \tau) = (X, \tau^{\#}, \leq_\tau)$, where the *patch topology* $\tau^{\#}$ is the topology with subbasis

$$\{V : V \in \mathcal{CO}(X, \tau) \text{ or } V^c \in \mathcal{CO}(X, \tau)\}.$$

The collection $\{U \cap W^c : U, W \in \mathcal{CO}(X,\tau)\}$ is a basis for $\tau^\#$ since $\mathcal{CO}(X,\tau)$ is closed under finite intersections. The topology $\tau^\#$ is a finer topology than τ; this follows from the fact that $\mathcal{CO}(X,\tau)$ is a basis for τ. If f is a spectral map, we define $\mathcal{P}(f) = f$. For the functor $\mathcal{S}$, if $(X,\rho,\leq)$ is a Priestley space, then $\mathcal{S}(X,\rho,\leq) = (X,\rho_s)$, where ρ_s is the set of open upsets of $(X,\rho,\leq)$. We will prove that **Spec** $\cong$ **PS** by showing that $\mathcal{P}$ and $\mathcal{S}$ are inverse functors. We start off by considering spectral spaces and $\mathcal{P}$. The hardest result to prove in this section is that the patch topology $\tau^\#$ is compact; this is done in the following.

Proposition 6.1. *Let (X,τ) be a spectral space. Then $(X,\tau^\#,\leq_\tau)$ is a Priestley space.*

Proof. Let (X,τ) be spectral. The relation $\leq_\tau$ is a partial order since τ is T_0. To show that the Priestley separation axiom holds, suppose that $x \not\leq y$. Then $x \notin \overline{\{y\}}$. Therefore, there is a compact open set U with $x \in U$ and $y \notin U$. Then U is clopen in $\tau^\#$, and is an upset with respect to the specialization order. So, we have separated x and y by a clopen upset in $\tau^\#$. Thus, the axiom holds. We next show that $(X,\tau^\#)$ is a Stone space. By construction, it is clear that $(X,\tau^\#)$ has a basis of clopen sets; this shows that $(X,\tau^\#)$ is 0-dimensional. It is also T_0 since $\tau^\#$ is finer than the T_0-topology τ. However, 0-dimensionality then forces $\tau^\#$ to be T_2. Finally, we need to show that $(X,\tau^\#)$ is compact. Let K be a collection of closed sets in $\tau^\#$ with the finite intersection property (FIP). We need to show that $\bigcap K$ is nonempty. Because each closed set is the intersection of basic closed sets, the collection of all basic closed sets containing some element of K is a collection of closed sets satisfying the FIP and whose intersection is equal to $\bigcap K$. Thus, we may assume K contains just basic clopen sets. This means each $C \in K$ is of the form $U \cap V$ with U and V^c compact open in τ. Then the collection of all such U and V is also a collection of closed sets with the FIP and having the same intersection as K. So, we replace K by this new collection. Finally, to prove that $\bigcap K$ is nonempty, it suffices to replace K by a collection containing K and maximal with respect to containing only compact open sets and/or closed sets and having the FIP. So, now assume that K is a collection of compact opens and closed sets of τ maximal with respect to having the FIP. Let C be the intersection of all closed sets in K. Then C is nonempty since this collection clearly has the FIP and since (X,τ) is compact. Note that $K \cup \{C\}$ has the FIP, so maximality shows that $C \in K$. We prove that C is irreducible. Suppose that $C = A \cup B$, the union of closed subsets. If $K \cup \{A\}$ and $K \cup \{B\}$ do not have the FIP, then there are $C_i \in K$ with $C_1 \cap \cdots C_n \cap A = \varnothing$ and $C_{n+1} \cap \cdots \cap C_m \cap B = \varnothing$. Then $(C_1 \cap \cdots \cap C_m) \cap C = \varnothing$, a contradiction. Thus, one of these collections has the FIP. By maximality, one of A or B is in K. The definition of C then shows that $C = A$ or $C = B$. Thus, C is irreducible. Let $x \in C$ with $C = \overline{\{x\}}$. Then x is contained in any closed set in K by definition of C. The point x is in each open set in K by the FIP: if $U \in K$ is open, then as $C \in K$, we have $U \cap C$ is nonempty. So, $U \cap \overline{\{x\}}$ is nonempty, forcing $x \in U$. Thus, $x \in \bigcap K$, proving that this intersection is nonempty. Therefore, $(X,\tau^\#)$ is compact. $\square$

Proposition 6.2. *Let (X,τ) be a spectral space. Then $\tau = (\tau^\#)_s$.*

Proof. Let (X, τ) be spectral. If U is open in τ, then U is the union of compact opens. Each compact open is a clopen upset in $(X, \tau^{\#}, \leq)$ by definition, and by the trivial fact that an open set is an upset with respect to the specialization order. So, U is an open upset in $(X, \tau^{\#}, \leq)$. Conversely, let U be an open upset in $(X, \tau^{\#}, \leq)$, and take $x \in U$. If $y \notin U$, then $x \not\leq y$ since U is an upset. So, there is a compact open set V_y containing x but not y. Note that V_y is closed in the patch topology, as is U^c. The collection $\{V_y : y \notin U\} \cup \{U^c\}$ has empty intersection, by construction of the V_y. By compactness, there are $y_1, \ldots, y_n \in U^c$ with $V_{y_1} \cap \cdots \cap V_{y_n} \cap U^c = \varnothing$. Thus, $V_{y_1} \cap \cdots \cap V_{y_n} \subseteq U$. The left hand side is an open neighborhood of x contained in U. This proves that U is open in τ. $\square$

Lemma 6.3. *Let (X, τ) be a spectral space. A subset U is compact open if and only if U is a clopen upset in $\tau^{\#}$.*

Proof. Let (X, τ) be spectral. If U is compact open in τ, then U is clopen in $\tau^{\#}$ by definition. Furthermore, any $U \in \tau$ is an upset with respect to $\leq_\tau$. Thus, U is a clopen upset in $(X, \tau^{\#}, \leq_\tau)$. Conversely, let U be a clopen upset in $(X, \tau^{\#}, \leq_\tau)$. We know that U is open in τ by Proposition 6.2. Moreover, U is compact in $\tau^{\#}$ since it is closed and $(X, \tau^{\#})$ is compact. Since τ is a smaller topology, U is also compact in τ. $\square$

Lemma 6.4. *Let (X, τ) and (Y, σ) be spectral spaces. A continuous map $f : X \to Y$ preserves the specialization orders. Furthermore, f is a spectral map if and only if f is continuous with respect to the patch topologies.*

Proof. Let $f : X \to Y$ be a spectral map. It automatically preserves the specialization order; for, if $x_1 \leq x_2$ in X, then $x_1 \in \overline{\{x_2\}}$. This is equivalent to: For every open set U, if $x_1 \in U$, then $x_2 \in U$. Let V be open in Y with $f(x_1) \in V$. Then $x_1 \in f^{-1}(V)$. Thus, $x_2 \in f^{-1}(V)$, so $f(x_2) \in V$. Therefore, $f(x_1) \in \overline{\{f(x_2)\}}$, so, $f(x_1) \leq f(x_2)$. This shows that f is order preserving. To see that f is continuous in the patch topology, let U be a subbasic open set in Y in the patch topology. Then $U = V \cap W^c$ with V and W compact open subsets of Y in the original topology. Then $f^{-1}(U) = f^{-1}(V) \cap f^{-1}(W)^c$ is open in the patch topology, since $f^{-1}(V)$ and $f^{-1}(W)$ are compact open. Thus, f is continuous. Conversely, if f is continuous and order preserving in the patch topologies, let V be open in (Y, σ). Then V is an open upset in $\sigma^{\#}$ by Proposition 6.2. So, $f^{-1}(V)$ is an open upset in $\tau^{\#}$ since f is a continuous and order-preserving map, which yields $f^{-1}(V) \in \tau$, again by the proposition. Thus, f is continuous in the spectral topology. Finally, let V be compact open in σ. We need to show that $f^{-1}(V)$ is compact. However, by Lemma 6.3, V is a clopen upset in $\sigma^{\#}$, and so $f^{-1}(V)$ is a clopen upset in $\tau^{\#}$. Thus, by that lemma, $f^{-1}(V)$ is compact open in τ. $\square$

Corollary 6.5. *$\mathcal{P}$ is a functor from* **Spec** *to* **PS**.

Proof. Let (X, τ) be a spectral space. By Proposition 6.1, $(X, \tau^{\#}, \leq_\tau)$ is a Priestley space. From the definition of how $\mathcal{P}$ acts on maps, along with Lemma 6.4, we see that $\mathcal{P}$ is indeed a functor. $\square$

We now investigate Priestley spaces and the functor $\mathcal{S}$.

Lemma 6.6. *Let $(X, \rho, \leq)$ be a Priestley space. A subset U of X is a clopen upset if and only if U is compact open in ρ_s.*

Proof. Let $(X, \rho, \leq)$ be Priestley and let U be a clopen upset in ρ. Then U is open in ρ_s by definition. To see that U is compact in ρ_s, suppose that $\{V_i : i \in I\}$ is an open cover of U in ρ_s. Then it is also an open cover of U in ρ. Since (X, ρ) is compact and U is clopen, $U = V_1 \cup \cdots \cup V_n$ for some i. Thus, U is compact in ρ_s. Conversely, let U be compact open in ρ_s. Then U is an open upset in ρ by definition. To see that U is clopen, note that U^c is a closed downset. An argument analogous to the proof that a compact Hausdorff space is normal shows that if X is a Priestley space, whenever A is a closed upset and B is a closed downset with $A \cap B = \varnothing$, then there is a clopen upset V with $A \subseteq V$ and $B \cap V = \varnothing$. So, for each $x \in U$, there is a clopen upset V_x containing x and missing U^c. This gives $x \in V_x \subseteq U$, so $U = \bigcup_{x \in U} V_x$. The V_x are open in ρ_s; thus, compactness of U in ρ_s implies that $U = V_{x_1} \cup \cdots \cup V_{x_n}$ for some $x_i \in U$. Thus, U is clopen in ρ. $\square$

Proposition 6.7. *Let $(X, \rho, \leq)$ be a Priestley space, then (X, ρ_s) is a spectral space.*

Proof. Let $(X, \rho, \leq)$ be Priestley. We need to prove that (X, ρ_s) is compact, T_0, has a basis of compact open sets, the set $\mathcal{CO}(X, \rho_s)$ of compact opens is closed under finite intersections, and (X, ρ_s) is sober. Compactness and T_0 are easy: First, if $\{V_i\}$ is a cover of X with each $V_i \in \rho_s$, then each V_i is open in ρ. Since (X, ρ) is compact, there is a finite subcover. For T_0, if $x \neq y$, suppose that $x \not\leq y$. By the Priestley separation axiom, there is a clopen upset V containing x but not y. This set is open in ρ_s, which is what we need. Next, we claim that an open upset is the union of clopen upsets, which says that an open set in ρ_s is the union of compact open sets in ρ_s; this latter statement follows from Lemma 6.6. To prove the claim, if U is an open upset, then U^c is an open downset. For each $x \in U$, the Priestley separation axiom yields a clopen upset V_x containing x and missing U^c. Then $V_x \subseteq U$. The set U is the union of the V_x, which means U is a union of compact opens in ρ_s. So, the claim is proved. Closure of $\mathcal{CO}(X, \rho_s)$ under finite intersections is also easy by Lemma 6.6 since the finite intersection of clopen upsets is a clopen upset. Finally, to prove that (X, ρ_s) is sober, let C be a closed irreducible subset of X with respect to ρ_s. Consider the collection $\{U_i : i \in I\}$ of all compact open sets in ρ_s which intersect C. This collection consists of clopen sets in ρ. We show that it has the FIP. If $U_1 \cap \cdots \cap U_n = \varnothing$, then $C = (U_1^c \cap C) \cup \cdots \cup (U_n^c \cap C)$. However, this is a finite union of closed subsets of C. By irreducibility, $C = U_i^c \cap C$ for some i. However, this forces $C \cap U_i = \varnothing$, a contradiction to the assumption. By compactness of (X, ρ), the intersection of the collection is nonempty; let x lie in the intersection. Thus, for every compact open U in ρ_s intersecting C, we have $x \in C$. We claim that $C = \overline{\{x\}}$. All we need to see is that every open set W in ρ_s which meets C contains x. However, any such W is the union of compact opens; this is because W is an open upset in ρ, and so is the union of clopen upsets (see the proof above), and so is the union of compact opens in ρ_s, by Lemma

6.6. Thus, if W meets C, then some compact open subset meets C, and then contains x. So, $x \in W$. This proves $C = \overline{\{x\}}$. Therefore, (X, ρ_s) is spectral. $\square$

Lemma 6.8. *Let $(X, \rho, \leq_X)$ and $(Y, \phi, \leq_Y)$ be Priestley spaces. If $f : X \to Y$ is a Priestley morphism, then $f : (X, \rho_s) \to (Y, \phi_s)$ is a spectral map.*

Proof. Let $f : X \to Y$ be a Priestley map. Then f^{-1} sends upsets to upsets and open sets to open sets. So, if V is an open upset in Y, then $f^{-1}(V)$ is an open upset in X. Thus, f is continuous with respect to the spectral topologies on X and Y. $\square$

Corollary 6.9. *$\mathcal{S}$ is a functor from* **PS** *to* **Spec**.

Proof. This follows by Proposition 6.7 and Lemma 6.8. $\square$

Proposition 6.10. *If $(X, \rho, \leq)$ is a Priestley space, then $\rho = (\rho_s)^\#$ and $\leq$ is the specialization order for ρ_s.*

Proof. Let $(X, \rho, \leq)$ be Priestley. We first prove that $\leq$ is the specialization order for ρ_s. That is, we need to show that $x \leq y$ if and only if $x \in \overline{\{y\}}$, where the closure is measured with respect to ρ_s. First, suppose that $x \leq y$. Let U be an open neighborhood of x in ρ_s. Then U is an open upset in ρ. Thus, $y \in U$. This proves $x \in \overline{\{y\}}$. Conversely, suppose that $x \not\leq y$. Then there is a clopen upset V with $x \in V$ and $y \notin V$. Then V is an open set in ρ_s, and so $x \notin \overline{\{y\}}$. Thus, $\leq$ is the specialization order for ρ_s.

We now prove that $\rho = (\rho_s)^\#$. A basis for $(\rho_s)^\#$ is the collection of compact open sets in ρ_s and their compliments. Any compact open in ρ_s is clopen in ρ by Lemma 6.6. Thus, the basic open sets of $(\rho_s)^\#$ are all open in ρ. This proves that $(\rho_s)^\# \subseteq \rho$. For the converse, it is enough to prove that each clopen set in ρ is also in $(\rho_s)^\#$ since ρ has a basis of clopens. So, let U be a clopen in ρ. Let $x \in U$, and let $y \in U^c$. If $x \not\leq y$, then there is a clopen upset V_y with $x \in V_y$ and $y \notin V_y$. If $x < y$, then $y \not\leq x$, so there is a clopen upset W_y with $y \in W_y$ and $x \notin W_y$. We set $V_y = W_y^c$. Then, for each $y \in U^c$, we have a clopen set V_y which is either an upset or a downset, and with $x \in V_y$ and $y \notin V_y$. The collection $\left\{V_y^c : y \in U^c\right\}$ is an open cover of U^c. Since U^c is closed and (X, ρ) is compact, there are y_i with $U^c \subseteq V_{y_1}^c \cup \cdots \cup V_{y_n}^c$. Thus, $V_{y_1} \cap \cdots \cap V_{y_n} \subseteq U$. Moreover, this intersection contains x and is open in $(\rho_s)^\#$. Therefore, for each $x \in U$ we have produced an open neighborhood, with respect to $(\rho_s)^\#$, containing x and contained in U. This proves that U is open in $(\rho_s)^\#$. This finishes the proof that $\rho = (\rho_s)^\#$. $\square$

Corollary 6.11. *The functors $\mathcal{P}$ and $\mathcal{S}$ are inverses to each other. Thus,* **Spec** *and* **PS** *are isomorphic categories.*

Proof. The functor $\mathcal{S} \circ \mathcal{P} = \mathrm{id}_{\mathbf{Spec}}$ by Proposition 6.2 and since both $\mathcal{S}$ and $\mathcal{P}$ send maps to themselves. Also, $\mathcal{P} \circ \mathcal{S} = \mathrm{id}_{\mathbf{PS}}$ by Proposition 6.10. $\square$

By considering the composition of functors **BDL** $\to$ **PS** $\to$ **Spec** and **Spec** $\to$ **PS** $\to$ **BDL**, we rephrase Priestley duality in terms of spectral spaces.

Theorem 6.12. *The functors $\mathcal{PF} \circ \mathcal{S}$ and $\mathcal{CU} \circ \mathcal{P}$ give a co-equivalence of categories between* **BDL** *and* **Spec***.*

Proof. This follows from Theorem 4.6 and the previous corollary. $\square$

Remark 6.13. We point out that the isomorphism $\mathbf{PS} \cong \mathbf{Spec}$ given above can be extended. We call a quasi-ordered topological space (X, τ, R) a quasi-Priestley space if (X, τ) is compact and 0-dimensional, R satisfies the Priestley separation axiom, and if U is an open set and $x \in U$, then for any y with xRy and yRx, we have $y \in U$. Next, a topological space X is quasi-spectral if it is compact, the set of compact open subsets is a basis and is closed under finite intersections, and X is *supersober*. This latter notion means that for every ultrafilter $\mathfrak{p}$, there is $x \in X$ with $\bigcap_{A \in \mathfrak{p}} \overline{A} = \overline{\{x\}}$. It is not hard to prove that a spectral space is quasi-spectral. In fact, if X is a topological space, then the T_0-reflection of X is the quotient X/E, where xEy if $\overline{\{x\}} = \overline{\{y\}}$. It then is not hard to show that X is quasi-spectral if and only if the T_0-reflection of X is spectral. The functors described above yield an isomorphism of categories between quasi-spectral spaces with spectral maps and quasi-Priestley spaces with continuous order-preserving maps. The the final condition in the definition of quasi-Priestley is needed in the proof of Proposition 6.10; in the second paragraph of the argument, we have $x \in U$ and $y \in U^c$. If our relation is a partial order, then either $x \not\leq y$ or $y \not\leq x$. However, we cannot say this, in general, for a quasi-order, since we may have xRy and yRx. However, the assumption of quasi-Priestley rules these out when $x \in U$ and $y \in U^c$.