Group Extensions and H^3

In this note we describe the theory of group extensions in full generality. We see how the cohomology groups $H^2(G, Z(A))$ and $H^3(G, Z(A))$ arise in describing extensions of A by G. Let G and A be groups. As earlier, an extension of A by G is a short exact sequence

$$1 \to A \to E \to G \xrightarrow{\pi} 1.$$

We view $A \subseteq E$; therefore, A is a normal subgroup of E since $A = \ker(\pi)$. We go through ideas analogous to those when we considered A Abelian and see what is different. For $g \in G$ choose $x_g \in E$ with $\pi(x_g) = g$. Note that x_g is determined only up to multiplication by an element of $\ker(\pi) = A$. Since A is normal in E, conjugation by x_g restricts to an automorphism of A. In other words, we have an automorphism ω_g of A given by $\omega_g(a) = x_g a x_g^{-1}$. The function $\omega : G \to \mathrm{Aut}(A)$ need not be a group homomorphism (nor the function $x \mapsto x_g$); this is different than the case when A is Abelian. In fact, $\omega_g \omega_h \omega_{gh}^{-1}$ is conjugation by $x_g x_h x_{gh}^{-1}$, which need not be the identity on A. However, we set $f(g, h) = x_g x_h x_{gh}^{-1}$, which is an element of A since $\pi(x_g x_h x_{gh}^{-1}) = 1$. Thus, we have a function $\omega : G \to A$ and a function $f : G \times G \to A$. These functions are tied together by the formula

$$\omega_g \omega_h = \mathrm{Int}(f(g, h))\omega_{gh}. \tag{1}$$

Here we write $\mathrm{Int}(a)$ for the inner automorphism $t \mapsto ata^{-1}$. Furthermore, associativity in E leads to the generalized cocycle condition

$$f(g, h)f(gh, k) = \omega_g(f(h, k))f(g, hk). \tag{2}$$

To get another view of the function ω, we consider the *outer automorphism group*

$$\mathrm{Out}(A) = \mathrm{Aut}(A)/\,\mathrm{Int}(A),$$

where $\mathrm{Int}(A) = \{\mathrm{Int}(a) : a \in A\}$ is the normal subgroup of inner automorphisms. Thus, equation (1) leads to a group homomorphism $\omega : G \to \mathrm{Out}(A)$; we write ω for this function for convenience. Thus, given a group extension, we get a pair (ω, f) of functions satisfying Equations (1) and (2) above, and that ω induces a group homomorphism $G \to \mathrm{Out}(A)$. We can define an equivalence relation on such pairs based on changing the x_g to $y_g = a_g x_g$ for $a_g \in A$. With this new change, if we define (ω', f') by $\omega'_g = \mathrm{Int}(a_g x_g) = \mathrm{Int}(a_g)\omega_g$ and

$f'(g,h) = y_g y_h y_{gh}^{-1}$, then we get $f'(g,h) = a_g \omega_g(a_h) f(g,h) a_{gh}^{-1}$, as is easy to check. Thus, these formulas lead to an equivalence relation on such pairs: two pairs (ω, f) and (ω', f') are equivalent if there are $a_g \in A$ with $\omega'_g = \mathrm{Int}(a_g)\omega_g$ and $f'(g,h) = a_g \omega_g(a_h) f(g,h) a_{gh}^{-1}$. This relation is an equivalence relation, and by similar arguments to those in Chapter 6.6 of [2], there is a 1–1 correspondence between equivalence classes of extensions of A by G and equivalence classes of pairs (ω, f) satisfying the relations above. We refer to the pair (ω, f) as a generalized cocycle.

Example 1. Let $\omega : G \to \mathrm{Out}(A)$ be a homomorphism. For ease of notation we write $\omega : G \to \mathrm{Aut}(A)$ for any function lifting ω. So, for each $g \in G$ we have $\omega_g \in \mathrm{Aut}(A)$. If we set $f(g,h) = 1$ for all g, h, then (ω, f) is a generalized cocycle only when $\omega : G \to \mathrm{Aut}(A)$ is a group homomorphism. For, the condition $\omega_g \omega_h = \mathrm{Int}(f(g,h))\omega_{gh}$ and $f(g,h) = 1$ forces $\omega_g \omega_h = \omega_{gh}$. This indicates that we may not have any generalized cocycles at all; in fact, this can happen. If $\omega : G \to \mathrm{Aut}(A)$ is a group homomorphism and $f(g,h) = 1$ for all g, h, then (ω, f) corresponds to the semidirect product of A and G with respect to ω.

Example 2. Consider the group extension $1 \to A_n \to S_n \stackrel{\mathrm{sgn}}{\to} \mathbb{Z}_2 \to 1$. We may choose the transposition (12) for $x_{\overline{1}}$ and id for $x_{\overline{0}}$. With these choices, we see that, as $(12)^2 = \mathrm{id}$, that the corresponding function $\omega : \mathbb{Z}_2 \to \mathrm{Aut}(A_n)$ is a group homomorphism. Moreover, the cocycle f is trivial. This tells us that S_n is the semidirect product of A_n and $\langle (12) \rangle$.

Let A and G be groups and suppose there is a group homomorphism $\omega : G \to \mathrm{Out}(A)$. There are two natural questions. First, when is there a group extension of A by G inducing the map ω? Second, if there is an extension of A by G inducing ω, can we classify extensions of A by G with a more understandable object than the set of equivalence classes of generalized cocycles?

To answer the first question, we choose, for each $g \in G$, a lift $\xi_g \in \mathrm{Aut}(A)$ of ω_g. In other words, the coset of ξ_g in $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Int}(A)$ is equal to ω_g. Note that we may choose $\xi_g = \mathrm{id}$ when $g = 1$. Then $\xi_g \xi_h \xi_{gh}^{-1} \in \mathrm{Int}(A)$ since it is a lift of $\omega_g \omega_h \omega_{gh}^{-1} = 1 \in \mathrm{Out}(A)$. We choose an element $f(g,h) \in A$ with $\xi_g \xi_h \xi_{gh}^{-1} = \mathrm{Int}(f(g,h))$; we may choose $f(g,h) = 1$ if $g = 1$ or $h = 1$ since in either case $\xi_g \xi_h \xi_{gh}^{-1} = \mathrm{id}$. If (ω, f) is a generalized cocycle, then we may produce an extension E of A by G as follows. We set $E = A \times G$ as sets, and define an operation on E by

$$(a, g)(b, h) = (a\omega_g(b) f(g,h), gh).$$

An elementary calculation shows that E is indeed a group. Furthermore, the maps $a \mapsto (a, 1)$ and $(a, g) \mapsto g$ are group homomorphisms, and $1 \to A \to E \to G \to 1$ is an extension of A by G; we need to choose $\xi_1 = \mathrm{id}$ and $f(g,h) = 1$ if $g = 1$ or $h = 1$ in order to show that the map $A \to E$ is a group homomorphism. However, f may not satisfy the generalized cocycle condition. What we can say is this:

$$\mathrm{Int}(f(g,h)f(gh,k)) = \xi_g \xi_h \xi_{gh}^{-1} \xi_{gh} \xi_k \xi_{ghk}^{-1} = \xi_g \xi_h \xi_k \xi_{ghk}^{-1}.$$

Also,

$$\begin{aligned}\mathrm{Int}(\omega_g(f(h,k))f(g,hk)) &= \xi_g(\xi_h\xi_k\xi_{hk}^{-1}\xi_g^{-1})\xi_g\xi_{hk}\xi_{ghk}^{-1} \\ &= \xi_g\xi_h\xi_k\xi_{ghk}^{-1}.\end{aligned}$$

Therefore, $f(g,h)f(gh,k)$ and $\omega_g(f(h,k))f(g,hk)$ conjugate A in the same manner. Since $\mathrm{Int}(a) = \mathrm{Int}(b)$ if and only if $a \equiv b \bmod Z(A)$, where $Z(A)$ is the center of the group A, we see that there is an element $c(g,h,k) \in Z(A)$ satisfying

$$f(g,h)f(gh,k) = c(g,h,k)\omega_g(f(h,k))f(g,hk).$$

A messy calculation shows that c is a 3-cocycle; it then represents an element of $H^3(G,Z(A))$. Furthermore, different choices of the ξ_g and $f(g,h)$ correspond to changing c to an equivalent cocycle in $H^3(G,Z(A))$. Therefore, the cocycle class of c is uniquely determined in $H^3(G,Z(A))$.

Proposition 3. *Let c be the 3-cocycle defined above. Then there is a group extension of A by G inducing the map $\omega : G \to \mathrm{Out}(A)$ if and only if $c = 0$ in $H^3(G,Z(A))$.*

Proof. If c is a 3-coboundary, then there are elements $a_{g,h} \in Z(A)$ with

$$c(g,h,k) = g(a_{h,k})a_{gh,k}^{-1}a_{g,hk}a_{g,h}^{-1}.$$

We may then replace $f(g,h)$ by $f'(g,h) = f(g,h)a_{g,h}^{-1}$; since $a_{g,h}$ is central, $\mathrm{Int}(f'(g,h)) = \mathrm{Int}(f(g,h))$. With this change, one can calculate that the resulting pair (ω, f') is a generalized cocycle, and so we can use it to produce a group extension of A by G that induces the map $\omega : G \to \mathrm{Out}(A)$. Conversely, if we have a group extension of A by G that induces (ω, f), then this pair is a generalized cocycle, so we may choose $c(g,h,k) = 1$ for all triples (g,h,k). Thus, $c = 0$ in $H^3(G,Z(A))$. □

By the previous proposition, given $\omega : G \to \mathrm{Out}(A)$, we have an "obstruction" in $H^3(G,Z(A))$ whose triviality determines when there is a group extension of A by G inducing ω. We next assume that there is a group extension of A by G inducing a given homomorphism $\omega : G \to \mathrm{Out}(A)$, and we show that $H^2(G,Z(A))$ classifies all such extensions.

Proposition 4. *Let $\omega : G \to \mathrm{Out}(A)$ be a homomorphism. If there is a group extension of A by G, then $H^2(G,Z(A))$ classifies the group extensions of A by G that induce ω.*

Proof. The equivalence classes of group extensions of A by G that induce the map ω are the generalized cocycles of the form (ω, f); we are using ω for both the map $G \to \mathrm{Out}(A)$ and for a lift $\omega : G \to \mathrm{Aut}(A)$. Since we are assuming there is an extension of A by G, there is a generalized cocycle (ω, f_0) corresponding to it. We define a map from $H^2(G,Z(A))$ to the set of equivalence classes of generalized cocycles by $c \mapsto (\omega, cf_0)$. An elementary calculation shows that this pair is indeed a generalized cocycle. It is also not hard to show that this is well defined; if c and c' are cocycles representing the same class in $H^2(G, Z(A))$, then (ω, cf_0) and $(\omega, c'f_0)$ are equivalent. To show surjectivity, let (ω, f) be a generalized cocycle. Then

$$\mathrm{Int}(f(g,h) = \omega_g \omega_h \omega_{gh}^{-1} = \mathrm{Int}(f_0(g,h))$$

for any pair (g, h). So, since the elements $f(g,h)$ and $f_0(g,h)$ induce the same inner automorphism on A, there is an element $c(g,h) \in Z(A)$ with $f(g,h) = c(g,h)f_0(g,h)$. Using that both f and f_0 satisfy the generalized cocycle condition, and that $c(g,h) \in Z(A)$, we see that c is a 2-cocycle. This proves that (ω, f) is equivalent to (ω, cf_0). For injectivity, suppose that (ω, cf_0) and $(\omega, c'f_0)$ are equivalent. Then there are $a_g \in A$ with

$$\begin{aligned} \omega_g &= \mathrm{Int}(a_g)\omega_g, \\ c(g,h)f_0(g,h) &= a_g\omega_g(a_h)c'(g,h)f_0(g,h)a_{gh}^{-1}. \end{aligned}$$

The first equation says that $\mathrm{Int}(a_g) = \mathrm{id}$; this forces $a_g \in Z(A)$. The second equation then can be written as

$$c(g,h) = a_g g(a_h) a_{gh}^{-1} c'(g,h),$$

which shows that c and c' are equal in $H^2(G, Z(A))$. This finishes the proof. □

Example 5. Let A be a group with $Z(A) = 1$. Then $\mathrm{Int}(A) \cong A$. Via this identification, we have a group extension $1 \to A \to \mathrm{Aut}(A) \to \mathrm{Out}(A) \to 1$. If we set $G = \mathrm{Out}(A)$, then the homomorphism $\omega : G \to \mathrm{Out}(A)$ induced by this extension is the identity map. Since $H^n(G, Z(A)) = 0$ for all n by the assumption that $Z(A) = 1$, by the two propositions, we see that this extension is the unique extension of A by G that induces the map ω.

Example 6. Let $E = \mathrm{Aut}(S_6)$. It is known, and a fairly easy calculation to prove, that $Z(S_6) = 1$, so $\mathrm{Int}(S_6) \cong S_6$. Furthermore, by viewing $S_6 \subseteq \mathrm{Aut}(S_6)$, the alternating group A_6 is actually normal in E. So, we have a group extension $1 \to A_6 \to \mathrm{Aut}(S_6) \to G \to 1$ for some G. We have $Z(A_6) = 1$ since the center is a normal subgroup of a group and A_6 is simple. So, this extension is the unique, up to equivalence, extension of A by G (inducing the same map $G \to \mathrm{Out}(A_6)$ as the given extension). However, it is also known (see [1, Cor. 3.9]) that this extension is not split. Therefore, even when there is only one extension of A by G, the middle group need not be a semidirect product of A and G.

References

[1] T. Y. Lam and D. B. Leep, *Combinatorial structure on the automorphism group of S_6*, Expo. Math. **11** (1993), 289–308.

[2] Charles A. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.