

Locally Free Sheaves

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In these talks we will discuss several important examples of locally free sheaves and see the connection between locally free sheaves and finitely generated projective modules. In addition, we will see the connection between the divisor class group and the Picard group (aka ideal class group) of a domain. The bulk of this talk is taken from Sections 5 and 6 of Chapter II of [3]. Proofs of the various facts about commutative rings can be found in one or more of the following sources: Atiyah and Macdonald [1], Eisenbud [2], or Kunz [4].

1 Motivations

In this section we consider three situations, each of which gives rise to locally free sheaves. We describe these situations in some detail.

1.1 The ideal class group of an integral domain

Let A be an integral domain, and let F be its quotient field. A nonzero A -submodule I of F is said to be a *fractional ideal* if there is a nonzero $a \in A$ with $aI \subseteq A$. We note that a finitely generated A -submodule of F is a fractional ideal, as is any ordinary ideal of A . In particular, if $a \in F^*$, then aA is a fractional ideal. For I a fractional ideal, define $I^{-1} = \{x \in F : xI \subseteq A\}$. Then it is easy to see that I^{-1} is a nonzero A -submodule of F , and that it is also a fractional ideal since $xI^{-1} \subseteq A$ for any nonzero $x \in I$. We also have $II^{-1} \subseteq A$. We say that I is *invertible* if $II^{-1} = A$. As an example, we note that $(a)^{-1} = (a^{-1})$, so any principal (fractional) ideal is invertible. However, there can be invertible ideals that are not principal. This notion is of particular importance for Dedekind domains, since an integral domain is a Dedekind domain if and only if every fractional ideal is invertible [1, Theorem 9.8]. We define the ideal class group $\text{ICl}(A)$ by

$$\text{ICl}(A) = \{\text{invertible fractional ideals}\} / \{\text{principal fractional ideals}\},$$

and define a group operation by ordinary multiplication of ideals. The idea of the ideal class group is to measure how far A is from being a PID. If A is a Dedekind domain, then every fractional ideal is invertible. Thus, a Dedekind domain A is a PID if and only if $\text{ICl}(A) = 0$.

Moreover, since a Dedekind domain is a PID if and only if it is a UFD, we see that a Dedekind domain is a UFD exactly when its class group is 0. We will see later that the ideal class group helps to determine when a more general integral domain A is a UFD.

We now give a module-theoretic interpretation of invertible ideal. Suppose that I is an invertible ideal. Then $II^{-1} = A$, so $1 = \sum_i x_i y_i$ with $x_i \in I$ and $y_i \in I^{-1}$. We see that $I = (x_1, \dots, x_n)$ since for any $a \in A$, we have $a = \sum_i x_i (y_i a)$, and if $a \in I$, then each $y_i a \in II^{-1} \in A$. Furthermore, let $f_i : I \rightarrow A$ be the A -module homomorphism $t \mapsto y_i t$. We then have $a = \sum_i f_i(a) x_i$ for all $a \in I$. Recall that this set of pairs (f_i, x_i) is said to be a projective basis for I ; more generally, if M is an A -module, a set $\{(f_i, x_i) : 1 \leq i \leq n\}$ with $x_i \in M$ and $f_i \in \text{hom}_A(M, A)$ is a (finite) projective basis if $a = \sum_i f_i(a) x_i$ for each $a \in M$. An A -module M has a finite projective basis if and only if it is finitely generated and projective. Without giving the full argument, we indicate why this is true. If we have a projective basis for a module M , then we define a map $A^n \rightarrow M$ by $(a_1, \dots, a_n) \rightarrow \sum a_i x_i$. This map is surjective since the x_i generate M . To produce a one-sided inverse, we define $M \rightarrow A^n$ by $m \mapsto (f_1(m), \dots, f_n(m))$. Then we have $m \mapsto (f_1(m), \dots, f_n(m)) \mapsto \sum_i f_i(m) x_i = m$, which shows that this map splits $A^n \rightarrow M$, and so M is a summand of A^n , which shows that M is projective. Conversely, if M is a direct summand of A^n , let f_i be the restriction to M of the i -th projection map $A^n \rightarrow A$. Writing $A^n = M \oplus N$, if x_i is the first component of the i -th standard basis vector of A^n , then a short argument shows that $\{(f_i, x_i) : 1 \leq i \leq n\}$ is a projective basis of M .

From this we see that if I is invertible, then it is a finitely generated projective A -module of rank 1. Conversely, if I is a finitely generated projective A -module of rank 1, then we may view $I \subseteq F$ since $I \otimes_A F \cong F$. We then view I as a fractional ideal. If $\{(f_i, x_i)\}$ is a projective basis for I , then $f_i : I \rightarrow A$ are A -homomorphisms. Then $f_i \otimes \text{id} : I \otimes_A F \rightarrow A \otimes_A F$ is an F -vector space homomorphism. By identifying both as F , we see that this map is multiplication by some $y_i \in F$. Then $f_i(x) = y_i x$ for all $x \in I$. This forces each $y_i \in I^{-1}$. We then see that, for all $a \in I$, we have $a = \sum_i f_i(a) x_i = \sum_i a y_i x_i$. Therefore, $1 = \sum_i x_i y_i$. Thus, I is invertible. We have thus shown that a fractional ideal I is invertible if and only if it is a finitely generated projective A -module of rank 1. More generally, if I is a nonzero projective A -submodule of F , then we claim that I is invertible. For, if $\{(f_i, x_i) : i \in I\}$ is a projective basis, take $a \in I$ with $a \neq 0$. Then $a = \sum_i f_i(a) x_i$, and where the sum is finite. If we relabel those x_i that show up in the sum as x_1, \dots, x_n , and we write $f_i(a) = y_i a$ for some $y_i \in I^{-1}$, then $a = \sum_i a y_i x_i$, and so $1 = \sum_i y_i x_i$. It then follows that $I = (x_1, \dots, x_n)$ since if $b \in I$, then $b = \sum (b y_i) x_i \in (x_1, \dots, x_n)$.

We state what we have just done as a theorem.

Theorem 1.1. *Let A be an integral domain. Then a fractional ideal I is invertible if and only if I is a projective A -module. When this occurs, I is finitely generated.*

This connection yields another interpretation of $\text{Cl}(A)$. We define the *Picard group* $\text{Pic}(A)$ to be the group of isomorphism classes of rank 1 projective A -modules, with operation induced by $(P, Q) \mapsto P \otimes_A Q$. We note that the tensor product of two projective modules is

projective, which can be seen easily, for finitely generated modules, by writing $P \oplus P' = A^n$ and $Q \oplus Q' = A^m$, and then noting that

$$\begin{aligned} A^{nm} &= (P \oplus P') \otimes_A (Q \oplus Q') \\ &= P \otimes_A Q \oplus (P' \otimes_A (Q \oplus Q')) \oplus P \otimes_A Q', \end{aligned}$$

so $P \otimes_A Q$ is a direct summand of a finitely generated free module. The inverse of a rank 1 projective P is $\text{hom}_A(P, A)$. It is a short proof using projective bases to show that $P \otimes_A \text{hom}_A(P, A) \cong A$, under the map $p \otimes f \mapsto f(p)$, if P is a finitely generated projective module of rank 1.

For another interpretation, we can talk about the divisor class group of a Noetherian domain A . First, assume that, for each height 1 prime ideal P , that A_P is a UFD. Then A_P is an integrally closed local Noetherian domain of dimension 1, so it is a DVR. Such a ring is called *locally factorial*. Denote by v_P the valuation associated to A_P . Let $\text{Div}(A)$ be the free abelian group on the height 1 prime ideals of A . Elements of $\text{Div}(A)$ are called *Weil divisors* on A . If $a \in A$ is nonzero, define $(a) = \sum_P v_P(a)P$. Then this is well defined (the sum is not infinite), and is an element of $\text{Div}(A)$. The set of all these principal divisors is a subgroup $P(A)$ of $\text{Div}(A)$, and the quotient is the *divisor class group* $\text{Cl}(A)$ of A . If A is a Dedekind domain, then the divisor class group is isomorphic to the Picard group. Moreover, since ideals factor uniquely into products of prime ideals, we see that the group of fractional ideals is the free Abelian group on the nonzero prime ideals of A . This will allow us to help understand the connection between the class group and Weil divisors. We will see the connection between the divisor class group and the Picard group later.

One of the main points of these talks is to make the connection between finitely generated projective modules over a ring A and locally free sheaves over $\text{Spec}(A)$.

1.2 The group law on an elliptic curve

One of the classical facts of number theory is the group law on the set of (rational) points on an elliptic curve. A nice reference for this material is the book by Silverman and Tate [6]. To simplify things a bit we work over an algebraically closed field k of characteristic not 2. Roughly speaking, an elliptic curve E is a curve in projective space \mathbf{P}^2 given by an affine polynomial equation of the form $y^2 = f(x)$ for some cubic $f(x)$ with no repeated roots. Alternatively, it is the set of solutions to $y^2 = f(x)$, together with an extra point, the point at infinity.

Geometrically, one adds points in the following way. Let $P_\infty = (0 : 1 : 0)$ be the point at infinity. Since E is a cubic curve, a line will intersect E in three points, counting multiplicity. If P, Q are points on the curve, let R be the third point of intersection with the line passing through P and Q . Then $P + Q$ is the point on the curve that is also on the line passing through P_∞ and R . While it is tedious to do so, one can show that this defines a group structure on E whose identity is P_∞ . A more detailed, but not complete, description of how to verify this can be found in Chapter 1.2 of [6].

For example, consider the elliptic curve E with affine formula $y^2 = x^3 + 17$. Homogenizing this gives the projective equation $y^2z = x^3 + 17z^3$. From this we see that $(0 : 1 : 0)$ is indeed on the curve. We can easily check that $P = (-1 : 4 : 1)$ and $Q = (2 : 5 : 1)$ are on the curve. Computing the line through these two points yields the affine equation $y = \frac{1}{3}x + \frac{13}{3}$, or the homogeneous equation $x - 3y + 13z = 0$. A short calculation yields the third point $R = (-8/9 : 109/27 : 1)$. Now, if we consider the line through R and $P_\infty = (0 : 1 : 0)$, the line must be vertical; that is, the y coefficient is 0. Thinking about what this means, we quickly note that $(a : b : 1) \in E \implies (a : -b : 1) \in E$ implies that the third point on the curve that is on this vertical line is then $(-8/9 : 109/27 : 1)$. In other words, under this operation, $(-1 : 4 : 1) + (2 : 5 : 1) = (-8/9 : 109/27 : 1)$. Similarly, if we want to find $2P$, we find the tangent line at P . This is the line

$$\frac{\partial f}{\partial x}(P)(x+1) + \frac{\partial f}{\partial y}(P)(y-4) = 0,$$

which simplifies to $-3x + 8y = 35$. If we solve simultaneously this and $f = 0$, we get the point $(137/64, 2651/512)$, yielding $2P = (137/64, -2651/512)$.

After we define the Picard group for a variety, we will see that $E \cong \text{Pic}_0(E)$, the classes of degree 0 in $\text{Pic}(E)$. This will give a different, and perhaps more algebraic, interpretation of the group law.

1.3 The problem of Riemann

Let X be a nonsingular curve over k , and let K be its function field, the field of rational functions on k . Elements of K are then interpreted as functions defined on an open set of X . If $f \in K$ and $P \in X$, we say that P is a zero of f if $f(P) = 0$ and that P is a pole of f if $f(P)$ is not defined. As with polynomials or complex analytic functions, we can define the notion of the order of a zero or a pole. This is not so trivial to do without the theory of discrete valuation rings, so we use this notion without defining it. Roughly, what happens is that for every P there is a rational function g_P such that $g_P(P) = 0$, and if f is a rational function with $f(P) = 0$, then $f = g_P h$ for some h defined at P . Thus, to define order, P is a zero of order n at f if $f = g_P^n h$ for some h with $h(P) \neq 0$. In terms of discrete valuation rings, each local ring \mathcal{O}_P of X is a DVR since X is nonsingular. The function g_P is nothing more than a uniformizer for \mathcal{O}_P ; that is, g_P is an element of value 1 in \mathcal{O}_P . If v_P is the valuation on K corresponding to \mathcal{O}_P , we may define v_P in an ad-hoc way by

$$v_P(f) = \begin{cases} n > 0 & \text{if } P \text{ is a zero of } f \text{ of order } n \\ -n < 0 & \text{if } P \text{ is a pole of } f \text{ of order } n \\ 0 & \text{if } P \text{ is neither a zero nor a pole of } f. \end{cases}$$

The problem of Riemann can be described as follows. Let $P_1, \dots, P_n, Q_1, \dots, Q_m$ be points on X . Also, let $a_1, \dots, a_n, b_1, \dots, b_m$ be positive integers. Determine those functions $f \in K$ which have a zero at P_i of order at least a_i and whose set of poles lie inside $\{Q_1, \dots, Q_m\}$ and such that the order of the pole at Q_i is at most b_i .

To give this problem a convenient framework, we define a *Weil divisor* on X to be a formal integral linear combination of the points of X . The group $\text{Div}(X)$ of divisors on X is then the free abelian group on the points of X . If $D = \sum n_P P$ is a divisor, we define

$$L(D) = \{f \in K^* : v_P(f) + n_P \geq 0\} \cup \{0\},$$

to be the space of D . The problem of Riemann can be phrased as: if $D = -\sum_i a_i P_i + \sum_i b_i Q_i$, then determine $L(D)$. We see that $f \in L(D)$ if and only if P_i is a root of f of order at least a_i , and the only poles of f can be the Q_i , and since $v_{Q_i}(f) + b_i \geq 0$, the order of Q_i is at most b_i . We will see that $L(D)$ is in fact the global sections of a locally free sheaf of rank 1.

If $f \in K^*$, then we may define a *principal divisor* (f) by $(f) = \sum v_P(f)P$. This is a divisor since every function has only finitely many zeros and poles, a fact that is not obvious. The set $P(X)$ of principal divisors is a subgroup of $\text{Div}(X)$ since $(f) + (g) = (fg)$ and $-(f) = (f^{-1})$. We then define the *divisor class group* $\text{Cl}(X) = \text{Div}(X)/P(X)$. If $X = \text{Spec}(A)$, then it is easy to see that $\text{Cl}(X) = \text{Cl}(A)$. We will see how this group is related to the ideal class group of an integral domain below.

2 Introduction to Sheaves

Let X be a topological space. A *presheaf* \mathcal{F} on X is an association such that for every open set U of X , we have an Abelian group (or ring, or module, etc.) $\mathcal{F}(U)$ together with “restriction” maps φ_{UV} for every pair of open sets $V \subseteq U$ satisfying (i) $\mathcal{F}(\emptyset) = (0)$, (ii) $\varphi_{UU} = \text{id}_{\mathcal{F}(U)}$, and (iii) If $W \subseteq V \subseteq U$ are open, then $\varphi_{UW} = \varphi_{VW} \circ \varphi_{UV}$. We will write $f|_V$ for $\varphi_{UV}(f)$ and think about it as restricting f to V . For another way to view presheaves, let $\mathfrak{Top}(X)$ be the category where the objects are open sets in X , and $\text{hom}(U, V) = \emptyset$ unless $V \subseteq U$, and $\text{hom}(U, V) = \{\text{res}_{UV}\}$ if $V \subseteq U$. A presheaf on X is then a contravariant functor from $\mathfrak{Top}(X)$ to the category of Abelian groups. A presheaf \mathcal{F} is a *sheaf* if the following two properties hold: for every open set U and every open cover $\{V_\alpha\}$ of U , we have (iv) if $f \in \mathcal{F}(U)$ with $f|_{V_\alpha} = 0$ for all α , then $f = 0$, and (v) if $g_\alpha \in \mathcal{F}(V_\alpha)$ with $g_\alpha|_{V_\alpha \cap V_\beta} = g_\beta|_{V_\alpha \cap V_\beta}$, then there is an $f \in \mathcal{F}(U)$ with $f|_{V_\alpha} = g_\alpha$ for all α .

If \mathcal{F} is a presheaf on X , then the stalk \mathcal{F}_P of \mathcal{F} at a point $P \in X$ is the direct limit

$$\mathcal{F}_P = \varinjlim \mathcal{F}(U),$$

where the limit is taken over all open sets containing P . Besides the functorial properties of stalks, two of the main things to know about them is that there are maps $\mathcal{F}(U) \rightarrow \mathcal{F}_P$ for every U containing P , where we write the image of $s \in \mathcal{F}(U)$ in \mathcal{F}_P as s_P , such that \mathcal{F}_P is the union of the images of the $\mathcal{F}(U)$, and if $s \in \mathcal{F}(U)$ satisfies $s_P = 0$, then there is an open set V with $P \in V \subseteq U$ with $s|_V = 0$.

Example 2.1. Let X be a topological space, and set $\mathcal{O}_X(U)$ be the set of continuous functions from U to \mathbb{R} (or \mathbb{C}). If $V \subseteq U$ are open in X , we let φ_{UV} be the usual restriction map. It is then clear that \mathcal{O}_X is a sheaf.

Example 2.2. Let X be an algebraic variety. Set $\mathcal{O}_X(U)$ to be the ring of functions defined and regular on U . Then \mathcal{O}_X is a sheaf, and is called the structure sheaf on X .

Example 2.3. If \mathcal{F} is a presheaf on X , then we may define a sheaf \mathcal{F}^+ , called the *sheafification* of \mathcal{F} . This sheaf comes together with a presheaf morphism $i : \mathcal{F} \rightarrow \mathcal{F}^+$. If U is open in X , then the group $\mathcal{F}^+(U)$ consists of all functions $f : U \rightarrow \bigsqcup_{P \in X} \mathcal{F}_P$ such that $f(P) \in \mathcal{F}_P$, and for every P there is a neighborhood V of P inside U and an $s \in \mathcal{F}(V)$ such that $f(Q) = s_Q$ for all $Q \in V$. The main property of \mathcal{F}^+ is the following universal mapping property: if $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a presheaf morphism into a sheaf \mathcal{G} , then there is a unique sheaf morphism $\varphi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ such that $\varphi = \varphi^+ \circ i$. Furthermore, for every $P \in X$, the stalks \mathcal{F}_P and \mathcal{F}_P^+ are isomorphic via the map induced by i .

Example 2.4. If X is a topological space, and A is any Abelian group, then the association $U \mapsto A$ is a presheaf, where the restriction maps are just the identity on A . It is not a sheaf in general, although it is a sheaf if X is connected. Its sheafification \mathcal{A} is an example of what is called a constant sheaf.

If \mathcal{F} is a sheaf, then \mathcal{F} is canonically isomorphic to \mathcal{F}^+ . From this it follows that we may assume that the elements of $\mathcal{F}(U)$ are functions on U , and that res_{UV} is the restriction of function map from U to V .

If \mathcal{F} is a sheaf on X such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module for every U , and the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is an $\mathcal{O}(U)$ -module homomorphism for every $V \subseteq U$ (where we view $\mathcal{F}(V)$ as an $\mathcal{O}_X(U)$ -module via the map $\text{res} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$), then we call \mathcal{F} a *sheaf of \mathcal{O}_X -modules*, or an \mathcal{O}_X -module.

Example 2.5. Let X be a topological space, and let \mathcal{O}_X be the sheaf defined above. If $p : E \rightarrow X$ is a vector bundle on X , then we define a sheaf \mathcal{F} on X as follows. If $U \subseteq X$ is open, define

$$\mathcal{F}(U) = \{s : U \rightarrow E : s \circ p = \text{id}, s \text{ is continuous}\}.$$

It is easy to verify, with usual restriction of functions, that \mathcal{F} is a sheaf on X .

To further hint at the connection between locally free sheaves and projective modules, we recall Swan's theorem [5, Theorem 1.6.3]. If X is compact Hausdorff, set $A = \mathcal{O}_X(X)$, the ring of globally defined continuous functions on X . If E is a (locally trivial) bundle on X , then $\mathcal{F}(X)$ is a finitely generated projective A -module. Moreover, $\mathcal{F} \mapsto \mathcal{F}(X)$ induces an isomorphism of categories from the category of vector bundles over X to the category of finitely generated projective A -modules. In fact, what we do later in these talks should lead to a proof of Swan's theorem.

A sheaf homomorphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a collection of group homomorphisms $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) \\ \text{res} \downarrow & & \downarrow \text{res} \\ \mathcal{F}(V) & \xrightarrow{f_V} & \mathcal{G}(V) \end{array}$$

commutes. If, in addition, \mathcal{F} and \mathcal{G} are \mathcal{O}_X -modules and φ_U is a morphism of $\mathcal{O}_X(U)$ -modules for every open set U , then we will call φ a morphism of \mathcal{O}_X -modules. If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves, then we have an induced map $\varphi_P : \mathcal{F}_P \rightarrow \mathcal{G}_P$, given by $\varphi_P(s_P) = \varphi(s)_P$ for $s \in \mathcal{F}(U)$ for some U containing P . One of the main uses of stalks is in determining if a morphism φ of sheaves is bijective (resp. injective or surjective) if and only if the stalk map φ_P is bijective (resp. injective or surjective) for every $P \in X$. This does not hold true for presheaves, since, for example, if \mathcal{F} is a presheaf that is not a sheaf, the morphism $i : \mathcal{F} \rightarrow \mathcal{F}^+$ is not an isomorphism even though $i_P : \mathcal{F}_P \rightarrow \mathcal{F}_P^+$ is an isomorphism for each $P \in X$. The module-theoretic analogue of this property of stalks is that if A is a ring and $\varphi : M \rightarrow N$ is an A -module homomorphism, then φ is an isomorphism if and only if the induced map $\varphi_P : M_P \rightarrow N_P$ is an isomorphism of A_P -modules for each maximal ideal P of A .

Example 2.6. Let $f : X \rightarrow Y$ be a morphism of varieties. This means that for every open set $V \subseteq Y$ and every $\sigma \in \mathcal{O}_Y(V)$, the function $f \circ \sigma$ is regular on $f^{-1}(V)$. Thus, f induces a sheaf homomorphism $\mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$, which is defined by $V \mapsto \mathcal{O}_X(f^{-1}(V))$, and is called the direct image sheaf.

Example 2.7. Let $f : X \rightarrow Y$ be a continuous map of topological spaces. Then f induces a sheaf morphism $\mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ just as in the previous example.

It is clear how to define direct sums of sheaves; we then have the sheaf $\mathcal{O}_X^n = \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X$ for any n . We call a sheaf of \mathcal{O}_X -modules *free* on X of rank n if $\mathcal{F} \cong \mathcal{O}_X^n$ for some n . It is easy to prove that $\mathcal{F} \cong \mathcal{O}_X^n$ if and only if \mathcal{F} is free on a set of n elements in the category of \mathcal{O}_X -modules.

Let \mathcal{F} be a sheaf on a space X , and let U be an open subset of X . The restricted sheaf $\mathcal{F}|_U$ on U is defined by $V \mapsto \mathcal{F}(V)$ for all open subsets of U . We define an \mathcal{O}_X -module \mathcal{F} to be *locally free* if there is an open cover $\{U_\alpha\}$ of X such that $\mathcal{F}|_{U_\alpha}$ is free for every α . If \mathcal{F} is locally free, then the stalks \mathcal{F}_P are free \mathcal{O}_P -modules for every $P \in X$; this is because the stalk at P of \mathcal{F} is equal to the stalk at P of $\mathcal{F}|_{U_\alpha}$ whenever $P \in U_\alpha$, and if $\mathcal{F} = \mathcal{O}_X^n$, then $\mathcal{F}_P = \mathcal{O}_P^n$. Thus, locally free sheaves have free stalks.

3 Examples

In this section we give some important examples of locally free sheaves. Just as the 1-dimensional F -vector space F is the basic building block of all vector spaces, the structure sheaf $\mathcal{O}_X = \mathcal{O}_A$ on $X = \text{Spec}(A)$, where A is a commutative ring, is the basic building block of free sheaves on X . We recall its definition. If U is an open subset of X , then $\mathcal{O}(U)$ is the set of functions $f : U \rightarrow \bigcup_{P \in U} A_P$ such that $f(P) \in A_P$ for all $P \in U$, and for each $P \in U$, there is a neighborhood $V \subseteq U$ of P such that there are $a \in A$ and $s \in \bigcap_{Q \in V} (A - Q)$ such that $f(Q) = a/s \in A_Q$ for each $Q \in V$.

Another way to think about this is that if U is an open subset of X , then we have a multiplicative set $S_U = \bigcap_{P \in U} (A - P)$. The sheaf \mathcal{O}_X is the sheafification of the presheaf $U \mapsto S_U^{-1}A$. In particular, if U is an affine open set in X , then $\mathcal{O}(U) = S_U^{-1}A$.

3.1 The sheaf \widetilde{M}

Let M be an A -module. Set $X = \text{Spec}(A)$. We define a presheaf on X by $U \mapsto M \otimes_A \mathcal{O}(U)$. We refer to the sheafification of this sheaf by \widetilde{M} . Alternatively, we can describe \widetilde{M} in much the same way as we define the structure sheaf on $\text{Spec}(A)$. That is, if U is an open set of X , we set $\widetilde{M}(U)$ to be the functions f from U to the disjoint union $\bigcup_{P \in U} M_P$ such that $f(P) \in M_P$ for each $P \in U$, and for each such P there is a neighborhood $V \subseteq U$ of P such that there is an $m \in M$ and $s \in A - Q$ for each $Q \in V$ such that $f(Q) = m/s \in M_Q$. We leave it as an exercise that we obtain a sheaf this way, where restriction maps are ordinary restriction of functions, and that this is the sheafification of the presheaf M' , given by $U \mapsto M \otimes_A \mathcal{O}(U)$. We note that in the case $M = A$, the structure sheaf \mathcal{O}_A is the sheafification of the sheaf $A(U) = S^{-1}A$, where $S = \bigcap_{P \in U} (A - P)$.

If M and N are A -modules, and if $\varphi : M \rightarrow N$ is an A -module homomorphism, then φ induces a sheaf morphism $\widetilde{M} \rightarrow \widetilde{N}$ as follows. If $U \subseteq \text{Spec}(A)$ is open, define $\widetilde{\varphi} : \widetilde{M}(U) \rightarrow \widetilde{N}(U)$ by $\widetilde{\varphi}(f)(P) = \varphi_P(f(P))$, where we denote by φ_P the induced A_P -homomorphism $M_P \rightarrow N_P$. A bit of tedious calculation shows that $\widetilde{\varphi}$ is a sheaf morphism. Alternatively, φ induces a presheaf morphism $M' \rightarrow N'$, since for $U \subseteq X$, we have the map $\varphi \otimes \text{id} : M \otimes_A \mathcal{O}(U) \rightarrow N \otimes_A \mathcal{O}(U)$. Composing this with the canonical map $i : N' \rightarrow \widetilde{N}$, the universal mapping property gives a map $\widetilde{\varphi} : \widetilde{M} \rightarrow \widetilde{N}$, which is the same as $\widetilde{\varphi}$ described above.

Example 3.1. The reason for the somewhat complicated definition of \widetilde{M} is that the simple idea of the presheaf $U \mapsto M \otimes_A \mathcal{O}(U)$ is not a sheaf, in general. For example, let $A = k[x, y]$ and $X = \text{Spec}(A)$. Let $U = D(x) \cup D(y) = \mathbf{A}^2 - \{(0, 0)\}$. Then $\mathcal{O}(U) = A$. If M is an A -module, then $\widetilde{M}(X) = M$ in addition to $\widetilde{M}(D(x)) = M_x$ and $\widetilde{M}(D(y)) = M_y$. moreover, $D(x) \cap D(y) = D(xy)$, so the module of sections on the intersection is M_{xy} . We then have the diagram

$$\begin{array}{ccc} M & \longrightarrow & M_x \\ \downarrow & & \downarrow \\ M_y & \longrightarrow & M_{xy} \end{array}$$

We give two examples to show that each of the two sheaf properties can be violated. First, let $M = A/(x, y) = k$. If $1 \in M$, then its image in M_x and M_y is 0 since x and y kill everything. Thus, 1 restricts to 0 on the open cover, but it is not 0. For a second example, let $M = A/(x) \oplus A/(y)$. If $m = (0, 1)$ and $n = (1, 0)$, we see that m/x and n/y both map to 0 in $M_{xy} = 0$. However, they do not glue. For, if there is a $p = (a, b) \in M$ with $p/1 = m/x \in M_x$, then there is an r with $x^r(m - xp) = 0$ in M . This means $(0, x^r) = (0, x^{r+1}b)$, or $x^r = x^{r+1}b + fy$ for some f . Then $x^r(1 - xb) = fy$. This is impossible

in $k[x, y]$, since if we evaluate at $(\alpha, 0)$, we get $\alpha^r(1 - \alpha b(\alpha, 0)) = 0$, which is not true for all α .

The following lemma gives the basic properties of the sheaf \widetilde{M} . We do not prove it; instead, we refer to [3, Chap. II, Prop. 5.1].

Lemma 3.2. *Let M be a module over a ring A .*

1. *If $P \in \text{Spec}(A)$, then $\widetilde{M}_P = M_P$.*
2. *If $f \in A$, then $\widetilde{M}|_{D(f)} = \widetilde{M}_f$, where $M_f = M \otimes_A A_f$ and $D(f) = Z(f)^c$.*
3. *$\widetilde{M}(X) = M$.*

In fact, one can show that $\widetilde{M}(U) = M \otimes_A \mathcal{O}(U)$ for any affine open set U in $\text{Spec}(A)$.

We can define morphisms from \widetilde{M} to another sheaf fairly easily. Let \mathcal{F} be a sheaf on X and U an open set of X . Suppose that $\sigma : M \rightarrow \mathcal{F}(X)$ is an A -module homomorphism. We then get a sheaf map $\varphi : \widetilde{M} \rightarrow \mathcal{F}$ such that $\sigma = \varphi_X$ as follows. First, consider the presheaf $V \mapsto M \otimes_A \mathcal{O}(V)$. We have an $\mathcal{O}(U)$ -module homomorphism $M \otimes_A \mathcal{O}(U) \rightarrow \mathcal{F}(U)$ given by $\sigma \otimes \text{id}_U$; that is, a generator $m \otimes f$ is sent to $f \cdot \sigma(m)|_U$. This presheaf morphism then induces a sheaf morphism $\varphi : \widetilde{M} \rightarrow \mathcal{F}$ by the universal mapping property for sheafification. We can use this to see when $\mathcal{F} \cong \widetilde{M}$ for $M = \mathcal{F}(X)$, since we only need to know that $\mathcal{F}_P \cong M_P$ for all P via φ_P to conclude that $\mathcal{F} \cong \widetilde{M}$.

3.2 Tangent Sheaves

To describe tangent sheaves, we first say a few words about tangent spaces. To keep things simple we consider $X = Z(f) \subseteq \mathbf{A}^2$, the zero set of an irreducible polynomial, and we work over a field k . If $P = (a, b) \in X$, then the *tangent space* $T_P(X)$ is the zero set of

$$d_P(f) = \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y.$$

This is a translation of the tangent line $\frac{\partial f}{\partial x}(P)(x - a) + \frac{\partial f}{\partial y}(P)(y - b) = 0$. The tangent space $T_P(X)$ is both an algebraic variety and a linear subspace of k^2 .

Let X be an algebraic variety and set $T = \{(P, Q) : P \in X, Q \in T_P(X)\}$. This is an ad-hoc description of the tangent bundle on X . If $p : T \rightarrow X$ is the natural projection, we define a sheaf, the *tangent sheaf*, on X by

$$\mathcal{T}(U) = \{s : U \rightarrow T : s \text{ is a morphism, } s \circ p = \text{id}\}.$$

It is easy to check that \mathcal{T} is a sheaf. Moreover, $\mathcal{T}(U)$ is an $\mathcal{O}(U)$ -module via $(fs)(Q) = f(Q)s(Q)$ for all $f \in \mathcal{O}(U)$, all $s \in \mathcal{T}(U)$, and all points $Q \in U$. The stalk of \mathcal{T}_P is the tangent space $T_P(X)$.

The set T is an algebraic variety. Instead of giving a formal proof, we illustrate it and the sheaf \mathcal{T}_X with two examples.

Example 3.3. Let $X = Z(y - x^2)$, the affine parabola. If $f = y - x^2$ and $P = (a, b) \in X$, then $d_P(f) = y - 2ax$. Thus,

$$\begin{aligned} T &= \{(a, b, c, d) \in k^4 : b = a^2, d = 2ac\} \\ &= \{(a, a^2, c, 2ac) \in k^4 : a, c \in k\}. \end{aligned}$$

This is then the zero set of $\{y - x^2, w - 2xz\}$, a subvariety of affine 4-space. We claim that $\mathcal{T} \cong \mathcal{O}_X$. To prove this, let U be an open subset of X . We define a map $\varphi : \mathcal{O}_X \rightarrow \mathcal{T}$ as follows. If U is open in X , then the map $\varphi_U : \mathcal{O}(U) \rightarrow \mathcal{T}(U)$ is given by $\varphi_U(f)(P) = (P, f(P), 2af(P))$. Then $\varphi_U(f) \in \mathcal{T}(U)$ since it is regular on U and $\varphi_U(f) \circ p = \text{id}$. It is trivial to check that φ is a sheaf morphism. Its inverse is given by $s \mapsto \pi_3 \circ s$, where π_3 picks the third component of a point of T . We have $f = \pi_3 \circ \varphi_U(f)$, which is the motivation for this definition. Again, it is easy to show that this is a sheaf morphism, and that it is the inverse φ . This proves $\mathcal{T} \cong \mathcal{O}_X$, so \mathcal{T} is free of rank 1.

Example 3.4. Let $X = Z(x^2 + y^2 + z^2 - 1)$, the affine 2-sphere, a subvariety of affine 3-space. Then

$$T = \{(a, b, c, u, v, w) : a^2 + b^2 + c^2 = 1, au + bv + cw = 0\}.$$

To show that \mathcal{T} is locally free of rank 2, consider the open subset U of X on which $x \neq 0$. To show that $\mathcal{T}|_U \cong (\mathcal{O}_X|_U)^2$, for every open set $V \subseteq U$, define a map $\varphi_V : \mathcal{O}(V) \times \mathcal{O}(V) \rightarrow \mathcal{T}(V)$ by

$$\varphi_V((f, g))(P) = \left(P, -\frac{y}{x}f(P), \frac{-z}{x}g(P), f(P), g(P) \right).$$

The inverse morphism is given by $s \mapsto (s_4, s_5)$, where s_i is the i -th component function of s . Similarly, if W is the open set on which $y \neq 0$, then $\mathcal{T}|_W \cong (\mathcal{O}_X|_W)^2$, and if U is the open set on which $z \neq 0$, then $\mathcal{T}|_U \cong (\mathcal{O}_X|_U)^2$. The three open sets U, V, W cover X , and so \mathcal{T} is locally free. However, we indicate that \mathcal{T} is not free, in the case the base field is \mathbb{R} . If $\mathcal{T} \cong \mathcal{O}_X^2$, then

$$\begin{aligned} \mathcal{T}(X) &\cong M := \{(f, g, h) \in A^3 : xf + yg + zh = 0\} \\ &\cong A^2, \end{aligned}$$

where $A = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ is the coordinate ring of X . If this happens, then a basis vector for $\mathcal{T}(X)$ would produce a globally defined tangent vector field, which cannot exist by the hairy ball theorem. Note that we have an exact sequence

$$0 \rightarrow M \rightarrow A^3 \rightarrow A \rightarrow 0$$

with $(f, g, h) \rightarrow xf + yg + zh$. Since A is free, $A^3 \cong A \oplus M$. Thus, M is projective, even though it is not free.

3.3 Sheaves on Nonsingular Curves

Let X be a nonsingular projective curve over a field k and let $K = k(X)$ be the function field of X . We produce locally free sheaves of rank 1 from divisors on X in the following way. Let D be a divisor on X . We define a sheaf \mathcal{L}_D on X as follows. Let v_P be the discrete valuation on K corresponding to a point $P \in X$. We use the notation $v_P(D)$ to denote the coefficient of P in D . For U open in X , set

$$\mathcal{L}_D(U) = \{f \in K : v_P(f) + v_P(D) \geq 0 \text{ for all } P \in U\},$$

and $\text{res}_{UV} : \mathcal{L}_D(U) \rightarrow \mathcal{L}_D(V)$ is the inclusion map. Note that since

$$\mathcal{O}(U) = \{f \in K : v_P(f) \geq 0 \text{ for all } P \in U\},$$

it is clear that $\mathcal{L}_D(U)$ is an $\mathcal{O}(U)$ -module. Moreover, since the restriction maps are inclusion maps, they are clearly module homomorphisms. So, \mathcal{L}_D is an \mathcal{O}_X -module.

Example 3.5. Suppose that $D = 0$. Then $\mathcal{L}_D(U) = \mathcal{O}_X(U)$ for any U , so $\mathcal{L}_D = \mathcal{O}_X$ is free of rank 1. If $D = (f)$, then $\mathcal{L}_D = f^{-1}\mathcal{O}_X$ is also free of rank one. In fact, if $D \sim E$, meaning $D = E + (f)$ for some $f \in K^*$, then $\mathcal{L}_D \cong \mathcal{L}_E$ via the map $g \mapsto fg$.

The sheaf \mathcal{L}_D need not be free, but it is locally free of rank 1, as we will prove shortly.

Example 3.6. Let $X = \mathbf{P}^1 = k \cup \{\infty\}$. The function field of X is $k(x)$. Let $\mathcal{F}_n = \mathcal{L}_{n\infty}$, the sheaf associated to the divisor $n\infty$, for some $n > 0$. We point out that the valuation v_∞ is given by $v_\infty(f) = -\deg(f)$. Moreover, $k[x]$ is the set of rational functions for which $v_a(f) \geq 0$ for all $a \in k$, since if $f/g \in k(x) - k[x]$ is in reduced form, then f/g has a pole wherever g has a zero, so f/g has negative value at such a point. The space of global sections of \mathcal{F}_n is then

$$\begin{aligned} \mathcal{F}_n(X) &= \{f \in k(x) : v_Q(f) \geq 0 \text{ for all } Q \neq \infty, v_\infty(f) \geq -n\} \\ &= \{f \in k[x] : \deg(f) \leq n\} \\ &= k + kx + \cdots + kx^n, \end{aligned}$$

so $\dim_k(\mathcal{F}_n(X)) = n + 1$. Thus, $\mathcal{F}_n \not\cong \mathcal{O}_X$ since $\dim_k(\mathcal{O}_X(X)) = 1$. Therefore, \mathcal{F}_n is not free if $n > 0$. The stalks of \mathcal{F}_n are given by

$$(\mathcal{F}_n)_P = \{f \in k(x) : v_P(f) + v_P(n\infty) \geq 0\}.$$

Thus,

$$(\mathcal{F}_n)_P = \begin{cases} \mathcal{O}_P & \text{if } P \neq \infty \\ x^n \mathcal{O}_P & \text{if } P = \infty \end{cases}.$$

To see that \mathcal{F}_n is locally free, first let $U = X - \{\infty\}$. Then

$$\mathcal{F}_n(U) = \{f \in k(x) : v_Q(f) \geq 0 \text{ for all } Q \in U\} = k[x].$$

In fact, $\mathcal{F}_n|_U = \widehat{k[x]} = \mathcal{O}_X|_U$, since these two sheaves have the same stalks for each $P \in U$. On the other hand, if we take V to be the neighborhood $X - \{0\}$ of ∞ , then

$$\mathcal{O}_X(V) = \left\{ h(x) = \frac{f(x)}{g(x)} \in k(x) : h(x) \text{ is defined everywhere but at } 0 \right\}.$$

Thus, if we write elements as f/g in reduced terms, $g(x)$ can have no root other than 0. Thus, $g(x) = x^r$ for some $r \geq 0$. In order for $h(x)$ to be defined at ∞ , it must be that $\deg(f) \leq \deg(g)$. So,

$$\mathcal{O}_X(V) = \left\{ \frac{f(x)}{x^r} : \deg(f) \leq r \right\}.$$

Now, if $h \in \mathcal{F}_n(V)$, then $v_Q(h) \geq 0$ for all $Q \neq \infty$ and $Q \neq 0$, while $v_\infty(h) \geq -n$. This means f is defined at all points other than ∞ and 0. Thus, $h = f/x^r$ for some r by the argument above. Moreover, $v_\infty(h) = r - \deg(f) \geq -n$, so $\deg(f) \leq n + r$. Therefore, $f/x^{n+r} \in \mathcal{O}_X(V)$. Thus, $\mathcal{F}_n(V) = x^{-n}\mathcal{O}_X(V)$. Again, thinking about the stalks shows that $\mathcal{F}_n|_V = x^{-n}\mathcal{O}_X(V)$ is free. The open sets U and V cover X , so \mathcal{F}_n is locally free.

We point out now that the divisor class group $\text{Cl}(X)$ is isomorphic to the Picard group $\text{Pic}(X)$ if X is a nonsingular curve. With $X = \mathbf{P}^1$, it is known that $\text{Cl}(X) \cong \mathbb{Z}$ under the degree map. Moreover, every class is represented by the twisting sheaf $\mathcal{O}(n)$ for some n , and that this sheaf maps to n under the degree map. From this we can conclude that $\mathcal{F}_n \cong \mathcal{O}(n)$.

Example 3.7. Let a_1, \dots, a_r be distinct elements in $k \subseteq \mathbf{P}^1$, and consider a divisor $D = n_1a_1 + \dots + n_ra_r$ on \mathbf{P}^1 with positive coefficients. If $\varphi = f/g \in L(D)$ is written in reduced terms, then for each i , if v_i is the valuation corresponding to a_i , then $v_i(\varphi) + n_i \geq 0$. Therefore, $v_i(f) - v_i(g) + n_i \geq 0$. Therefore, if a_i is a root of g , then the multiplicity of a_i as a root is at most n_i . Also, no other element can be a root of g since $v_P(\varphi) \geq 0$ for any $a \neq a_i$. Thus, $(x - a_1)^{n_1} \dots (x - a_r)^{n_r} \varphi \in k[x]$. For $v_\infty(\varphi) \geq 0$, we must have $\deg(f) \leq \deg(g)$. From these facts we conclude that

$$L(D) = \left\{ \frac{f(x)}{(x - a_1)^{n_1} \dots (x - a_r)^{n_r}} : f(x) \in k[x] : \deg(f) \leq \sum_i n_i \right\}.$$

Therefore, $L(D)$ has dimension $1 + \sum_i n_i$.

What we saw occur in the example of $\mathcal{L}_{n\infty}$ is a special case of the following general result.

Proposition 3.8. *The sheaf \mathcal{L}_D is locally free of rank 1.*

Proof. Let $P \in X$. We produce a neighborhood U of P on which \mathcal{L}_D is free. Let $n_P = v_P(D)$ be the coefficient of P in D . Let g_P be a uniformizer of \mathcal{O}_P . Choose a neighborhood U of P that misses all poles of g_P and which does not contain any point $Q \neq P$ of U for which the coefficient $v_Q(D) \neq 0$. If $f \in \mathcal{L}_D(U)$, we have $v_P(f) \geq -n_P$ and $v_Q(f) \geq 0$ for any $Q \neq P$. Thus, as $v_Q(g_P) \geq 0$, we have $v_Q(g_P^{n_P} f) \geq 0$ for all $Q \in U$, including P . Thus, $g_P^{n_P} f \in \mathcal{O}(U)$. The converse is clear; if $f \in g_P^{-n_P} \mathcal{O}(U)$, then $f \in \mathcal{L}_D(U)$. To see that this

implies $\mathcal{L}_D|_U \cong \mathcal{O}_X|_U$, the same argument for V replacing U shows that if V is an open set in U , then $\mathcal{L}_D|_U(V) \cong g_P^{-n_P} \mathcal{O}(V)$ under the map $f \mapsto g_P^{n_P} f$. This then yields a sheaf isomorphism $\mathcal{L}_D|_U = g_P^{-n_P} \mathcal{O}|_U$, which shows that \mathcal{L}_D is locally free of rank 1. \square

4 Invertible Sheaves

A locally free sheaf \mathcal{F} is called an *invertible sheaf* if the rank of \mathcal{F} is 1. The reason for this terminology is that the isomorphism classes of invertible sheaves form a group, called the *Picard group* of X , where multiplication is given by tensor products. Since $\mathcal{O}_X^n \otimes \mathcal{O}_X^m \cong \mathcal{O}_X^{n+m}$, it follows that the tensor product of locally free sheaves is locally free. Moreover, if \mathcal{F} is an invertible sheaf, set $\mathcal{F}^\vee = \mathfrak{H}om(\mathcal{F}, \mathcal{O}_X)$. Then $\mathcal{F} \otimes \mathcal{F}^\vee \cong \mathcal{O}_X$; the argument for this is similar to the proof that if P is a finitely generated projective A -module, then $P \otimes_A \mathfrak{h}om_A(P, A) \cong A$, which is any easy consequence of having a projective basis. We then see that $\text{Pic}(X)$ is a group, where the inverse of the class of \mathcal{F} is the class of \mathcal{F}^\vee . See Chapter II, Section 6 of [3] for more details about inverses of invertible sheaves.

A related group is the divisor class group $\text{Cl}(X)$ of an irreducible variety. To define this group, we first take the free Abelian group $\text{Div}(X)$ on the irreducible codimension 1 closed subvarieties of X . We assume that all such varieties are nonsingular. If $X = \text{Spec}(A)$, then an irreducible codimension 1 subvariety has the form $Z(P)$ for P a height 1 prime. Elements of $\text{Div}(X)$ are called *Weil divisors*. Under this assumption, the local rings $\mathcal{O}_{X,Y}$, if Y is one of these varieties, is a DVR, so we have an associated valuation v_Y . If f is a rational function on X , we define the principal divisor of f to be $\sum v_Y(f)Y$. Then $\text{Cl}(X)$ is the quotient of $\text{Div}(X)$ modulo the subgroup of principal divisors. If $X = \text{Spec}(A)$, we see that $\text{Cl}(X) = \text{Cl}(A)$.

If X is a curve, then $\text{Div}(X)$ is the free Abelian group on the points of X . If $D = \sum n_P P \in \text{Div}(X)$, define $\deg(D) = \sum n_Y$. The map \deg is a group homomorphism from $\text{Div}(X)$ to \mathbb{Z} . It is known that principal divisors have degree 0. Thus, there is an induced map $\deg : \text{Cl}(X) \rightarrow \mathbb{Z}$. We denote by $\text{Cl}_0(X)$ the kernel, which is the subgroup of divisors of degree 0. We note that $\text{Cl}(X)/\text{Cl}_0(X) \cong \mathbb{Z}$. Thus, in some sense, to know $\text{Cl}(X)$, we then have to determine $\text{Cl}_0(X)$. If X is nonsingular, then $\text{Cl}(X) \cong \text{Pic}(X)$, which is proved in [3, Chap. II, Cor. 6.6]. In the next example, we show how to represent the group law on an elliptic curve with the group operation in $\text{Pic}(X)$, or, more accurately, in $\text{Cl}(X)$.

Example 4.1. If E is an elliptic curve, then we show that $\text{Cl}_0(E)$ is in 1-1 correspondence with the points on E , and this correspondence is a group isomorphism when one uses the group law on an elliptic curve. An elliptic curve has genus 1. In fact, an abstract definition of an elliptic curve is a nonsingular curve of genus 1. Riemann's theorem, a special case of the Riemann-Roch theorem, says that if D is a divisor on a curve of genus g , then

$$\dim(L(D)) \geq \deg(D) + 1 - g.$$

Thus, if D is a divisor with $\deg(D) > 0$, then $\dim(L(D)) = \deg(D)$. Let D have degree 0. Then $D + P_\infty$ has degree 1, so $L(D + P_\infty)$ contains a nonzero function f . Then $D + P_\infty + (f)$ is a divisor with nonnegative coefficients and has degree 1. Thus, there is a point P with $D + P_\infty + (f) = P$, which shows $D \sim P - P_\infty$. Therefore,

$$\text{Cl}_0(E) = \{[P - P_\infty] : P \in E\}.$$

Let $P, Q \in E$, let R be the third point on the curve and on the line L containing P and Q , and let S be the third point on the curve and on the line L' containing P_∞ and R . Write $L = Z(f)$ and $L' = Z(g)$ for some linear polynomials f and g . We then have $(f) = P + Q + R - 3P_\infty$ and $(g) = P_\infty + R + S - 3P_\infty = R + S - 2P_\infty$; the P_∞ terms are there to account for the degree of the principal divisor being 0. Then $(fg^{-1}) = (f) - (g) = P + Q - S - P_\infty$. Since this is a trivial divisor, we have $(P - P_\infty) + (Q - P_\infty) \sim S - P_\infty$. Since S is the point $P + Q$ under the group law, we see that the map $E \rightarrow \text{Cl}_0(E)$ given by $P \mapsto P - P_\infty$ is a group homomorphism. This is a way to interpret the group law on E .

Example 4.2. Sheaves, divisors, and the related groups are more subtle objects that can help to distinguish varieties. For example, how does one see that \mathbf{P}^2 is not isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$? We can see this with the help of the Picard group. We have $\text{Pic}(\mathbf{P}^n) \cong \mathbb{Z}$ for any n ; a proof can be found by combining Proposition 6.4 and Proposition 6.15 of Chapter II of [3]. Since $\text{Pic}(\mathbf{P}^1 \times \mathbf{P}^1) \cong \text{Pic}(\mathbf{P}^1) \times \text{Pic}(\mathbf{P}^1)$, a calculation done in Example 6.6.1 of [3], we see that this implies $\mathbf{P}^2 \not\cong \mathbf{P}^1 \times \mathbf{P}^1$, which is a surprisingly difficult fact. More generally, $\text{Pic}(X \times Y) \cong \text{Pic}(X) \times \text{Pic}(Y)$, which implies that \mathbf{P}^{n+m} is not isomorphic to $\mathbf{P}^n \times \mathbf{P}^m$ for any n, m .

5 Relations Between $\text{Pic}(A)$ and $\text{Cl}(A)$

Weil divisors and invertible sheaves arose when we discussed divisors on a curve. In this section we see the relation between the Picard group and the divisor class group of an integral domain. The geometric analogues of this material is in [3, Chap. II, Section 6]. In general, these groups are not isomorphic. Let A be a Noetherian integral domain. We say that A is *locally factorial* if A_M is a UFD for every maximal ideal of A . Since being integrally closed is a local property, we see that a locally factorial ring is integrally closed since a UFD is integrally closed. Examples of locally factorial rings include the polynomial ring $k[x_1, \dots, x_n]$ over a field, and, more generally, any regular ring, since regular local rings are UFD's. Furthermore, since a DVR is simply a Noetherian integrally closed local domain of dimension 1, we see that if A has dimension 1, then A is locally factorial if and only if it is integrally closed. In terms of geometry, if A is the coordinate ring of a variety X , then A is regular if and only if X is nonsingular [3, Chap. I, Theorem 5.1].

We will prove a few results to get the connection between Weil divisors and invertible ideals. We make heavy use of localization. Recall the following local-global facts. If A is a commutative ring and if M is an A -module, then $M = 0$ if and only if $M_P = 0$ for all P .

Consequently, if $m \in M$, then $m = 0$ if and only if $m/1 = 0$ in M_P for all P , by applying the previous result to Am . Also, if N is a submodule of M , then by considering M/N , we see that $N = M$ if and only if $N_P = M_P$ for all P .

Lemma 5.1. *Let I be a finitely generated fractional ideal of an integral domain A . Then I is invertible if and only if I_P is invertible for each prime ideal (resp. maximal ideal) P of A .*

Proof. We use the notation $(A : I)$ to represent I^{-1} . An easy argument shows that $(A : I)_P = (A_P : I_P)$ since I is finitely generated. If I is invertible, then $II^{-1} = I(A : I) = A$, so localizing gives $I_P(A_P : I_P) = A_P$, and so I_P is invertible. Conversely, if I_P is invertible for every P , then $I_P(A : I)_P = A_P$. Then $(I(A : I))_P = A_P$ for every P . This forces $I(A : I) = A$, and so I is invertible. \square

We will use the principal ideal theorem [2, Theorem 10.1] in two places below. Recall that this says if A is Noetherian, then any minimal prime lying over a principal ideal has height at most 1. We also recall that every ideal I of A has only finitely many associated primes, and these are all obtained as annihilators of elements of the A -module A/I . In fact, they are maximal among such annihilators.

Proposition 5.2. *If A is a Noetherian integrally closed domain, then (i) A_P is a DVR for each height one prime ideal P , (ii) $A = \bigcap_{\text{ht}(P)=1} A_P$, and (iii) any $a \in A$ lies in only finitely many P .*

Proof. If P has height 1, then A_P is a local Noetherian integrally closed domain, so A_P is a DVR. To prove (ii), let $x = a/b \in \bigcap_{\text{ht}(P)=1} A_P$. We want to show $x \in A$, which is equivalent to $a \in (b)$. Consider the module $A/(b)$. We want to know that $\bar{a} \in A/(b)$ is 0. Since the associated primes of (b) are the maximal annihilators of elements of $A/(b)$, if $\bar{a} \neq 0$, then $\text{ann}(\bar{a}) \subseteq P$ for some associated prime. But, $\text{ht}(P) = 1$ by the principal ideal theorem. However, $a/b \in A_P$ means $a/b = \alpha/s$ for some $\alpha \in A$ and $s \in A - P$. Then $s\bar{a} = 0$, so $s \in \text{ann}(\bar{a})$, a contradiction. Thus, $\bar{a} = 0$, as desired. Finally, $a \in P$ only when P is associated to (a) , and there are only finitely many such P . \square

Proposition 5.3. *If A is a Noetherian integral domain, then A is a UFD if and only if A is integrally closed and $\text{Cl}(A) = 0$.*

Proof. If A is a UFD, then it is integrally closed. Furthermore, since A is a UFD, we claim that every nonzero prime ideal contains a principal prime ideal. For, if P is prime, let $a \in P$ be nonzero. If we factor $a = \pi_1 \cdots \pi_n$ into prime elements, then $\pi_i \in P$ for some i . Then (π_i) is a principal prime ideal contained in P . This implies that height one primes are principal. Thus, every divisor is a principal divisor, since $\sum n_i(\pi_i)$ is the principal divisor of $(\pi_1^{n_1} \cdots \pi_r^{n_r})$. Therefore, $\text{Cl}(A) = 0$. Conversely, if A is integrally closed and $\text{Cl}(A) = 0$, then let P be a height one prime. We need to show that P is principal. We have $P = 0$ in $\text{Cl}(A) = 0$, so P is the divisor of an element (a) . Therefore, $v_P(a) = 1$ and $v_Q(a) = 0$ for

any $Q \neq P$. Since A is Noetherian and integrally closed $A = \bigcap_{\text{ht}(Q)=1} A_Q$, which we prove in Proposition 5.2. Therefore, we have

$$(a) = \bigcap_Q aA_Q = \bigcap_{Q \neq P} A_Q \cap PA_P = A \cap PA_P = P.$$

This finishes the proof. \square

Lemma 5.4. *Let A be a Noetherian locally factorial ring. If P is a height one prime, then P is invertible.*

Proof. Since A is Noetherian, we only need to prove that P is locally invertible, by Lemma 5.1. However, if M is a maximal ideal of A , then A_M is a UFD. If $P \not\subseteq M$, then $P_M = A_M$, so P_M is invertible. If $P \subseteq M$, then P_M is a height 1 prime of A_M , so it is principal as A_M is a UFD. Thus, P_M is invertible. Thus, since P_M is invertible for every M , and since it is finitely generated, P is invertible. \square

Lemma 5.5. *Let A be a Noetherian locally factorial ring. If I is an invertible ideal, then I is a product of height one prime ideals or their inverses.*

Proof. If I is invertible, then aI is an invertible ordinary ideal of A for some $a \in A$. As (a) is also invertible, it suffices to prove the lemma for ordinary ideals. We argue by contradiction. Let I be maximal among invertible ideals that are not products of height one prime ideals. Take a minimal prime P of I . Let M be a maximal ideal containing P . Then P_M is minimal over I_M , since primes of A_N are in 1-1 inclusion preserving correspondence with primes of A contained in M . Since I_M is invertible, it is finitely generated projective, so I_M is free, and hence principal. Thus, by the principal ideal theorem, $\text{ht}(P_M) = 1$, which forces P to have height 1. Now, we know that P is invertible by Lemma 5.4. Also, $P^{-1} \subseteq I^{-1}$, so $P^{-1}I \subseteq A$. Thus, $P^{-1}I$ is an invertible ideal at least as large as I . If $P^{-1}I = I$, then each element of P^{-1} is integral over A by the usual determinant trick: if $I = (a_1, \dots, a_n)$ and $xI \subseteq I$, then $xa_i = \sum_j \alpha_{ij}a_j$, and so $\sum_j (\alpha_{ij} - \delta_{ij}x)a_j = 0$. Taking determinants, if $d = \det(\alpha_{ij} - \delta_{ij}x)$, then the product of d with the vector of a_j is 0. Since $I \neq 0$, this forces $d = 0$. This shows x is integral over A . Since A is integrally closed, $x \in A$, an impossibility. So, $P^{-1}I$ is properly larger than I . Maximality of I shows $P^{-1}I$ is a product of height 1 primes, so I is also such a product. This contradiction proves the result. \square

Theorem 5.6. *If A is a Noetherian locally factorial ring, then $\text{Cl}(A) \cong \text{Pic}(A)$.*

Proof. Let P be a height one prime ideal. By Lemma 5.4, P is invertible, so represents an element of $\text{ICl}(A) \cong \text{Pic}(A)$. We then have a map from the free Abelian group $\text{Div}(A)$ on height 1 primes to $\text{Pic}(A)$, given by $\sum n_P P \mapsto \prod P^{n_P}$. By Lemma 5.5, if $a \in F^*$, then (a) is a product of height 1 prime ideals. An easy local-global argument shows that $(a) = \prod P^{v_P(a)}$. Therefore, principal divisors go to zero under this map. Thus, we have a well defined map from $\text{Pic}(A)$ to $\text{Cl}(A)$. It is also surjective by the lemma. Finally, for injectivity, we need

to see that the only divisors going to 0 are principal divisors. But, if $\sum n_P P$ goes to 0 in $\text{Pic}(A)$, then $\prod P^{n_P} = (a)$ for some a . Then $n_P = v_P(a)$, so $\sum n_P P$ is a principal divisor. Thus, $\text{Cl}(A) \cong \text{Pic}(A)$. \square

More generally, if X is a Noetherian integral separated locally factorial scheme, then $\text{Cl}(X) \cong \text{Pic}(X)$. A proof of this can be found in Section 6 of Chapter II of [3].

6 Locally Free Sheaves and Projective Modules

If \mathcal{F} is a locally free sheaf on X , then for each $P \in X$, there is a neighborhood U of P such that $\mathcal{F}|_U$ is a free sheaf on U . In other words, $\mathcal{F}|_U \cong (\mathcal{O}_X|_U)^n = \widetilde{\mathcal{O}(U)^n}$ for some n . Thus, \mathcal{F} can be written locally as \widetilde{M} for some M , which depends on the open set U . This is the idea of a quasi-coherent sheaf: A sheaf \mathcal{F} is said to be *quasi-coherent* if there is a cover of X by open affine sets $U_i = \text{Spec}(A_i)$ such that there is an A_i -module M_i with $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$. If, in addition, X is Noetherian and each M_i is a finitely generated A_i -module, then \mathcal{F} is said to be *coherent*. Note that if $\mathcal{F}|_U \cong (\mathcal{O}_X|_U)^n$, then $\mathcal{F}_P \cong \mathcal{O}_P^n$ for each $P \in U$. Therefore, the stalk \mathcal{F}_P of a locally free sheaf \mathcal{F} is a free \mathcal{O}_P -module. The goal of this section is to show that \widetilde{M} is locally free if and only if M is finitely generated and projective.

We start by proving module-theoretic results to help determine when a module is projective. A nice writeup of these results can be found in [4, Chapter IV].

Lemma 6.1. *Let A be a local ring. If P is a finitely generated projective A -module, then P is free.*

Proof. Let \mathfrak{m} be the maximal ideal of A . Let $\{m_1, \dots, m_n\}$ be a set of elements of P such that $\{\overline{m}_1, \dots, \overline{m}_n\}$ is a basis for the A/\mathfrak{m} -vector space $P/\mathfrak{m}P = P \otimes_A A/\mathfrak{m}$; such a set exists since $P/\mathfrak{m}P$ is finitely generated. If Q is the submodule generated by the m_i , then $P = Q + \mathfrak{m}P$. Thus, Nakayama's lemma implies that $P = Q$. To show that P is free, consider the surjective map $\varphi : A^n \rightarrow P$ that sends e_i to m_i ; we write e_i for the i -th standard basis vector of A^n . We then have an exact sequence $0 \rightarrow \ker(\varphi) \rightarrow A^n \rightarrow P \rightarrow 0$. Since P is projective, $A^n \cong P \oplus \ker(\varphi)$. So, there is an A -homomorphism $i : P \rightarrow A^n$ such that $\varphi \circ i = \text{id}_P$. Let $P' = i(P) \cong P$. Then $A^n = P' + \ker(\varphi)$. If $(a_1, \dots, a_n) \in \ker(\varphi)$, then $\sum_i a_i m_i = 0$, so $\sum_i \overline{a}_i \cdot \overline{m}_i = \overline{0}$, which forces $\overline{a}_i = \overline{0}$. Thus, all $a_i \in \mathfrak{m}$. This proves that $\ker(\varphi) \subseteq \mathfrak{m}A^n$. Therefore, $A^n = P' + \mathfrak{m}A^n$, which by Nakayama's lemma shows that $P' = A^n$. Thus, $P \cong P'$ is free. \square

An A -module M is said to be *finitely presented* if it can be represented as $M = F/K$ with F a finitely generated free A -module and K a finitely generated submodule of F . If A is Noetherian, then any finitely generated module is finitely presented. If M is an A -module for which M_P is a free A_P -module for each prime ideal P of A , then we say M is *locally free*.

Let M be a projective A -module. We note that M_P is a projective A_P -module for every prime P of A ; for, if $M \oplus Q = F$ is finitely generated and free, and since localization

commutes with direct sums, we have $F_P \cong M_P \oplus Q_P$, so M_P is projective. By Lemma 6.1, M_P is actually a free M_P -module.

Proposition 6.2. *A finitely generated A -module M is projective if and only if M is finitely presented and locally free.*

Proof. Suppose that M is finitely generated and projective. Let F be a free module of finite rank such that there is a surjective homomorphism $F \rightarrow M$. If K is the kernel, then there is an exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$, which splits since M is projective, so $F = M \oplus K$. Then K is a homomorphic image of F , so K is also finitely generated. Then M is finitely presented. If $P \in \text{Spec}(A)$, then M_P is finitely generated and projective over A_P , so it is free by Lemma 6.1. Thus, M is locally free.

For the converse, suppose that M is finitely presented and locally free. We need to prove that M is projective. There is a short exact sequence $0 \rightarrow K \rightarrow F \xrightarrow{g} M \rightarrow 0$ with F finitely generated free and K finitely generated. If P is a maximal ideal, then since localization is an exact functor, $0 \rightarrow K_P \rightarrow F_P \rightarrow M_P \rightarrow 0$ is exact, and it is split exact since M_P is free. We see that this forces the original sequence to be split. It is enough to show that $g_* : \text{hom}_A(M, F) \rightarrow \text{hom}_A(M, M)$ contains id in its image; this will show that $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ splits. However, for each prime, $\text{hom}_{A_P}(M_P, F_P) \rightarrow \text{hom}_{A_P}(M_P, M_P)$ is surjective, and $(\text{hom}_A(M, F))_P \cong \text{hom}_{A_P}(M_P, F_P)$. This forces $\text{hom}_A(M, F) \rightarrow \text{hom}_A(M, M)$ to be surjective, since if T is the cokernel, then $T_P = 0$ for all P , so $T = 0$. Thus, M is projective. \square

Proposition 6.3. *Let M be a finitely generated module over a Noetherian ring A . Then the sheaf \widetilde{M} is locally free if and only if M is projective.*

Proof. Suppose that \widetilde{M} is locally free. For every $P \in X = \text{Spec}(A)$, there is a neighborhood U of P such that $\widetilde{M}|_U$ is free. Then the stalk at P is a free module. However, this stalk is M_P . Thus, M is locally free. Since A is Noetherian, M is finitely presented. Thus, by Proposition 6.2, we see that M is projective. Conversely, suppose that M is projective. We prove that \widetilde{M} is locally free. Let $P \in X$. We produce an $s \in A - P$ such that M_s is free. There are $m_i \in M$ such that $m_1/1, \dots, m_n/1$ is a basis for M_P . Then we have an exact sequence $0 \rightarrow \ker(\alpha) \rightarrow A^n \xrightarrow{\alpha} M \rightarrow \text{coker}(\alpha) \rightarrow 0$, where $\alpha(e_i) = m_i$. If we localize this sequence at P , we get $0 \rightarrow \ker(\alpha)_P \rightarrow A_P^n \rightarrow M_P \rightarrow \text{coker}(\alpha)_P \rightarrow 0$, which shows that $\ker(\alpha)_P = 0$ and $\text{coker}(\alpha)_P = 0$ since M_P is free of rank n , generated by the $m_i/1$. The module $C = \text{coker}(\alpha)$ is finitely generated, since it is a quotient of M , and since $C_P = 0$, finite generation implies that there is an $f \in A - P$ with $C_f = 0$. Now, localizing at f and using $C_f = 0$ gives $0 \rightarrow \ker(\alpha)_f \rightarrow A_f^n \rightarrow M_f \rightarrow 0$. Since M is A -projective, M_f is A_f -projective, so this latter sequence splits. Thus, $\ker(\alpha)_f$ is finitely generated. Therefore, there is a $g \in A - P$ with $\ker(\alpha)_{fg} = 0$. Set $s = fg$. Then $\ker(\alpha)_s = \text{coker}(\alpha)_s = 0$, so $A_s^n \cong M_s$. Thus, M_s is free. Then $\widetilde{M}|_{D(s)} = \widetilde{M}_s$ is free. Thus, for any point $P \in X$, we can produce an open neighborhood $D(s)$ of P on which \widetilde{M} is free. Thus, \widetilde{M} is locally free. \square

Corollary 6.4. *A coherent sheaf \mathcal{F} on a Noetherian scheme X is locally free if and only if \mathcal{F}_P is a free \mathcal{O}_P -module for every $P \in X$. This occurs if and only if for every open affine subset U of X , we have $\mathcal{F}|_U \cong \widetilde{M}$ for some finitely generated projective $\mathcal{O}(U)$ -module M .*

Proof. Suppose \mathcal{F}_P is free for every P . Since \mathcal{F} is coherent, if $P \in X$, then there is an affine neighborhood U of P and a finitely generated $\mathcal{O}(U)$ -module M for which $\mathcal{F}|_U \cong \widetilde{M}$. If $Q \in U$, then $\mathcal{F}_Q = M_Q$. Then M is projective by Proposition 6.2 since all of its stalks are free. We are using that $\mathcal{O}(U)$ is Noetherian to see that M is finitely presented. Then \widetilde{M} is locally free by Proposition 6.3. Thus, since X is covered by these various neighborhoods, \mathcal{F} is also locally free. The final statement follows from [3, Chap. II, Prop. 5.4] and Proposition 6.3. \square

Example 6.5. Quasi-coherence is necessary in the result above. Fix an open set $U \subseteq X = \text{Spec}(A)$ with A an integral domain and consider the sheaf \mathcal{F} associated to the presheaf

$$V \mapsto \begin{cases} \mathcal{O}(V) & \text{if } V \subseteq U \\ 0 & \text{if } V \not\subseteq U \end{cases} .$$

The sheaf \mathcal{F} is the sheaf obtained by extending $\mathcal{O}_X|_U$ by zero; see [3, Chap. II, Problem 1.19]. We consider the sheaf $\mathcal{G} = \mathcal{F} \oplus \mathcal{O}_X$ to avoid considering the trivial free module 0. Then $\mathcal{F}_P = \mathcal{O}_P$ if $P \in U$, and $\mathcal{F}_P = 0$ otherwise. Thus,

$$\mathcal{G}_P = \begin{cases} \mathcal{O}_P^2 & \text{if } P \in U \\ \mathcal{O}_P & \text{if } P \notin U \end{cases} .$$

Therefore, the stalks of \mathcal{G} are all free. Take $P \notin U$. If \mathcal{G} is locally free, then there is a neighborhood V of P with $\mathcal{G}|_V \cong (\mathcal{O}_X|_V)^n$ for some n . However, by considering the stalk at P , we see that $n = 1$. However, if $Q \in U \cap V$, which exists since X is irreducible and U, V are nonempty open sets, then $\mathcal{G}_Q = \mathcal{O}_Q^2$. This is impossible, since if $\mathcal{G}|_V \cong \mathcal{O}_X|_V$, then $\mathcal{G}_Q = \mathcal{O}_Q$. Thus, \mathcal{G} is not locally free.

By taking more care in the arguments above, we can actually obtain a category equivalence between the category of finitely generated projective A -modules and the category of locally free sheaves over $\text{Spec}(A)$.

Theorem 6.6. *There is an equivalence of categories between the category of A -modules and the category of quasi-coherent sheaves of $\mathcal{O}_{\text{Spec}(A)}$ -modules, given by $M \mapsto \widetilde{M}$. This restricts to an equivalence of categories between the category of finitely generated projective A -modules and the category of locally free sheaves of $\mathcal{O}_{\text{Spec}(A)}$ -modules.*

Sketch of Proof. Suppose that we have an A -module homomorphism $\varphi : M \rightarrow N$. Let M' and N' be the presheaves $U \mapsto M \otimes_A \mathcal{O}(U)$ and $U \mapsto N \otimes_A \mathcal{O}(U)$, respectively. We have a presheaf morphism $M' \rightarrow N'$ that on U is the natural map $\varphi \otimes \text{id} : M'(U) \rightarrow N'(U)$. Composing this with the natural map $N' \rightarrow \widetilde{N}$ yields a presheaf map $M' \rightarrow \widetilde{N}$,

and the universal mapping property for sheafification gives a unique morphism $\tilde{\varphi} : \tilde{M} \rightarrow \tilde{N}$ compatible with this map. That we have a functor from the two categories. The inverse morphism sends a sheaf morphism $\psi : \tilde{M} \rightarrow \tilde{N}$ to the A -module homomorphism $\psi_X : \tilde{M}(X) \rightarrow \tilde{N}(X)$, which is a map from M to N . \square

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