Rings with no Maximal Ideals

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In this note we give examples of a ring that has no maximal ideals. Recall that, by a Zorn's lemma argument, a ring with identity has a maximal ideal. Therefore, we need to produce examples of rings without identity. To help motivate our examples, let S be a ring without identity. We may embed S in a ring R with identity so that S is an ideal of R. Notably, set $R = \mathbb{Z} \oplus S$, as groups, and where multiplication is given by $(n, s) \cdot (m, t) = (nm, nt + ms + st)$. It is easy to show that R is indeed a ring, that $(1, 0)$ is the identity of this ring, and that $\{(0, s) : s \in S\}$ is an ideal of R that is isomorphic, as a ring, to S. Thus, any ring without identity may be viewed as an ideal in a ring with identity. We will then search for ideals in rings with identity as candidates for rings with no maximal ideals.

The most common example in textbooks of a ring with no maximal ideals is to start with the group $\mathbb{Z}(p^\infty)$, which is the subgroup of $\mathbb{Q}/\mathbb{Z}$ of elements of order a power of the prime p. It is known that the subgroups of $\mathbb{Z}(p^\infty)$ form an infinite increasing chain, and so there is no maximal subgroup. By defining multiplication in $\mathbb{Z}(p^\infty)$ by $x \cdot y = 0$, this group becomes a ring with no maximal ideals. However, there are simpler examples of groups with no maximal subgroups. In fact, we prove in the proposition below that any divisible group has no maximal subgroups. To prove the proposition, we prove the following lemma, which is a common problem in an introductory group theory course. A group is said to be *simple* if it has no nontrivial normal subgroups. If the group is Abelian, then this is equivalent to the group having no nontrivial subgroups. From the fundamental homomorphism theorems, if A is a subgroup of an Abelian group G, then the subgroups of G/A are in 1-1 correspondence with the subgroups of G that contain A. Thus, if A is a maximal subgroup of G, then G/A is simple.

Lemma 1. *Let G be a simple Abelian group. Then $|G|$ is prime.*

Proof. Let $a \in G$ be different from the identity of G. Then the cyclic group generated by a is a nonzero subgroup, so $G = \langle a \rangle$. This implies that $G \cong \mathbb{Z}$ if the order of a is infinite, or $G \cong \mathbb{Z}/n\mathbb{Z}$ if the order of a is n. However, $\mathbb{Z}$ has nontrivial subgroups, so the order of a must be finite. If this order n is not prime, and $n = rs$ with $1 < r, s$, then $\langle a^r \rangle$ is a subgroup of order s. This forces n to be prime. □

Recall that an Abelian group G is said to be *divisible* if for any $a \in G$ and any positive integer n, there is an element $b \in G$ with $a = nb$.

Proposition 2. *Let G be a divisible Abelian group. Then G has no maximal subgroups.*

Proof. Let A be a subgroup of G. If A is a maximal subgroup, then G/A is a simple group. However, the only simple Abelian groups are of prime order by Lemma 1. Thus, $[G : A] = p$ for some prime p, which yields $pG \subseteq A$. Let $a \in G \setminus A$. Since G is divisible, there is a $b \in G$ with $pb = a$. This is a contradiction since $a \in pG$ but $a \notin A$. Therefore, A is not a maximal subgroup. □

Corollary 3. *Let F be a field of characteristic 0. Then $(F, +)$ has no maximal subgroups.*

Proof. Let F be a field of characteristic 0. Then F contains an isomorphic copy of $\mathbb{Q}$ as a subfield. From this we see that $(F, +)$ is a divisible group, since if $\alpha \in F$ and $n \in \mathbb{N}$, then $\alpha/n \in F$ since F is a field and n is a nonzero element of F. Therefore, there is a $\beta \in F$, namely α/n, such that $n\beta = \alpha$. The proposition then shows that $(F, +)$ has no maximal subgroups. □

We can use the corollary to obtain more interesting examples of rings without maximal ideals. We consider a discrete valuation ring R with maximal ideal $M = (x)$ such that R contains a field F of characteristic 0 such that $R = M + F$. In other words, R is a discrete valuation ring containing an isomorphic copy of its residue field R/M. For example, we could take $R = F[[x]]$, the ring of power series in x over F, or $R = F[x]_{(x)}$, the localization of the polynomial ring $F[x]$ at the maximal ideal (x). We show that M has no maximal ideals. We point out that since R is a local ring with maximal ideal M, the group of units of R is $R \setminus M$. Furthermore, since $R = F + M$, every unit of R is of the form $\alpha + xf$ for some $f \in R$ and nonzero $\alpha \in F$, and any element of this form is a unit. Also, since R is a discrete valuation ring with maximal ideal (x), every element of R can be written uniquely in the form $x^r u$ for some $r \geq 0$ and unit u.

Theorem 4. *Let F be a field of characteristic 0. If R is a discrete valuation ring with maximal ideal M such that $F \subseteq R$ and $R = F + M$, then M, viewed as a ring, has no maximal ideals.*

Proof. Let N be a proper ideal of M. First suppose that $(x^2) \subseteq N$, and let

$$A = \{\alpha \in F : \alpha x \in N\}.$$

It is easy to see that A is an additive subgroup of F. Moreover, we claim that $N = (x^2) + Ax$. Since $(x^2) \subseteq N$, we have $(x^2) + Ax \subseteq N$ by definition of A. For the reverse inclusion, let $t \in N$. Since R is a discrete valuation ring, we may write $t = x^r u$ for some positive integer r and unit u. If $r \geq 2$, then $t \in (x^2) \subseteq (x^2) + Ax$, so suppose that $r = 1$. Then $t = xu = x(\alpha + xf)$ for some nonzero $\alpha \in F$ and $f \in R$ by the description above of the units of R. Therefore, as $t = \alpha x + x^2 f$ and $(x^2) \subseteq N$, we have $\alpha x = t - x^2 f \in N$, so $\alpha \in A$, and then $t = \alpha x + x^2 f \in (x^2) + Ax$. By Proposition 2, there is a subgroup A' of F with $A \subset A' \subset F$. Then $N \subset (x^2) + A'x$. It is easy to see that $(x^2) + A'x$ is an ideal of M since

$x(A'x) \subseteq (x^2)$. Therefore, N is not a maximal ideal of M. Next, suppose that $(x^2) \nsubseteq N$. If $(x^2) + N \subset M$, then N is not a maximal ideal of M. Suppose that $(x^2) + N = M$. Write $x = x^2 f + xg$ with $xg \in N$; we can write any element of N in this form since $N \subseteq M$. Then $1 = xf + g$, so $g = 1 - xf$. Therefore, g is a unit in R. Then for any $h \in R$, $x^2 h = (xg)(g^{-1}xh) \in NM \subseteq N$, a contradiction to the assumption that $(x^2) \nsubseteq N$. Thus, $(x^2) + N \neq M$, so it is a proper ideal of M properly containing N. This proves that N is not a maximal ideal in any case. Therefore, M has no maximal ideals. □

The following result shows that we had to work a bit to produce an example.

Proposition 5. *Let R be a commutative ring with 1, and let S be an ideal of R. If R contains a maximal ideal M with $S \nsubseteq M$, then S, viewed as a ring, has a maximal ideal.*

Proof. Since M is a maximal ideal of R not containing S, we have $R = M + S$. Write $1 = m + s$ with $m \in M$ and $s \in S$. For any $x \in S$, we have $x = (mx) + (sx)$, so $x \equiv sx \bmod MS$. In particular, $s \equiv s^2 \bmod MS$. Therefore, $s + MS$ is an identity of S/MS. The ring S/MS then has a maximal ideal N. The ideals of S/MS are in 1-1 inclusion preserving correspondence with the ideals of S that contain MS. Therefore, the preimage $N' = \{s \in S : s + MS \in N\}$ of N is an ideal of S containing MS, and it is maximal since N is maximal in S/MS. □

Recall that the Jacobson radical $J(R)$ of a commutative ring is the intersection of all maximal ideals of R. The proposition above shows that if $J(R) = 0$, then any nonzero ideal S of R is not contained in some maximal ideal of R, and so S has a maximal ideal. Similarly, if R is any commutative ring, and if S is an ideal of R not contained in $J(R)$, then S has maximal ideals. In particular, any nonzero ideal of the polynomial ring $k[x_1, \ldots, x_n]$ over a field k has maximal ideals, since it is known that Jacobson radical of this ring is (0).

We now give a second construction of a ring with no maximal ideals. This construction also comes from valuation theory, but we do not need to assume that the ring contains a field. In return for this, our valuation ring is not discrete. A *valuation* v on a field F is a function $v : F^* \to \Gamma$, where $F^* = F \setminus \{0\}$ and Γ is a totally ordered Abelian group, such that $v(ab) = v(a) + v(b)$ for all $a, b \in F^*$ and $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in F^*$ with $a + b \neq 0$. The image of v is a totally ordered subgroup of Γ, which is called the value group of v. Without loss of generality, we may assume that v is surjective, and so Γ is the value group of v. Let F be a field with a valuation v whose value group is a subgroup Γ of $\mathbb{R}$ having no smallest positive element, and let V be the valuation ring of v. That is,

$$V = \{x \in F^* : v(x) \geq 0\} \cup \{0\}.$$

Also, let M be the maximal ideal of V. Then

$$M = \{x \in F : v(x) > 0\} \cup \{0\}.$$

Then M is a ring under the induced operations of F. We claim that M does not have any maximal ideals.

Lemma 6. *The ideal M is not a principal ideal of V.*

Proof. If $x \in M$ is nonzero, then $v(x) > 0$. Since Γ has no smallest positive element, there is a y with $0 < v(y) < v(x)$. Then $y \in M$ and $y \notin xV$. Thus, $M \neq xV$. Since this is true for all $x \in M$, we see that M is not principal. □

Lemma 7. *If I is an ideal of V with $I \subset M$, then there is an ideal J of V with $I \subset J \subset M$.*

Proof. Let I be an ideal of V with $I \subset M$. Then there is an $x \in M$ with $x \notin I$. Then xV is an ideal of V. We claim that $I \subset xV$. Let $a \in I$ be nonzero. Since $x \notin I$, we have $xa^{-1} \notin V$. Therefore, $v(xa^{-1}) < 0$, so $v(x) < v(a)$. Then $v(ax^{-1}) \geq 0$, so $ax^{-1} \in V$, which gives $a \in xV$. This yields $I \subseteq xV$ as desired. Moreover, $I \subset xV$ since $x \notin I$. Finally, since M is not principal, by Lemma 6, we have $xV \subset M$. □

Lemma 8. *Let N be an ideal of M. If the set $\{v(x) : x \in N, x \neq 0\}$ has no least element, then N is an ideal of V.*

Proof. Suppose that N has no element of smallest value, and let $x \in N$. Then there is a $y \in N$ with $v(y) < v(x)$. For any $r \in V$ we have $rx = (ry)(xy^{-1})y \in N$ since $ry \in M$ and $xy^{-1} \in M$. Therefore, N is an ideal of V. □

We are now able to prove that M has no maximal ideals.

Theorem 9. *The ring M has no maximal ideals.*

Proof. Let $N \subset M$ be an ideal. First suppose that N has no element of smallest value. Then N is an ideal of V by Lemma 8, and so N is not a maximal ideal of M, by Lemma 7, since there is an ideal J of V with $N \subset J \subset M$, and J is also an ideal of M. On the other hand, suppose that N has an element x of least value. This implies that $N \subseteq xV$ since if $y \in N$, then $v(x) \leq v(y)$, so $y = x(yx^{-1}) \in xV$. Since Γ has no smallest positive element, there is a z with $0 < v(z) < v(x)$. Then $N \subseteq xV \subset zV \subset M$. In this case, we also see that N is not a maximal ideal of M. Thus, M cannot have a maximal ideal. □

We give two constructions of a field with a valuation v where the value group and residue field of v can be chosen arbitrarily. The residue field of v is the quotient ring V/M. Let k be a field and let Γ be a totally ordered Abelian group. We write Γ multiplicatively for the moment. The group ring $k[\Gamma]$ is the set of formal finite sums

$$k[\Gamma] = \left\{ \sum_{g \in \Gamma} a_\gamma \gamma : a_\gamma \in k, |\gamma : a_\gamma \neq 0| < \infty \right\}$$

with componentwise addition, and multiplication coming from the equation $(a\gamma)(b\delta) = (ab)(\gamma\delta)$. This is a ring with identity 1_Γ such that Γ is a subgroup of the group of units $k[\Gamma]^*$. Since Γ is a totally ordered group and k is a field, a short argument will show that the group ring $k[\Gamma]$ is an integral domain. Thus, it has a quotient field, which we denote by $k(\Gamma)$. We define a valuation $v : k(\Gamma)^* \to \Gamma$ by

$$v\left(\sum_{\gamma\in\Gamma} a_\gamma \gamma\right) = \min\{\gamma : a_\gamma \neq 0\}$$

for an element in $k[\Gamma]$, and if $\varphi \in k[\Gamma]$, writing $\varphi = f/g$ with $f, g \in k[\Gamma]$, we set $v(\varphi) = v(f) - v(g)$. A short calculation shows that v is well defined and a valuation. Its value group is Γ since $v(\gamma) = \gamma$, so v is surjective. The valuation ring V is

$$\begin{aligned} V &= \{\varphi \in k(\Gamma) : v(\varphi) \geq 0\} \\ &= \{f/g : v(f) \geq v(g)\} \\ &= \left\{ \frac{\sum_{\gamma\geq 0} a_\gamma \gamma}{\sum_{\gamma\geq 0} b_\gamma \gamma} : b_0 \neq 0 \right\} \end{aligned}$$

and

$$M = \left\{ \frac{\sum_{\gamma\geq 0} a_\gamma \gamma}{\sum_{\gamma\geq 0} b_\gamma \gamma} : a_0 = 0, b_0 \neq 0 \right\}.$$

From this description, we see that the residue field V/M is k, under the isomorphism induced from the map $V \to k$ given by

$$\frac{\sum_{\gamma\geq 0} a_\gamma \gamma}{\sum_{\gamma\geq 0} b_\gamma \gamma} \mapsto a_0.$$

For a second example, again with k an arbitrary field and Γ a totally ordered Abelian group, let F be the set of formal series

$$F = \left\{ \sum_{\gamma\in\Gamma} a_\gamma x^\gamma : a_\gamma \in k, \{\gamma : a_\gamma \neq 0\} \text{ is well ordered} \right\}.$$

We claim that F is a field under the obvious operations. Given this, define $v : F^* \to \Gamma$ by $v\left(\sum_{\gamma\in\Gamma} a_\gamma x^\gamma\right) = \min_\gamma \{a_\gamma \neq 0\}$. This exists by the definition of F. It is easy to see that v is a valuation with value group Γ. Moreover, Its valuation ring V is

$$V = \left\{ \sum_{\gamma\geq 0} a_\gamma x^\gamma \in F : a_\gamma \in k \right\}$$

and the maximal ideal is

$$M = \left\{ \sum_{\gamma> 0} a_\gamma x^\gamma \in F : a_\gamma \in k \right\}.$$

Furthermore, the map $\varphi : V \to k$ given by $\sum_{\gamma\geq 0} a_\gamma x^\gamma \to a_0$ is a surjective ring homomorphism with kernel M, so the residue field V/M is isomorphic to k. Therefore, F is a valued field with value group Γ and residue field k.

We now prove that F is a field. We prove a couple of lemmas about well ordered subsets of Γ to help to do this.

Lemma 10. *Let Γ be a totally ordered Abelian group.*

1. *If S and T are well ordered subsets of Γ, then $S \cup T$ is a well ordered subset of Γ.*
2. *If S and T are well ordered subsets of Γ, and if $S + T = \{s + t : s \in S, t \in T\}$, then $S + T$ is a well ordered subset of Γ.*

Proof. Let S and T be well ordered subsets. To show that $S \cup T$ is well ordered, let A be a nonempty subset of $S \cup T$. Then $A = S' \cup T'$ for some subsets $S' \subseteq S$ and $T' \subseteq T$. Let s be the least element of S' and t be the least element of T'. Then $\min\{s, t\}$ is the least element of $S' \cup T'$. Thus, A has a least element, and so $S \cup T$ is well ordered. For (2), suppose that $S + T$ is not well ordered. Then there is a strictly decreasing sequence $\{s_n + t_n : n \geq 1\}$ with $s_n \in S$ and $t_n \in T$. We note that, for any $t \in \{t_n : n \geq 1\}$, the number of n for which $t_n = t$ is finite. For, if we have a subsequence $\{s_{n_i} + t : i \geq 1\}$ of the original sequence, then $\{s_{n_i}\}$ is a decreasing sequence of elements of S. This cannot happen since S is well ordered. Let $t'_1 = \min\{t_n : n \geq 1\}$, and let $n_1 = \max\{n : t_n = t'_1\}$. Next, let $t'_2 = \min\{t_n : n \geq n_1, t_n \neq t'_1\}$ and $n_2 = \max\{n : t_n = t'_2\}$. Continuing gives an infinite sequence $n_1 < n_2 < \cdots$ of integers and terms $s_{n_1} + t_{n_1} > s_{n_2} + t_{n_2} > \cdots$ such that $t_{n_1} < t_{n_2} < \cdots$. Therefore, $s_{n_1} > s_{n_2} > \cdots$ is a decreasing sequence in S, which is impossible. This contradiction shows that $S + T$ is well ordered. □

From the lemma we can see that addition is well defined since given $a = \sum_\gamma a_\gamma x^\gamma$ and $b = \sum_\gamma b_\gamma x^\gamma$ in F, as $\{\gamma : a_\gamma \neq 0\}$ and $\{\gamma : b_\gamma \neq 0\}$ are both well ordered subsets of Γ, their union is also well ordered, and so any subset of the union is also well ordered. Since $\{\gamma : a_\gamma + b_\gamma \neq 0\}$ is such a subset, it is well ordered, and so $a + b = \sum_\gamma (a_\gamma + b_\gamma) x^\gamma \in F$. To see that multiplication is well defined, let $a = \sum_\gamma a_\gamma x^\gamma$ and $b = \sum_\gamma b_\gamma x^\gamma$. Then $ab = \sum_\gamma \left(\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right) x^\gamma$. To see that this is well defined and is an element of F, we first show that, for all γ, the sum $\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta$ is a finite sum. Let $T = \{\alpha : a_\alpha \neq 0, b_{\gamma-\alpha} \neq 0\}$, a subset of the support of a. Then T is well ordered since the support of a is well ordered. If T is not finite, then there is an infinite sequence $\alpha_1 < \alpha_2 < \cdots$ in T. If $S = \{\gamma - \alpha_i : i \geq 1\}$, a subset of the support of b, then S must have a minimum element. This would imply that T the sequence has a maximum element. This contradiction shows that T is finite, and so the sum is indeed finite. The next thing we need to check is that the support of ab is well ordered. However, it is clear that if $S = \mathrm{supp}(a)$ and $T = \mathrm{supp}(b)$, then the support of ab is contained in $S + T$. Since this set is well ordered by the lemma, so is the support of ab. Therefore, $ab \in F$. Now that our operations are well defined, it is formal to show that F is a commutative ring.

To finish the argument, we must show that every nonzero element has a multiplicative inverse. Let $a = \sum_\gamma a_\gamma x^\gamma \in F$. If $\gamma_0 = \min\{\gamma : a_\gamma \neq 0\}$, then by multiplying by $x^{-\gamma_0}$, we may assume that $a = \sum_{\gamma \in S} a_\gamma x^\gamma$, where S is a well ordered subset of Γ such that every element of S is nonnegative, and that $a_0 \neq 0$. We set $T = S + S$, a well ordered subset of Γ. Finally, we construct, by transfinite induction, $b = \sum_{\gamma \in T} b_\gamma x^\gamma$ such that $ab = 1$. Note that 0 is the least element of T. Since $ab = a_0 b_0 + \cdots$, we set $b_0 = a_0^{-1}$. Now, assume that $\gamma > 0$ and that b_β has been constructed for all $\beta < \gamma$. The coefficient of x^γ in ab is $\sum_{\alpha+\beta=\gamma} a_\alpha b_\beta$, and for this to be 0, we see that

$$b_\gamma = -a_0^{-1} \sum_{\substack{\alpha+\beta=\gamma, \\ \alpha>0}} a_\alpha b_\beta.$$

This is a finite sum by the same argument as showed that multiplication is well defined earlier. Then, by transfinite induction, we have an element $b \in F$ with $ab = 1$. This finishes the proof that F is a field.