Normed Vector Spaces and Double Duals

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In this note we look at a number of infinite-dimensional $\mathbb{R}$-vector spaces that arise in analysis, and we consider their dual and double dual spaces. As an application, we give an example of an infinite-dimensional vector space V for which the natural map $\eta : V \to V^{**}$ is not an isomorphism. In analysis, duals and double duals of vector spaces are often defined differently than in algebra, by considering continuity. We will be more specific shortly.

Let V be an $\mathbb{R}$-vector space. We say that V is a *normed vector space* if there is a function $\|\cdot\| : V \to \mathbb{R}$ which satisfies

- $\|v\| \geq 0$ for all $v \in V$, and $\|v\| = 0$ if and only if $v = 0$.
- $\|\alpha v\| = |\alpha| \cdot \|v\|$ for all $\alpha \in \mathbb{R}$ and $v \in V$.
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

The function $\|\cdot\|$ is called a *norm* on V. If V is a normed vector space, then the function $\|\cdot\|$ allows us to define a metric on V by $d(v, w) = \|v - w\|$, just as we do for $\mathbb{R}$ or $\mathbb{R}^n$. We can then talk about functions $f : V \to \mathbb{R}$ being continuous at $v_0 \in V$: if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|v - v_0\| < \delta$ implies $\|f(v) - f(v_0)\| < \varepsilon$, then f is continuous at v_0.

Example 1. If $V = \mathbb{R}^n$, then V is a normed vector space under the usual Euclidean metric $\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Recall that the set S of all real valued sequences is an $\mathbb{R}$-vector space under pointwise addition and scalar multiplication. The next example gives a collection of subspaces of this sequence space.

Example 2. For $x = \{x_n\} \in S$, define, for $p \geq 1$,

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}$$

and

$$\|x\|_\infty = \sup_n \{|x_n|\}.$$

If $p = 1$, then $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$. By the convention that $\infty + \infty = \infty$ and $r < \infty$ for each $r \in \mathbb{R}$, we see that each of these three functions satisfy all properties of a norm except possibly for $\|x\| < \infty$. Set

$$l^p = \left\{ x \in S : \|x\|_p < \infty \right\},$$
$$l^\infty = \{ x \in S : \|x\|_\infty < \infty \}.$$

Then each of these are normed vector spaces. Furthermore, let

$$c_0 = \left\{ \{x_n\} \in l^\infty : \lim_{n\to\infty} x_n = 0 \right\}.$$

Then c_0 is a subspace of l^∞, and so c_0 is a normed vector space with respect to the norm $\|\cdot\|_\infty$. A short argument shows that if $p \le p'$, then $l^p \subseteq l^{p'}$. We then have the containments $l^1 \subseteq l^p \subseteq c_0 \subseteq l^\infty$for each $p \ge 1$.

Example 3. If (X, μ) is a measure space, $p \ge 1$, and V is the vector space of all measurable functions $X \to \mathbb{R}$, then V is a normed vector space under the norm

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

To make $\|\cdot\|_p$ a norm, we must identify functions that agree almost everywhere, since $\|f\|_p = 0$ if and only if $f = 0$ a.e. A proof of the triangle inequality can be found in [1, Thm 3.5].

Let V be a normed vector space. If $T : V \to \mathbb{R}$ is a linear functional, define

$$\|T\| = \sup \{ |T(x)| : x \in V, \|x\| = 1 \}. \tag{1}$$

We say T is *bounded* if $\|T\| < \infty$. We note that if T is linear and $x \in V$ is nonzero, then we may write $x = \alpha y$ with $\alpha = \|x\|$ and $y = x/\|x\|$. Then $T(x) = \alpha T(y) = \|x\| T(y)$. Consequently, $|T(x)| / \|x\| = |T(y)|$. Thus,

$$\|T\| = \sup \left\{ \frac{|T(x)|}{\|x\|} : x \in V, x \neq 0 \right\}.$$

As a consequence, $|T(x)| \le \|T\| \cdot \|x\|$ for all $x \in V$. More generally, this calculation shows that if $x = \alpha y$ for any nonzero $\alpha \in \mathbb{R}$, then $|T(x)| / \|x\| = |T(y)| / \|y\|$.

Lemma 4. *Let $T : V \to \mathbb{R}$ be a linear functional. Then the following statements are equivalent.*

(1) *T is bounded.*

(2) *T is uniformly continuous.*

(3) *T is continuous.*

(4) *T is continuous at some $v_0 \in V$.*

Proof. (1) implies (2): Suppose that T is bounded. Let $\varepsilon > 0$ and take $v, w \in V$. Then $|T(w) - T(v)| = |T(w - v)| \leq \|T\| \cdot \|w - v\|$. Thus, if we define $\delta = \varepsilon / \|T\|$, this calculation shows that $\|w - v\| < \delta$ implies $|T(w) - T(v)| < \varepsilon$. Therefore, T is uniformly continuous.

(2) implies (3) and (3) implies 4) are both trivial.

(4) implies (1): Suppose T is continuous at v_0. Then for $\varepsilon = 1$, there is a $\delta > 0$ such that if $\|v - v_0\| < \delta$, then $\|T(v) - T(v_0)\| < 1$. By setting $x = v - v_0$ and noting that $T(v) - T(v_0) = T(x)$, we see that if $\|x\| < \delta$, then $\|T(x)\| < 1$. Let $v \in V$ be nonzero and let $x = \frac{\delta}{2\|v\|} v$. Then $\|x\| = \delta/2$, so $|T(x)| < 1$. Since $|T(x)| / \|x\| = |T(v)| / \|v\|$, we see that$|T(v)| / \|v\| \leq 2/\delta$. This implies that $\|T\| \leq 2/\delta$, so T is bounded. □

Let
$$\hom_{\mathrm{b}}(V, \mathbb{R}) = \{T \in \hom(V, \mathbb{R}) : \|T\| < \infty\}.$$
The set $\hom_{\mathrm{b}}(V, \mathbb{R})$ consists of all bounded linear functionals on V. By the lemma, this set is the same as the set of all continuous linear functionals on V. Since the sum and difference of continuous maps is continuous, and any scalar multiple of a continuous map is continuous, we see that $\hom_{\mathrm{b}}(V, \mathbb{R})$ is a subspace of $\hom(V, \mathbb{R})$. We consider $\hom_{\mathrm{b}}(V, \mathbb{R})$ to be the analytic dual space of V.

Lemma 5. *If V is a normed vector space, then $\hom_{\mathrm{b}}(V, \mathbb{R})$ is a normed vector space under the definition of the norm $\|T\|$ given in Equation (1) above.*

Proof. It is clear that $\|T\| \geq 0$ for any $T \in \hom_{\mathrm{b}}(V, \mathbb{R})$, and that if $\|T\| = 0$, then $T(x) = 0$ for all $x \in V$, so $T = 0$. Next, let $\alpha \in \mathbb{R}$. Then
$$\begin{aligned} \|\alpha T\| &= \sup\{|\alpha T(x)| : \|x\| = 1\} = \sup\{|\alpha| \cdot |T(x)| : \|x\| = 1\} \\ &= |\alpha| \cdot \sup\{|T(x)| : \|x\| = 1\} = |\alpha| \cdot \|T\|. \end{aligned}$$
Finally, if $S, T \in \hom_{\mathrm{b}}(V, \mathbb{R})$, then
$$\begin{aligned} \|S + T\| &= \sup\{|S(x) + T(x)| : \|x\| = 1\} \leq \sup\{|S(x)| + |T(x)| : \|x\| = 1\} \\ &\leq \sup\{|S(x)| : \|x\| = 1\} + \sup\{|T(x)| : \|x\| = 1\} \\ &= \|S\| + \|T\| \end{aligned}$$
since $|S(x) + T(x)| \leq |S(x)| + |T(x)|$. Thus, $\hom_{\mathrm{b}}(V, \mathbb{R})$ is a normed vector space. □

The analytic double dual of a normed vector space is $\hom_{\mathrm{b}}(\hom_{\mathrm{b}}(V, \mathbb{R}), \mathbb{R})$. As with the double dual of an arbitrary vector space, we have a natural map $\eta' : V \to \hom_{\mathrm{b}}(\hom_{\mathrm{b}}(V, \mathbb{R}), \mathbb{R})$, defined by $\eta'(v)(f) = f(v)$.

Let W be a subspace of a vector space V, and let $T : W \to U$ be a linear transformation. By using bases, we can produce a subspace W' of V for which $V = W \oplus W'$. We can then extend T to V by defining $T(w + w') = T(w)$ for each $w' \in W'$. This argument shows that we can always extend linear transformations on a subspace to the space itself. The following is the analogue of this result in analysis.

Theorem 6 (Hahn-Banach). *Let V be a normed vector space. If W is a subspace of V and $f : W \to \mathbb{R}$ is a bounded linear functional, then there is a linear functional $F : V \to \mathbb{R}$ with $F|_W = f$ and $\|F\| = \|f\|$.*

We refer to analysis texts for a proof of this theorem; see, for example, [1, Thm. 5.6]. Its proof uses a Zorn's lemma argument, as does the abstract vector space analogue we mentioned earlier.

Lemma 7. *Let V be a normed vector space, and let $\eta' : V \to \mathrm{hom_b}(\mathrm{hom_b}(V, \mathbb{R}), \mathbb{R})$ be the map defined above by $\eta'(v)(f) = f(v)$. Then $\|\eta'(v)\| = \|v\|$. Thus, $\eta'(v)$ is a bounded linear functional on $\mathrm{hom_b}(V, \mathbb{R})$, and so $\eta'(v) \in \mathrm{hom_b}(\mathrm{hom_b}(V, \mathbb{R}), \mathbb{R})$. Moreover, η' is an injective linear transformation.*

Proof. It is very easy to prove that $\eta'(v)$ is a linear functional, so the only issue is to prove that it is bounded. We have

$$\|\eta'(v)\| = \sup \{|f(v)| : f \in \mathrm{hom_b}(V, \mathbb{R}), \|f\| = 1\}.$$

Since $|f(v)| \le \|f\| \cdot \|v\| = \|v\|$ for f with $\|f\| = 1$, we see that $\|\eta'(v)\| \le \|v\|$. This is enough to prove that $\eta'(v)$ is bounded. To prove equality, define $f_0 : \mathbb{R}v \to \mathbb{R}$ by $f_0(\alpha v) = \|\alpha v\|$. It is trivial to see that f_0 is a bounded linear functional on $\mathbb{R}v$ with $\|f_0\| = 1$. By the Hahn-Banach theorem, there is a bounded linear functional $f : V \to \mathbb{R}$ such that $f|_{\mathbb{R}v} = f_0$ and $\|f\| = 1$. Since $f(v) = \|v\|$, we see that $\|\eta'(v)\| \ge |f(v)| = \|v\|$. This gives the reverse inequality, and so $\|\eta'(v)\| = \|v\|$.

It is an easy argument to see that η' is a linear map. Another application of the Hahn-Banach theorem shows that η' is injective: If $v \neq 0$, define $f_0 : \mathbb{R}v \to \mathbb{R}$ by $f_0(\alpha v) = \alpha$. By the Hahn-Banach theorem, there is $f \in \mathrm{hom_b}(V, \mathbb{R})$ with $f(v) = f_0(v) = 1$. Then $\eta'(v)(f) = 1$, so $\eta'(v) \neq 0$. Thus, $\ker(\eta') = \{0\}$, so η' is injective. □

We now consider the spaces l^p, l^∞, and c_0. We show how to obtain bounded linear functionals on them in the following lemma.

Lemma 8. *Let $x = \{x_n\}, y = \{y_n\}$ be sequences of real numbers. Define T_y by $T_y(x) = \sum_{n=1}^\infty x_n y_n$.*

(1) *Let $y \in l^\infty$. Then T_y is a well-defined linear functional on l^1 with $\|T_y\| = \|y\|_\infty$.*

(2) *Let $p, q \ge 1$ with $1/p + 1/q = 1$, and let $y \in l^q$. Then T_y is a well-defined linear functional on l^p with $\|T_y\| = \|y\|_q$.*

(3) *Let $y \in l^1$. Then T_y is a well-defined linear functional on l^∞ with $\|T_y\| = \|y\|_1$.*

Proof. Once we know that T_y is well-defined; that is, the sequence $\sum_{n=1}^{\infty} x_n y_n$ is convergent for each appropriate x, the linearity is easy to prove. For, let $\{x_n\}, \{z_n\}$ be sequences in the appropriate space, and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} T_y(\alpha\{x_n\} + \beta\{z_n\}) &= \sum_{n=1}^{\infty} y_n(\alpha x_n + \beta z_n) = \sum_{n=1}^{\infty} y_n \alpha x_n + y_n \beta z_n \\ &= \alpha \sum_{n=1}^{\infty} y_n x_n + \beta \sum_{n=1}^{\infty} y_n z_n = \alpha T_y(\{x_n\}) + \beta T_y(\{z_n\}). \end{aligned}$$

(1). Let $y \in l^\infty$, and let $x \in l^1$. Then $\sum_{n=1}^{\infty} |x_n y_n| \le \sum_{n=1}^{\infty} \|y\|_\infty \cdot |x_n| = \|y\|_\infty \cdot \|x\|_1 < \infty$, so $\sum_{n=1}^{\infty} x_n y_n$ is an absolutely convergent series. Thus, $T_y(x) \in \mathbb{R}$, so T_y is well-defined. Moreover, this shows that $\|T\| \le \|y\|_\infty$. For the reverse inequality, let e_n be the sequence whose n-th term is 1 and all other terms 0. Then $e_n \in l^1$ and $T_y(e_n) = y_n$. Thus, $|y_n| = |T(e_n)| \le \|T\| \|e_n\|_1 = \|T\|$. Thus, $\|y\|_\infty = \sup\{|y_n|\} \le \|T\|$. Therefore, $\|T\| = \|y\|_\infty$.

(2). We recall the *Holder inequality* [1, Thm. 3.5], which says that $\sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_p \|y\|_q < \infty$. Therefore, T_y is well-defined and $\|T\| \le \|y\|_q$. For the reverse inequality, let

$$s_N = (\operatorname{sgn}(y_1)|y_1|^{q-1}, \ldots, \operatorname{sgn}(y_n)|y_n|^{q-1}, 0, \ldots) \in l^p.$$

Then $T(s_N) = \sum_{n=1}^{N} |y_n|^q$. Since $p + q = pq$, the inequality $|T(s_N)| \le \|T\| \cdot \|s_N\|_p$ says

$$\sum_{n=1}^{N} |y_n|^q \le \|T\| \cdot \left(\sum_{n=1}^{N} \left(|y_n|^{q-1}\right)^p\right)^{1/p} = \|T\| \cdot \left(\sum_{n=1}^{N} |y_n|^q\right)^{1/p} = \|T\| \cdot \left(\sum_{n=1}^{N} |y_n|^q\right)^{1-1/q},$$

so

$$\left(\sum_{n=1}^{N} |y_n|^q\right)^{1/q} \le \|T\|.$$

Letting $N \to \infty$, we obtain $\|y\|_q \le \|T\|$, and so $\|T\| = \|y\|_q$.

(3). The argument is virtually identical to that in (1): Let $y \in l^1$, and let $x \in l^\infty$. Then $\sum_{n=1}^{\infty} |x_n y_n| \le \sum_{n=1}^{\infty} |y_n| \cdot \|x\|_\infty = \|y\|_1 \cdot \|x\|_\infty < \infty$, so $\sum_{n=1}^{\infty} x_n y_n$ is an absolutely convergent series. Thus, $T_y(x) \in \mathbb{R}$, so T_y is well-defined and $\|T\| \le \|y\|_1$. For the reverse inequality, let $s_N = (\operatorname{sgn}(y_1), \ldots, \operatorname{sgn}(y_N), 0, \ldots) \in l^\infty$. We have $\|s_N\|_\infty = 1$. Therefore, $\sum_{n=1}^{N} |y_n| = T_y(s_n) \le \|T\|$. Letting $N \to \infty$, we get $\|y\|_1 \le \|T\|$. □

By restricting the domain, we see that for any $y \in l^1$, the map T_y yields a bounded linear functional $c_0 \to \mathbb{R}$. We will see below that Lemma 8 describes all bounded linear functionals on l^p for all $p \ge 1$ and on c_0. To help us do this, recall that a linear transformation is determined by its action on a basis. We need an analogue of this fact for continuous linear transformations. If V is a normed vector space, we call a sequence $\{v_n\}_{n=1}^{\infty}$ of elements of V a *topological basis* of V if each $x \in V$ can be written in the form $x = \sum_{n=1}^{\infty} a_n v_n$ for some $a_n \in \mathbb{R}$. This means that, for each $x \in V$, there is a sequence of real numbers $\{a_n\}$ such that $\lim_{N\to\infty} \sum_{n=1}^{N} a_n v_n \to x$ with respect to the norm on V. The existence of a nice topological basis of l^p for $p \ge 1$ and of c_0 will be a key for us in determining their dual spaces.

Lemma 9. *Let e_n be the sequence whose n-th term is 1 and all of whose other terms are 0. Then $\{e_n\}$ is a topological basis for l^p for each $p \geq 1$ and for c_0.*

Proof. The e_n are elements of all the sequence spaces we have discussed. Let $x = \{x_n\}$. We claim that $x = \sum_{n=1}^{\infty} x_n e_n$ for all x in the spaces stated in the lemma. We must prove that if $s_N = \sum_{n=1}^{N} x_n e_n$, then $\lim_{N\to\infty} s_N = x$, the convergence taking place in the given space we are considering. That is, we must prove, for each $p \geq 1$, if $x \in l^p$, then $\|s_N - x\|_p \to 0$ as $N \to \infty$, and if $x \in c_0$, then $\|s_N - x\|_\infty \to 0$. First let $x \in l^p$. Then $\sum_{n=1}^{\infty} |x_n|^p < \infty$. We have $s_N = \{x_1, x_2, \ldots, x_n, 0, \ldots\}$, so $\|s_N - x\|_p^p = \sum_{n>N} |x_n|^p$. Because $\sum_{n=1}^{\infty} |x_n|^p$ is a convergent series, $\sum_{n>N} |x_n|^p \to 0$ as $N \to \infty$, which shows that $s_N \to x$ in l^p. Now suppose that $x \in c_0$. Then $\lim_{n\to\infty} x_n \to 0$. We have $\|s_N - x\|_\infty = \sup\{|x_n| : n > N\} = 0$ since $x_n \to 0$. Thus, $s_n \to x$ in c_0. $\square$

We now determine the analytic dual space $\hom_{\mathrm{b}}(l^p, \mathbb{R})$ and $\hom_{\mathrm{b}}(c_0, R)$. To give some terminology, if V, W are normed vector spaces, then we say that $V \cong W$ as normed spaces if there is a vector space isomorphism $\varphi : V \to W$ with $\|\varphi(v)\| = \|v\|$ for all $v \in V$.

Proposition 10.

(1) *We have* $\hom_b(l^1, \mathbb{R}) = \{T_y : y \in l^\infty\}$, *and* $\hom_b(l^1, \mathbb{R}) \cong l^\infty$ *as normed spaces.*

(2) *If* $1/p + 1/q = 1$, *then* $\hom_{\mathrm{b}}(l^p, \mathbb{R}) = \{T_y : y \in l^q\}$, *and* $\hom_{\mathrm{b}}(l^p, \mathbb{R}) \cong l^q$ *as normed spaces.*

(3) *We have* $\hom_b(c_0, \mathbb{R}) = \{T_y|_{c_0} : y \in l^1\}$, *and* $\hom_b(c_0, \mathbb{R}) \cong l^1$ *as normed spaces.*

Proof. The main idea in all three statements is the following: suppose that $\{v_n\}$ is a topological basis for a normed vector space V, and let T be continuous linear functional on V. If $x = \sum_{n=1}^{\infty} a_n v_n$, then

$$T(x) = T\left(\lim_{N\to\infty} \sum_{n=1}^{N} a_n v_n\right) = \lim_{N\to\infty} T\left(\sum_{n=1}^{N} a_n v_n\right) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n T(v_n) \qquad (2)$$
$$= \sum_{n=1}^{\infty} a_n T(v_n).$$

Thus, T is uniquely determined by the sequence $\{T(v_n)\}$. By Lemma 9, we may work with the topological basis $\{e_n\}$ in all three cases; Equation (2) shows that if $x = \{x_n\}$, then $T(x) = \sum_{n=1}^{\infty} x_n T(e_n)$ for all x in any of the spaces under consideration. Moreover, whenever T_y and T_z are defined, it is a trivial argument to prove that $T_{\alpha y + \beta z} = \alpha T_y + \beta T_z$ for any $\alpha, \beta \in \mathbb{R}$. Thus, the map $y \mapsto T_y$ is linear. Note that if $y = \{y_n\}$, then $T_y(e_n) = y_n$. Thus, if $T_y = 0$, then each $y_n = 0$, so $y = 0$.

(1). Let $T \in \hom_{\mathrm{b}}(l^1, \mathbb{R})$. Since T is bounded, $|T(e_n)| \leq \|T\| \cdot \|e_n\|_1$. Thus, $y = \{T(e_n)\} \in l^\infty$, and from the description of T above, we see that $T(x) = \sum_{n=1}^{\infty} x_n T(e_n) = T_y(x)$. Thus,

$T = T_y$. This yields $\hom_b(l^1, \mathbb{R}) = \{T_y : y \in l^\infty\}$. As we pointed out in general, the map $y \mapsto T_y$ is an injective linear map, and so is an isomorphism. Moreover, since $\|T_y\| = \|y\|_\infty$, it is an isomorphism of normed spaces.

(2) Let $T \in \hom_b(l^p, \mathbb{R})$ and set $y = \{T(e_n)\}$. Since T is bounded, the argument in Statement (2) of Proposition 10 used to prove $\|y\|_q \leq \|T\|$ shows that $y \in l^q$. Thus, as in (1), we obtain the result.

(3). Let $T \in \hom_b(c_0, \mathbb{R})$ and set $y = \{T(e_n)\}$. The argument in Statement (3) of Proposition 10 shows that $y \in l^1$, and as before, we get $\hom_b(c_0, \mathbb{R}) = \{T_y|_{c_0} : y \in l^1\}$, and $\hom_b(c_0, \mathbb{R}) \cong l^1$ as normed spaces. $\square$

Unlike the case for the spaces l^p, we have $\hom_b(l^\infty, \mathbb{R}) \neq \{T_y : y \in l^1\}$, as we will see shortly. The problem is that l^∞ does not have a topological basis. To help see this, we recall that a topological space X is said to be *separable* if X contains a countable dense subset.

Proposition 11. *Let V be a normed vector space. If V has a topological basis, then V is separable.*

Proof. Let $\{v_n\}$ be a topological basis for V. Then $A := \left\{\sum_{n=1}^N q_n v_n : q_n \in \mathbb{Q}, N \geq 1\right\}$ is a countable set. We claim that it is dense in V. To see this, take $x \in V$, and write $x = \sum_{n=1}^\infty a_n v_n$ for some $a_n \in \mathbb{R}$. Let $\varepsilon > 0$. Then there is an N such that $\left\|x - \sum_{n=1}^N a_n v_n\right\| < \varepsilon/2$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find $q_n \in \mathbb{Q}$ such that

$$|a_n - q_n| < \frac{\varepsilon}{2^{n+1}\|v_n\|}$$

for each n. Then

$$\left\|\sum_{n=1}^N a_n v_n - \sum_{n=1}^N q_n v_n\right\| \leq \sum_{n=1}^N |a_n - q_n| \|v_n\| \leq \sum_{n=1}^N \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2}.$$

Consequently, $\left\|x - \sum_{n=1}^N q_n v_n\right\| < \varepsilon$. This proves that A is a countable dense subset of V. Therefore, V is separable. $\square$

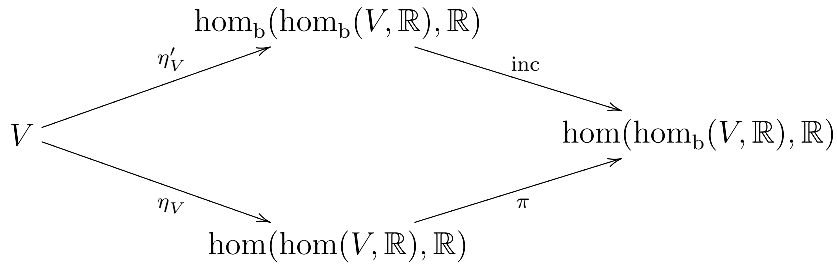
As a consequence of the proposition, l^p for each $p \geq 1$ and c_0 are separable, since we showed in Lemma 9 that each has a topological basis.

Example 12. The space l^∞ is not separable, and so does not have a topological basis; for if $\{v_n\}_{n=1}^\infty$ is a countable subset of l^∞, define a sequence $x = \{x_n\}$ by

$$x_n = \begin{cases} 0 & \text{if} \quad v_{n,n} \geq \frac{1}{2} \\ 1 & \text{if} \quad v_{n,n} < \frac{1}{2} \end{cases}.$$

Then $|x_n - v_{n,n}| \geq 1/2$. Clearly $x \in l^\infty$ and $\|x - v_n\|_\infty \geq 1/2$. This proves that $\{v_n\}_{n=1}^\infty$ is not dense in l^∞.

We finish this note by proving that the canonical maps η_V and η'_V are not always surjective, unlike the case of finite dimensional vector spaces. Consider the following diagram



where $\mathrm{inc} : \hom_{\mathrm{b}}(\hom_{\mathrm{b}}(V, \mathbb{R}), \mathbb{R}) \to \hom(\hom_{\mathrm{b}}(V, \mathbb{R}), \mathbb{R})$ is the inclusion map, and $\pi : \hom(\hom(V, \mathbb{R}), \mathbb{R}) \to \hom(\hom_{\mathrm{b}}(V, \mathbb{R}), \mathbb{R})$ is the restriction of domain map.

Proposition 13. *The maps* $\eta_{l^1} : l^1 \to \hom(\hom(l^1, \mathbb{R}), \mathbb{R})$ *and* $\eta'_{l^1} : l^1 : \hom_{\mathrm{b}}(\hom_{\mathrm{b}}(l^1, \mathbb{R}), \mathbb{R})$ *are not surjective.*

Proof. An easy calculation shows that $\pi \circ \eta_V = \mathrm{inc} \circ \eta'_V$. The map π is surjective since every linear functional on $\hom_{\mathrm{b}}(V, \mathbb{R})$ can be extended to a linear functional on $\hom(V, \mathbb{R})$. If η_V were surjective, then $\pi \circ \eta_V$ would be surjective, and this would force inc to be surjective. We show this is not true. By Proposition 10, we identify l^∞ with $\hom_{\mathrm{b}}(l^1, \mathbb{R})$ by identifying y with T_y. Define a linear transformation S on the span of $\{e_n : n \geq 1\}$ by $S(e_n) = n$, and extend S in any way to all of l^∞. Then S is not bounded, so S does lie in the image of inc. Thus, η_{l^1} is not surjective.

To see that η'_{l^1} is not surjective we need an analytic variant of the argument in the previous paragraph. Note that each $T_y \in \hom_{\mathrm{b}}(l^\infty, \mathbb{R})$ coming from a nonzero element $y \in l^1$ has the property that T_y is a nonzero operator on c_0. This is clear since if $y = \{y_n\}$ with $y_m \neq 0$, then $e_m \in c_0$ and $T_y(e_m) = y_m \neq 0$. We will produce a nonzero $T \in \hom_{\mathrm{b}}(l^\infty, \mathbb{R})$ for which $T|_{c_0} = 0$. Let v be the constant sequence whose n-th term is 1 for each n. Then $v \in l^\infty$ but $v \notin c_0$. The sum $c_0 + \mathbb{R}v$ is direct since clearly the only sequence in $\mathbb{R}v$ converging to 0 is the zero sequence. Consider the linear transformation $T_0 : c_0 + \mathbb{R}v \to \mathbb{R}$ defined by $T_0(w + rv) = r$ for all $r \in \mathbb{R}$ and $w \in c_0$. Then T_0 is bounded, since if $\{w_n + r\} \in c_0 + \mathbb{R}v$ has norm 1, then $|w_n + r| \leq 1$ for each n. This forces $|r| \leq 1$, and so $|T_0(w + rv)| = |r| \leq 1$. Thus, $\|T_0\| \leq 1$. By the Hahn-Banach theorem, we can extend T_0 to a bounded linear functional T on l^∞. Since $T|_{c_0} = 0$, the functional $T \neq \eta'(y)$ for every $y \in l^1$. □

References

[1] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, 1987.