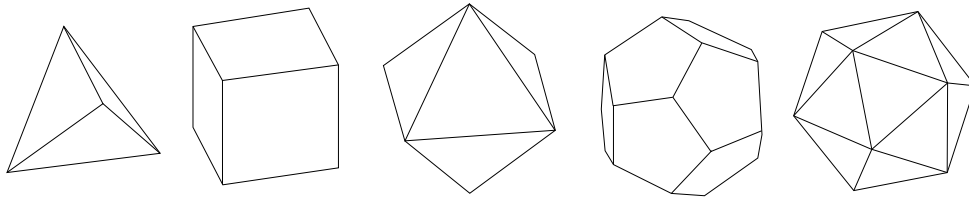


# Computing the Symmetry Groups of the Platonic Solids With the Help of Maple

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In this note we will determine the symmetry groups of the Platonic solids by a combination of some elementary group theory and use of the computer algebra package Maple. The five Platonic solids are the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. By determining a symmetry group, we mean not just to determine its elements but to identify it, up to isomorphism, with a well-known group, such as a symmetric or alternating group. As we will see, we can use Maple not just to determine the elements of a symmetry group but to identify the group, once we apply the appropriate group theory.

The symmetry group of a 3-dimensional figure is the set of all distance preserving maps, or isometries, of  $\mathbb{R}^3$  which map the figure to itself, and with composition as the operation. To make this more concrete, we view each Platonic solid centered at the origin. An isometry which sends the solid to itself then must fix the origin. We recall (see [1, Ch. 4, Prop. 5.16]) that an isometry which preserves the origin is a linear transformation, and is represented by a matrix  $A$  satisfying  $A^T A = I_3$ . Thus, the symmetry group of a Platonic solid is isomorphic to a subgroup of the orthogonal group  $O_3(\mathbb{R}) = \{A \in GL_3(\mathbb{R}) : A^T A = I\}$ . Elements of  $O_3(\mathbb{R})$  are either rotations or reflections across a plane, depending on whether the matrix has determinant 1 or  $-1$ . The set of rotations is then the subgroup  $SO_3(\mathbb{R})$  of  $O_3(\mathbb{R})$ .

Let  $G$  be the symmetry group of a Platonic solid, viewed as a subgroup of  $O_3(\mathbb{R})$ . If  $R = G \cap SO_3(\mathbb{R})$ , then  $R$  is the subgroup of rotations in  $G$ . We note that if  $z : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is defined by  $z(x) = -x$  for all  $x \in \mathbb{R}^3$ , then  $z$  is a reflection,  $z$  is a central element in  $O_3(\mathbb{R})$ , and  $O_3(\mathbb{R}) = SO_3(\mathbb{R}) \times \langle z \rangle \cong SO_3(\mathbb{R}) \times \mathbb{Z}_2$ . These facts are all easy to prove. Thus,

$[O_3(\mathbb{R}) : SO_3(\mathbb{R})] = 2$ . As a consequence,  $[G : R] \leq 2$ . The element  $z$  is a symmetry of all the Platonic solids except for the tetrahedron, and there are reflections which preserve the tetrahedron. Therefore,  $[G : R] = 2$  in all cases, and  $G \cong R \times \mathbb{Z}_2$  for the four largest solids. Thus, for them, it will be sufficient to determine the rotation subgroup  $R$ .

Our approach to computing the symmetry groups of the Platonic solids is twofold. The first step is to determine the order of the group. We will do this by using a simple counting argument to find an upper bound for the order of the symmetry group  $G$ . We will then choose two rotations in  $G$  and use Maple to show that the group  $R_0$  generated by them has order exactly equal to half this upper bound. Since  $|R| = \frac{1}{2}|G|$ , this calculation yields  $R = R_0$  and that  $|G|$  is equal to our upper bound.

To come up with our counting argument, we first identify a symmetry group as a subgroup of the permutation group of the solid's vertices. By numbering the vertices, we can then view the symmetry group as a subgroup of the symmetric group  $S_n$ , for some  $n$ . We point out that any symmetry of a Platonic solid is determined by its action on three vertices, as long as they do not lie on a plane through the origin; this is because any such symmetry is a linear transformation, and so is determined by its action on three linearly independent vectors. We will use this fact without further comment.

Once we have the order of a symmetry group, we will use a result from the theory of group actions to determine the group up to isomorphism. If  $S$  is a set, we denote by  $\text{Perm}(S)$  the group of permutations of  $S$ . If  $R$  is a group and  $H$  a subgroup of  $R$ , then there is a homomorphism  $\varphi : R \rightarrow \text{Perm}(R/H)$ , where  $R/H$  is the set of left cosets of  $H$  in  $R$ , defined as follows. For  $g \in R$ , the permutation  $\varphi(g)$  is defined by  $xH \mapsto gxH$ . The kernel of  $\varphi$  is the largest normal subgroup of  $R$  contained in  $H$ . If  $[R : H] = n$ , then  $\text{Perm}(R/H) \cong S_n$ , so  $\varphi$  yields a homomorphism from  $R$  to  $S_n$ . Proofs of these facts can be found in [3, Theorem 2.4.2] or [2, Theorem 2.9.2]. Maple can calculate the kernel of  $\varphi$ ; this normal subgroup is called the *core* of  $H$  in  $R$ .

The second step of our calculation of the symmetry groups is, for each solid, to find an appropriate subgroup  $H$  of the rotation group  $R$  in order to apply the result of the previous paragraph. For each of the four largest Platonic solids, we will let  $H$  be the normalizer of a  $p$ -Sylow subgroup of  $R$ , with  $p = 3$  for the cube and the octahedron, and  $p = 2$  for the dodecahedron and icosahedron. In all cases the core of  $H$  in  $R$  is the identity subgroup. For the former two solids we get, from Maple, that  $|R| = 24$  and  $[R : H] = 4$ , and so we have an injective homomorphism  $R \rightarrow S_4$ . We then conclude  $R \cong S_4$ . For the latter two solids we have, from Maple, that  $|R| = 60$  and  $[R : H] = 5$ , and so  $R$  is isomorphic to a subgroup of  $S_5$  of order 60. Since  $A_5$  is the only such subgroup, we get  $R \cong A_5$ . To help to understand our choice of  $H$ , suppose we are considering the cube. We need, if we are to use the result of the previous paragraph, to find a subgroup  $H$  of  $R$  with  $[R : H] = 4$  and whose core in  $R$  is trivial in order to get an injective map into  $S_4$ . Such concerns led to the choice of  $H$  in each case.

We summarize our calculations in the following table.

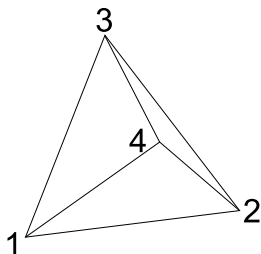
| Solid        | Rotation Group | Symmetry Group            |
|--------------|----------------|---------------------------|
| tetrahedron  | $A_4$          | $S_4$                     |
| cube         | $S_4$          | $S_4 \times \mathbb{Z}_2$ |
| octahedron   | $S_4$          | $S_4 \times \mathbb{Z}_2$ |
| dodecahedron | $A_5$          | $A_5 \times \mathbb{Z}_2$ |
| icosahedron  | $A_5$          | $A_5 \times \mathbb{Z}_2$ |

It is no coincidence that the symmetry group of the octahedron and the symmetry group of the cube are isomorphic, as are the groups for the dodecahedron and icosahedron. There is a notion of *duality* of Platonic solids. If we take a Platonic solid, put a point in the center of each face, and connect all these points, we get the skeleton of another Platonic solid. The resulting solid is called the *dual* of the first. For instance, the dual of the octahedron is the cube, and the dual of the cube is the octahedron. By viewing the dual solid as being built from another in this way, any symmetry of the solid will yield a symmetry of its dual, and vice-versa. Thus, the symmetry groups of a Platonic solid and its dual are isomorphic.

In the following calculations, we denote by  $G$  the symmetry group of a given Platonic solid and  $R = G \cap \text{SO}_3(\mathbb{R})$  its subgroup of rotations. We choose two rotations  $a, b$  and denote by  $R_0$  the subgroup of  $R$  generated by  $a$  and  $b$ . We will show  $R = R_0$  by finding an upper bound for  $|G|$  and by having Maple calculate  $|R_0|$ . We will see that  $|R_0|$  is equal to half of the upper bound for  $|G|$ . Since  $[G : R] = 2$ , we then conclude that  $R = R_0$  and that  $|G|$  is equal to this upper bound.

To help understand the Maple commands we use, we point out that cycle notation in Maple is similar, but not the same as, usual notation for cycles. For example, the transposition which interchanges 1 and 2 is denoted by  $[[1, 2]]$ , and  $[[1, 2], [3, 4]]$  represents the product of transpositions (1 2) and (3 4). Maple uses the command `permgrou(n, {a, b})` to denote the subgroup of  $S_n$  generated by  $a, b$ . Finally, Maple denotes the identity subgroup of  $S_n$  as generated by the empty set. In other words, `permgrou(n, {})` represents the identity subgroup of  $S_n$ . The meaning of the other commands we use will be clear from their syntax.

## 1 The Tetrahedron



The tetrahedron is a solid with four faces and four vertices. Thus, we identify  $G$  with a subgroup of  $S_4$ . It is clear that  $|G| \leq |S_4| = 24$ ; this is the one case in which we don't need a counting argument to get an upper bound for  $|G|$ . It is easier to determine the symmetry

group of the tetrahedron than for the other solids; in particular, we do not need to apply the technique mentioned in the introduction.

We now show that  $G$  is isomorphic to  $S_4$ . Let  $a$  be the counterclockwise rotation of  $120^\circ$  that fixes vertex 4 and let  $b$  be the counterclockwise rotation of  $120^\circ$  that fixes vertex 1, and let  $c$  be the reflection across the plane containing the center and vertices 3 and 4. We now have Maple determine the group  $G_0$  generated by  $a, b, c$  and the group  $R_0$  generated by  $a, b$ .

```
> with(group):
> a := [[1,2,3]]:
> b := [[4,2,3]]:
> c := [[1,2]]:
> G0 := permgroup(4, {a,b,c}):
> grouporder(G0);
```

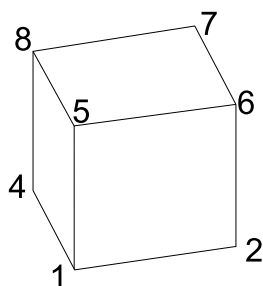
24

```
> R0 := permgroup(4, {a,b}):
> grouporder(R0);
```

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From the Maple output and our upper bound, we see that  $|G| = 24$  since  $24 = |G_0| \leq |G| \leq 24$ . Since  $G$  is isomorphic to a subgroup of  $S_4$ , we conclude that  $G \cong S_4$ . Furthermore, the rotation group  $R$  of the tetrahedron is then isomorphic to  $A_4$ , since  $A_4$  is the only subgroup of  $S_4$  of index 2.

## 2 The Cube



The cube has six faces and eight vertices. We thus view  $G$  as a subgroup of  $S_8$ . We use a counting argument to get an upper bound for  $|G|$ . First, there are eight choices for where vertex 1 can be sent. Once vertex 1 has been sent somewhere, there are three choices for where vertex 2 can be sent because there are three vertices connected to any given vertex. Finally, there are two choices for where vertex 4 can be sent since it must be sent to a vertex

connected to the image of vertex 1 but not to the image of vertex 2. Since any isometry is determined by its action on three vertices, we see that  $|G| \leq 8 \cdot 3 \cdot 2 = 48$ . Since  $[G : R] = 2$ , we have the upper bound  $|R| \leq 24$ .

Let  $a$  be the counterclockwise rotation by  $90^\circ$  that fixes the top and bottom faces and let  $b$  be the rotation of  $90^\circ$  that sends the top face to the front face. We then use Maple to get the following output. As with the tetrahedron, we let  $R_0$  be the subgroup of  $R$  generated by  $a$  and  $b$ .

```
> a := [[1,2,3,4],[5,6,7,8]]:
> b := [[1,4,8,5],[2,3,7,6]]:
> R0 := permgroup(8,{a,b}):
> grouporder(R0);
```

24

```
> P := Sylow(R0,3):
> H := normalizer(R0,P):
> grouporder(H);
```

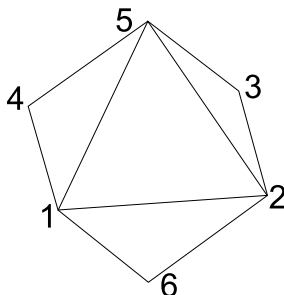
6

```
> core(H,R0);
```

permgroup(8, {})

From this and our upper bound, we see that  $R_0 = R$ ; therefore,  $|R| = 24$ . Furthermore, since  $H$  is a subgroup of  $R$  of order 6, and since the core of  $H$  in  $R$  is trivial, we see that the homomorphism  $R \rightarrow \text{Perm}(R/H) \cong S_4$  obtained from [3, Theorem 2.4.2.] is injective, and so  $R \cong S_4$  since these groups have the same order. We pointed out earlier that  $G \cong R \times \mathbb{Z}_2$ . Thus,  $G \cong S_4 \times \mathbb{Z}_2$ .

### 3 The Octahedron



The octahedron is a solid with eight faces and six vertices. We view its symmetry group as a subgroup of  $S_6$ . To get an upper bound for  $|G|$ , we note that there are six choices for where vertex 1 is sent, and once it is determined, there are four choices for where vertex 2 goes, since each vertex is connected to four other vertices. Finally, there are two choices for where vertex 5 is sent, since it must be sent to a vertex connected both to the image of 1 and to the image of 2. Thus,  $|G| \leq 6 \cdot 4 \cdot 2 = 48$ , and so  $|R| \leq 24$ .

Let  $a$  be the counterclockwise rotation of  $90^\circ$  that fixes the top and bottom vertices,  $b$  be the rotation of  $120^\circ$  that sends vertex 1 to 2 and vertex 2 to 5. We have the following Maple output.

```
> a := [[1,2,3,4]]:
> b := [[5,1,2],[6,3,4]]:
> R0 := permgp(6,{a,b}):
> grouporder(R0);
```

24

```
> P := Sylow(R0,3):
> H := normalizer(R0,P):
> grouporder(H);
```

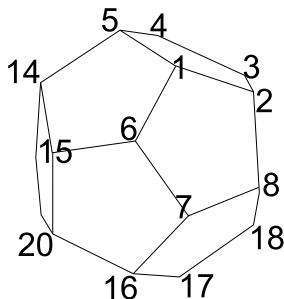
6

```
> core(H,R0);
```

permgp(6, {})

From the output, we get  $|R| = 24$ , and as with the case of the cube, we see that  $R$  is isomorphic to  $S_4$ . Therefore,  $G \cong R \times \mathbb{Z}_2 \cong S_4 \times \mathbb{Z}_2$ .

## 4 The Dodecahedron



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The dodecahedron has twelve faces and twenty vertices. We view its symmetry group as a subgroup of  $S_{20}$ . Because the picture above does not show all the vertices, we explain the numbering scheme. The vertices of the top face are numbered 1 through 5 in counterclockwise order, and the vertices of the bottom face are numbered 16 through 20. The “middle” ten vertices are then numbered 6 through 16 in counterclockwise order. To get an upper bound for  $|G|$ , we note that vertex 1 can be sent to any of the twenty vertices. Once the image of vertex 1 is determined, vertex 2 must be sent to one of three vertices since each vertex is connected to three others. Finally, there are two choices for where vertex 6 is sent since it must also go to a vertex connected to the image of vertex 1. Thus,  $|G| \leq 20 \cdot 3 \cdot 2 = 120$ , and so  $|R| \leq 60$ .

Let  $a$  be the counterclockwise rotation of  $72^\circ$  that fixes the top and bottom faces, let  $b$  the rotation of  $72^\circ$  sends vertex 1 to 6 and vertex 6 to 7. We have the following Maple output.

```
> a := [[1,2,3,4,5], [6,8,10,12,14], [7,9,11,13,15], [16,17,18,19,20]] :
> b := [[1,6,7,8,2], [3,5,15,16,9], [4,14,20,17,10], [12,13,19,18,11]] :
> R0 := permgroup(20, {a,b}) :
> grouporder(R0) ;
```

60

```
> P := Sylow(R0,2) :
> H := normalizer(R0,P) :
> grouporder(H) ;
```

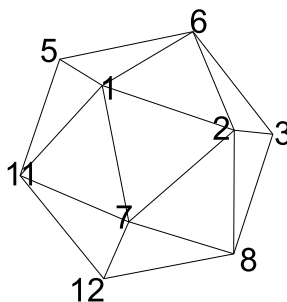
12

```
> core(H,R0) ;
```

permgroup(20, {})

We conclude that  $|R| = 60$ . Since  $H$  is a subgroup of  $R$  of index  $60/12 = 5$ , and the core of  $H$  in  $R$  is trivial, we have an injective homomorphism  $R \rightarrow S_5$ . We recall that the only subgroup of  $S_5$  of order 60 is  $A_5$ . Thus,  $R \cong A_5$ . As we mentioned earlier,  $G \cong R \times \mathbb{Z}_2$ , so  $G \cong A_5 \times \mathbb{Z}_2$ .

## 5 The Icosahedron



The icosahedron has twenty faces and twelve vertices, so we view its symmetry group as a subgroup of  $S_{12}$ . Let  $a$  be the counterclockwise rotation of  $72^\circ$  that fixes the top and bottom vertices, and let  $b$  be the  $120^\circ$  rotation that sends vertex 1 to 2 and vertex 2 to 6. We then have the following Maple output.

```
> a := [[1,2,3,4,5],[7,8,9,10,11]] :
> b := [[1,2,6],[3,5,7],[4,11,8],[9,10,12]] :
> R0 := permgrou(20,{a,b}) :
> grouporder(R0) ;
```

60

```
> P := Sylow(R0,2) :
> H := normalizer(R0,P) :
> grouporder(H) ;
```

12

```
> core(H,R0) ;
```

permgrou(20, {})

From the output we have the same conclusion as for the dodecahedron; the group  $R$  is isomorphic to  $A_5$ , and  $G$  is isomorphic to  $A_5 \times \mathbb{Z}_2$ .

## References

- [1] M. Artin, *Algebra*, Prentice Hall, Englewood Cliffs, NJ, 1991.
- [2] I. Herstein, *Topics in Algebra*, Xerox Coll. Publ., Lexington, MA, 1975.
- [3] E. Walker, *Introduction to Abstract Algebra*, Random House, New York, 1987.