

Sheaf Cohomology

In this note we give the background needed to define sheaf cohomology. In particular, we prove the following two facts. First, the category $\mathfrak{Ab}(X)$ of sheaves on a topological space X has enough injectives. Second, if Γ is the global function functor from $\mathfrak{Ab}(X)$ to the category of Abelian groups, we show that Γ is a left exact functor. Therefore, the right derived functors $R^n(\Gamma)$ exist, and we can then define the cohomology groups $H^n(X, \mathcal{F}) = R^n(\Gamma)(\mathcal{F})$ for any sheaf \mathcal{F} on X . Sheaf cohomology is one of the major tools of modern algebraic geometry, and it is a good example to help justify learning homological algebra over an arbitrary Abelian category instead of restricting to categories of modules. As we will see, several facts come from producing an adjoint pair of functors, including the proof that Γ is left exact. While one can prove that Γ is left exact in a straightforward manner, we choose to prove it with the help of adjoint functors to illustrate the power of adjoint pairs.

There are subtleties in working with sheaves and presheaves. Since every sheaf is a presheaf, we can view the category of sheaves as a subcategory of the category of presheaves. It is not hard to show that the global section functor is exact in the category of presheaves, but it is not exact in the category of sheaves. This is due to the fact that cokernels are different in the two categories even though morphisms are not. We will give an example to demonstrate this. In fact, sheaf cohomology is nontrivial exactly because the global section functor is not exact. Therefore, one can say that sheaf cohomology exists since cokernels are different in these two categories.

We now recall the definitions of presheaf and sheaf. Let $\text{Top}(X)$ be the category of open sets of a topological space X , where the morphisms are inclusions. That is, $\text{hom}(U, V)$ consists of the inclusion map $U \rightarrow V$ if $U \subseteq V$, and $\text{hom}(U, V)$ is empty otherwise. A *presheaf* (of Abelian groups) on X is a contravariant functor \mathcal{F} from $\text{Top}(X)$ to the category of Abelian groups with $\mathcal{F}(\emptyset) = 0$. Therefore, if \mathcal{F} is a presheaf on X , then for every open set U there is a group $\mathcal{F}(U)$, and for every inclusion $U \subseteq V$, there is a map $\text{res}_{U,V} : \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, such that for open sets $U \subseteq V \subseteq W$, the composition $\text{res}_{U,V} \circ \text{res}_{V,W} : \mathcal{F}(W) \rightarrow \mathcal{F}(V) \rightarrow \mathcal{F}(U)$ is equal to $\text{res}_{U,W}$. The maps $\text{res}_{U,V}$ is generally called *restriction maps*, and $\text{res}_{U,V}(f)$ is often written $f|_U$. This terminology is due to the following examples.

Example 1. Let X be a topological space. Define \mathcal{F} by $\mathcal{F}(U) = C(U, \mathbb{C})$, the group of continuous functions from U to \mathbb{C} , for any open set U of X . If $U \subseteq V$, then restriction of functions gives a group homomorphism $\mathcal{F}(V) \rightarrow \mathcal{F}(U)$. Thus, \mathcal{F} is a presheaf.

Example 2. Let $X = \mathbb{C}$. Define \mathcal{A} on X by $\mathcal{A}(U)$ is the set of all analytic complex valued functions defined on U . Again, for $V \subseteq U$, the map $\mathcal{A}(U) \rightarrow \mathcal{A}(V)$ is the restriction of functions map. Then \mathcal{A} is a presheaf.

Example 3. Let X be an algebraic variety over an algebraically closed field k . Define $\mathcal{O}(U)$ to be the ring of regular functions on U . That is, an element of $\mathcal{O}(U)$ is a function $f : U \rightarrow k$ such that for every $x \in U$ there is an open neighborhood V of x such that f is the quotient of polynomial functions on V . If $U \subseteq V$, then restriction of functions gives the map $\mathcal{O}(V) \rightarrow \mathcal{O}(U)$. The presheaf \mathcal{O} is called the sheaf of regular functions on X .

Example 4. Let $X = \{x\}$ be a one point topological space. If \mathcal{F} is a presheaf on X , then we have an Abelian group $A = \mathcal{F}(X)$, and no other groups associated to \mathcal{F} since the only nonempty open subset of X is X itself. Furthermore, the only nonidentity restriction map is the trivial map $A = \mathcal{F}(X) \rightarrow 0 = \mathcal{F}(\emptyset)$. Thus, a presheaf on \mathcal{F} is really nothing more than an Abelian group.

We now give the definition of a sheaf.

Definition 5. Let \mathcal{F} be a presheaf on X . Then \mathcal{F} is a sheaf provided that the following two conditions hold: (i) if U is an open set on X , if $\{U_i\}$ is an open cover of U , and if $f \in \mathcal{F}(U)$ such that $f|_{U_i} = 0$ for all i , then $f = 0$; and (ii) if $\{U_i\}$ is an open cover of U and if there are $f_i \in \mathcal{F}(U_i)$ satisfying $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j , then there is an $f \in \mathcal{F}(U)$ with $f|_{U_i} = f_i$ for all i .

It is not hard to see that the presheaves of the examples above are in fact sheaves. In fact, the properties of the definition of a sheaf are motivated by examples involving functions, since functions clearly have the properties of the definition. Not all presheaves are sheaves, as we can see in the following example.

Example 6. Let A be an Abelian group. If X is a topological space, then we may define a presheaf \tilde{A} by $\tilde{A}(U) = A$ for all U , and the restriction maps are all just the identity map on A . This presheaf is sometimes called the constant presheaf on X associated to A . It is clear that \tilde{A} is a presheaf. However, it need not be a sheaf. For example, suppose that X is the disjoint union of two open sets U_1 and U_2 ; that is, X is not connected. If a, b are distinct elements of A , then $a \in \tilde{A}(U_1)$ and $b \in \tilde{A}(U_2)$ do not glue to give an element of $\tilde{A}(X)$ even though the compatibility axiom $a|_{U_1 \cap U_2} = b|_{U_1 \cap U_2}$ is trivially true since $U_1 \cap U_2 = \emptyset$ and $A(\emptyset) = 0$. It is true, however, that if X is connected, then \tilde{A} is a sheaf.

Example 7. Let A be an Abelian group, made into a topological space by giving it the discrete topology. If X is a topological space, then we define the *constant sheaf* associated to A by $\mathcal{A}(U) = C(U, A)$, the group of continuous functions from U to A . The restriction maps are given by ordinary function restriction. It is straightforward from the definition to see that \mathcal{A} is a sheaf. Note that if $a \in A$, then the constant function $f_a : X \rightarrow A$ given by $f_a(x) = a$ is an element of $\mathcal{A}(X)$, and so $f_a|_U \in \mathcal{A}(U)$ for any U . In fact, if U is connected, then $\mathcal{A}(U) = \{f_a : a \in A\} \cong A$. It is this fact that gives the name constant sheaf to \mathcal{A} .

We will have categories of presheaves and of sheaves once we define morphisms of presheaves.

Definition 8. A morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ of presheaves is a natural transformation of functors. That is, for every open set U , there is a group homomorphism $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$, and for every inclusion $V \subseteq U$, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \text{res} \downarrow & & \downarrow \text{res} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array}$$

Example 9. Let U be an open set of a topological space X . If \mathcal{F} is a sheaf on X , we get another sheaf \mathcal{G} on X as follows. For V an open set in X , set $\mathcal{G}(V) = \mathcal{F}(U \cap V)$. If we define $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ by $\varphi_V : \mathcal{F}(V) \rightarrow \mathcal{G}(V) = \mathcal{F}(U \cap V)$ is the restriction map, then the compatibility of the restriction maps shows that φ is a morphism of sheaves, as is indicated by the following commutative diagram.

$$\begin{array}{ccc} \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \cap V) \\ \downarrow & & \downarrow \\ \mathcal{F}(W) & \longrightarrow & \mathcal{F}(U \cap W) \end{array}$$

Example 10. Let A be an Abelian group and X a topological space. Looking at the constant sheaves and presheaves of Examples 6 and 7, we have a natural morphism of presheaves $\varphi : \tilde{A} \rightarrow \mathcal{A}$, given by $\varphi_U(a) = f_a|_U$ for any $a \in A$ and any open U .

If \mathcal{F} and \mathcal{G} are sheaves, then a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ of sheaves is just a presheaf morphism. We will write $\text{hom}_X(\mathcal{F}, \mathcal{G})$ for the group of all sheaf homomorphisms from \mathcal{F} to \mathcal{G} . We thus have two categories associated to a topological space X , the category of presheaves on X and that of sheaves on X . There is the obvious inclusion functor i from sheaves on X to presheaves on X . It is clear that i is an additive functor.

Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of presheaves. Then we define the following presheaves

$$\begin{aligned} \ker(\varphi) : U &\mapsto \ker(\varphi_U), \\ \text{coker}(\varphi) : U &\mapsto \text{coker}(\varphi_U), \\ \text{im}(\varphi) : U &\mapsto \text{im}(\varphi_U). \end{aligned}$$

From these definitions it is easy to see that a sequence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ of presheaves is exact if and only if each sequence $0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{H}(U) \rightarrow 0$ is exact. In particular, if Γ is the global section functor on presheaves; i.e., $\Gamma(\mathcal{F}) = \mathcal{F}(X)$, then Γ is an exact functor. As we will see, the global section functor on sheaves is only left exact. The difference between these categories with respect to Γ comes from the following facts: if \mathcal{F} and \mathcal{G} are sheaves,

then $\ker(\varphi)$ is a sheaf; however, $\operatorname{coker}(\varphi)$ and $\operatorname{im}(\varphi)$ need not be sheaves. We will show this in Example 19 below. However, this is a subtle point, and we need to know more about sheaves in order to have the tools to understand. In particular, we need to discuss cokernels in the categories of sheaves. We need some preliminary concepts to do this.

Definition 11. *Let \mathcal{F} be a sheaf on X . If $x \in X$, then the stalk \mathcal{F}_x is the direct limit of the groups $\mathcal{F}(U)$ as U ranges over all open sets containing x .*

We point out that the collection of open neighborhoods of x forms a directed set by ordering by reverse inclusion: $U \leq V$ if $V \subseteq U$. This is a directed set since if U and V are open neighborhoods of x , then $U \cap V$ is a neighborhood of x contained in both U and V . Thus, the direct limit \mathcal{F}_x does exist. We recall the universal mapping property that arises from \mathcal{F}_x being the given direct limit. First of all, for each neighborhood U of x , there is a map $\sigma_{U,x} : \mathcal{F}(U) \rightarrow \mathcal{F}_x$ that is compatible with the restriction maps in that if $V \subseteq U$, then $\sigma_{U,x} = \sigma_{V,x} \circ \operatorname{res}_{U,V}$. Next, if B is an Abelian group and if for every U there is a map $\tau_U : \mathcal{F}(U) \rightarrow B$ such that $\tau_U = \tau_V \circ \operatorname{res}_{U,V}$ whenever $V \subseteq U$, then there is a unique map $\tau : \mathcal{F}_x \rightarrow B$ such that $\tau_U = \tau \circ \sigma_{U,x}$.

An alternative way to view the stalk \mathcal{F}_x is as follows. Consider pairs (U, f) with $f \in \mathcal{F}(U)$. Define an equivalence relation by $(U, f) \sim (V, g)$ if $f|_{U \cap V} = g|_{U \cap V}$. Then \mathcal{F}_x is the set of equivalence classes of such pairs. Furthermore, addition is given by $(U, f) + (V, g) = (U \cap V, f|_{U \cap V} + g|_{U \cap V})$. With this in mind, consider the sheaf \mathcal{A} of Example 2. Then elements of \mathcal{A}_x consist of functions analytic in a neighborhood of x . Similarly, in Example 3, the stalk \mathcal{O}_x consists of regular functions defined in a neighborhood of x . Common usage in complex analysis and algebraic geometry calls elements of stalks germs of functions.

For ease of notation, if $f \in \mathcal{F}(U)$, we will write f_x for the image of f in \mathcal{F}_x . The following lemma states two of the basic properties of direct limits in the case of stalks.

Lemma 12. *Let \mathcal{F} be a presheaf on X , and let $x \in X$.*

1. *Every $s \in \mathcal{F}_x$ can be written in the form f_x for some open neighborhood U of x and some $f \in \mathcal{F}(U)$.*
2. *If $f \in \mathcal{F}(U)$ with $f_x = 0$, then there is some $V \subseteq U$ with $f|_V = 0$.*

For every $x \in X$ we have the association $\mathcal{F} \mapsto \mathcal{F}_x$. This is in fact a functor from sheaves to Abelian groups. To see this, we first need to define, for a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$, a stalk map $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$. This is obtained by using the mapping property of direct limits. For each open neighborhood U of x , we have the canonical maps $\sigma_{U,x} : \mathcal{F}(U) \rightarrow \mathcal{F}_x$ and $\tau_{U,x} : \mathcal{G}(U) \rightarrow \mathcal{G}_x$ that sends $f \in \mathcal{F}(U)$ to f_x (and similarly for \mathcal{G}). We then have the map $\rho_{U,x} := \tau_{U,x} \circ \varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow \mathcal{G}_x$. If $V \subseteq U$ and $f \in \mathcal{F}(U)$, then the definition of a presheaf morphism gives

$$\begin{aligned} \rho_{V,x}(f|_V) &= \tau_{V,x}(\varphi_V(f|_V)) = \tau_{V,x}(\varphi_U(f)|_V) \\ &= \varphi_U(f)_x = \rho_{U,x}(f). \end{aligned}$$

The $\rho_{U,x}$ thus have the needed compatibility to get a map $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ satisfying $\varphi_x = \rho_{U,x} \circ \sigma_U$ for every neighborhood U of x . This definition says that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \sigma_{U,x} \downarrow & & \downarrow \tau_{U,x} \\ \mathcal{F}_x & \xrightarrow{\varphi_x} & \mathcal{G}_x \end{array}$$

It also give the following formula for φ_x : if $f \in \mathcal{F}(U)$, then $\varphi_x(f_x) = \rho_{U,x}(f) = \varphi_U(f)_x$. Now that we have the mapping φ_x , we can show that $\mathcal{F} \mapsto \mathcal{F}_x$ is a functor. All we need to prove that is if $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ and $\sigma : \mathcal{G} \rightarrow \mathcal{H}$ are morphisms of sheaves, then $(\sigma \circ \varphi)_x = \sigma_x \circ \varphi_x$. This is easy since

$$(\sigma \circ \varphi)_x(f_x) = \sigma_U(\varphi_U(f))_x = \sigma_x(\varphi_U(f)_x) = \sigma_x(\varphi_x(f_x)),$$

so $(\sigma \circ \varphi)_x = \sigma_x \circ \varphi_x$ as desired.

As an application of stalks in sheaf theory, we point out the following easy consequence of the definitions. Let \mathcal{F} be a sheaf on X , and let $f \in \mathcal{F}(U)$. If $f_x = 0$ for every $x \in U$, then $f = 0$. To prove this, we note that if $x \in U$, then there is an open neighborhood V_x of x contained in U with $f|_{V_x} = 0$. The set of all V_x is an open cover of U , and so the first sheaf axiom then yields $f = 0$. This is a convenient way to show that an element is zero.

Let \mathcal{F} be a presheaf. We define a sheaf \mathcal{F}' in the following manner. Let $\mathcal{F}'(U)$ be the set of functions f from U to the disjoint union of the stalks \mathcal{F}_x for $x \in U$ that satisfy the properties (i) $f(x) \in \mathcal{F}_x$, and (ii) for every $x \in U$, there is an open neighborhood $V \subseteq U$ of x and an element $s \in \mathcal{F}(V)$ such that for all $Q \in V$, we have $f(Q) = s_Q$. Then a short calculation shows that \mathcal{F}' is a sheaf. We then define a sheaf morphism $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$ by $\alpha_U(s) : U \rightarrow \bigsqcup_{x \in U} \mathcal{F}_x$ is the function given by $\alpha_U(s)(x) = s_x$ for all $x \in U$. We then have the following property of this construction.

Proposition 13. *Let \mathcal{F} be a presheaf on X . Then the sheafification \mathcal{F}' together with a morphism $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$ is the unique up to isomorphism sheaf satisfying the following universal mapping property: if \mathcal{G} is a sheaf and if $\sigma : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves, then there is a unique morphism $\sigma' : \mathcal{F}' \rightarrow \mathcal{G}$ with $\sigma = \sigma' \circ \alpha$.*

Proof. Let $\sigma : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism with \mathcal{G} a sheaf. In order to define a sheaf map $\sigma' : \mathcal{F}' \rightarrow \mathcal{G}$, for each open set U and each $f \in \mathcal{F}'(U)$, we must define $\sigma'_U(f)$. We do by using the glueing axiom of sheaves. By the definition of \mathcal{F}' , there is an open cover $\{U_i\}$ of U such that on U_i there is an $s_i \in \mathcal{F}(U_i)$ with $f(x) = (s_i)_x$ for all $x \in U_i$. Let $g_i = \sigma_{U_i}(s_i) \in \mathcal{G}(U_i)$. If we show that $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$ for each i, j , then there is a unique $g \in \mathcal{G}(U)$ with $g|_{U_i} = g_i$. If $x \in U_i \cap U_j$, then $f(x) = (s_i)_x = (s_j)_x$. Therefore, $s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}$ has zero stalk at each point in $U_i \cap U_j$. Therefore, applying σ , we get $g_i|_{U_i \cap U_j} - g_j|_{U_i \cap U_j}$ also has zero stalk at each point. By the application of stalks we pointed out earlier, this forces $g_i|_{U_i \cap U_j} - g_j|_{U_i \cap U_j} = 0$.

Thus, as \mathcal{G} is a sheaf, there is such a g . A routine check shows that, by defining $\sigma'_U(f) = g$, we do have a well defined sheaf morphism satisfying $\sigma = \sigma' \circ \alpha$. Finally, the formula for σ' is forced upon us by the requirement that $\sigma = \sigma' \circ \alpha$ since if $f = \alpha_U(s)$, then $\sigma'(f) = \sigma(s)$. Thus, if $f \in \mathcal{F}'(U)$, there is a cover $\{U_i\}$ of U and elements $s_i \in \mathcal{F}(U_i)$ with $f|_{U_i} = \alpha_{U_i}(s_i)$. Thus, $\sigma'_{U_i}(f|_{U_i}) = \sigma_{U_i}(s_i)$. Thus, $\sigma'_U(f)$ must be the unique element that is obtained by glueing the $\sigma_{U_i}(s_i)$. \square

As an example of the sheafification construction, if \mathcal{F} is the presheaf of Example 6 associated to an Abelian group A , then the constant sheaf \mathcal{A} associated to A is in fact the sheafification of \mathcal{F} . The verification of this statement is a short exercise.

As we will indicate below, one way of proving results about sheaves is to prove appropriate results about their stalks. The following lemma will help with this technique when working with the sheafification of a presheaf.

Lemma 14. *Let \mathcal{F} be a presheaf and let \mathcal{F}' be its sheafification. Then $\mathcal{F}'_x = \mathcal{F}_x$. More precisely, if $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$ is the canonical inclusion, then $\alpha_x : \mathcal{F}_x \rightarrow \mathcal{F}'_x$ is an isomorphism.*

Proof. This falls out of the definition of sheafification. The presheaf map $\alpha : \mathcal{F} \rightarrow \mathcal{F}'$ induces a map on stalks $\alpha_x : \mathcal{F}_x \rightarrow \mathcal{F}'_x$, which is given by $\alpha_x(f_x) = \alpha(f)_x$, for $f \in \mathcal{F}(U)$. Recall that $\alpha(f)$ is the map from U to the disjoint union of the stalks \mathcal{F}'_x for $x \in U$ given by $\alpha_U(f)(P) = f_P$ for any $P \in U$. So, if $\alpha_x(f_x) = 0$ with $f \in \mathcal{F}(U)$, then there is a neighborhood $V \subseteq U$ of x with $\alpha_V(f) = 0$. Thus, $f_P = 0$ for all $P \in V$. In particular, $f_x = 0$. Thus, α_x is injective. For surjectivity, if $s \in \mathcal{F}'_x$, then $s = t_x$ for some open neighborhood U of x and some $t \in \mathcal{F}'(U)$. Recalling the definition of \mathcal{F}' , we may shrink U to assume that there is a $g \in \mathcal{F}(U)$ with $t(P) = g_P$ for all $P \in U$. In other words, $t = \alpha_U(g)$. Thus, $\alpha_x(g_x) = t_x = s$. Therefore, α_x is an isomorphism. \square

Now that we have the notion of sheafification, we can define the cokernel and image of a sheaf morphism in the category of sheaves. If $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, then $\text{coker}(\varphi)$ is the sheafification of the presheaf cokernel, and $\text{im}(\varphi)$ is the sheafification of the presheaf image. A short argument shows that the sheaf $\text{coker}(\varphi)$ is a cokernel of φ in the category of sheaves on X .

If \mathcal{F} is a presheaf, write $j(\mathcal{F})$ for its Sheafification. We claim that j is actually a functor from presheaves to sheaves. To verify this, we need to show that if $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of presheaves, then there is a morphism $j(\varphi) : j(\mathcal{F}) \rightarrow j(\mathcal{G})$ with the appropriate properties. To define $j(\varphi)$, if $\alpha : \mathcal{G} \rightarrow j(\mathcal{G})$ is the canonical map, then composing with φ gives a morphism $\mathcal{F} \rightarrow j(\mathcal{G})$. The mapping property for sheafification then gives a unique map $j(\varphi) : j(\mathcal{F}) \rightarrow j(\mathcal{G})$ that factors through this map. To see that this gives a functor, suppose $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ and $\sigma : \mathcal{G} \rightarrow \mathcal{H}$ are morphisms of presheaves. Let $\alpha : \mathcal{F} \rightarrow j(\mathcal{F})$ (resp. β, γ) be the canonical map from \mathcal{F} (resp. \mathcal{G}, \mathcal{H}) to its sheafification. The map $j(\sigma \circ \varphi)$ is the unique map $j(\mathcal{F}) \rightarrow j(\mathcal{H})$ with $\gamma \circ (\sigma \circ \varphi) = j(\varphi \circ \sigma) \circ \alpha$. However,

$$(\gamma \circ \sigma) \circ \varphi = (j(\sigma) \circ \beta) \circ \varphi = j(\sigma) \circ j(\varphi) \circ \alpha.$$

Therefore, $j(\varphi \circ \sigma) = j(\varphi) \circ j(\sigma)$ as desired. We thus have two functors, the inclusion functor i from sheaves to presheaves, and the sheafification functor j from presheaves to sheaves. These functors form an adjoint pair, as we show below. This is essentially a restatement of the definition of the sheafification of a presheaf.

Proposition 15. *Let i be the inclusion functor from sheaves to presheaves and j the sheafification functor. Then $\text{hom}_X(\mathcal{F}, i(\mathcal{H})) \cong \text{hom}_X(j(\mathcal{F}), \mathcal{H})$ for any presheaf \mathcal{F} and any sheaf \mathcal{H} . Therefore, j is left adjoint to i , and i is right adjoint to j .*

Proof. Given a presheaf morphism $\varphi : \mathcal{F} \rightarrow \mathcal{H}$ where \mathcal{H} is a sheaf, the universal mapping property gives a unique sheaf morphism $\varphi' : j(\mathcal{F}) \rightarrow \mathcal{H}$. We view $\varphi : \mathcal{F} \rightarrow i(\mathcal{H})$. The map $\varphi \mapsto \varphi'$ gives a function $\text{hom}_X(\mathcal{F}, i(\mathcal{H})) \rightarrow \text{hom}_X(j(\mathcal{F}), \mathcal{H})$. Its inverse is the function that sends a sheaf morphism $j(\mathcal{F}) \rightarrow \mathcal{H}$ to the composition $\mathcal{F} \rightarrow j(\mathcal{F}) \rightarrow \mathcal{H}$. It is trivial to check that these are inverses to each other. \square

Corollary 16. *The inclusion functor i from sheaves to presheaves on X is left exact.*

Proof. This follows by [2, Thm. 2.6.1] since i is a right adjoint (to the sheafification functor). \square

It is not true, however, that the inclusion functor i is exact. In particular, a surjective morphism of sheaves is not necessarily a surjective morphism of presheaves. We will give an example of a surjection of sheaves that is not a surjection of presheaves. Since the image presheaf of a morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ is defined by $U \mapsto \text{im}(\varphi_U)$, a presheaf morphism is surjective if every map $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is surjective. However, this is not the same condition in the category of sheaves. The proposition below tells us that a morphism of sheaves is surjective exactly when each stalk map is surjective. This is the most convenient way to determine when a morphism of sheaves is surjective. It also illustrates the importance of stalks. We first prove a special case of this fact.

Lemma 17. *Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then φ is an isomorphism if and only if each $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is an isomorphism of groups.*

Proof. Suppose that φ is an isomorphism. Then $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ has an inverse φ_U^{-1} for every open set U . Let $f_x \in \ker(\varphi_x)$. We may represent f_x by an element f on an open set U , and $\varphi(f)_x = 0$ implies that there is an open set $V \subseteq U$ with $\varphi(f)|_V = 0$. Therefore, $\varphi_V(f|_V) = 0$, forcing $f|_V = 0$ since φ_V is injective. This implies that $f_x = 0$. Thus, φ_x is injective. Next, take $g_x \in \mathcal{G}_x$, and represent g_x with an element $g \in \mathcal{G}(U)$ for some open set U . Since φ_U is surjective, $g = \varphi_U(f)$ for some $f \in \mathcal{F}(U)$. Then $g_x = \varphi_U(f)_x = \varphi_x(f_x)$, so φ_x is surjective.

Conversely, suppose that each φ_x is an isomorphism. We show that each φ_U is an isomorphism. For injectivity, suppose that $f \in \mathcal{F}(U)$ with $\varphi_U(f) = 0$. Then $\varphi_x(f_x) = \varphi_U(f)_x = 0$ for all $x \in U$. Since each φ_x is injective, we get $f_x = 0$ for all x . Thus, for each $x \in U$ there is an open neighborhood $V_x \subseteq U$ with $f|_{V_x} = 0$. Since the sets V_x form an open

cover of U , the sheaf axiom implies that $f = 0$. Thus, φ_U is injective. For surjectivity, let $g \in \mathcal{G}(U)$. For $x \in X$ there is an element $s \in \mathcal{F}_x$ with $\varphi_x(s) = g_x$. Represent s by an element $f(x)$ on a neighborhood $V_x \subseteq U$ of x . Then $\varphi(f(x))_x = g_x$. By shrinking the neighborhood V_x , we may assume that $\varphi(f(x)) = g|_{V_x}$. Doing this for every $x \in U$, we have an open cover $\{V_x\}$ of U and elements $f(x) \in \mathcal{F}(V_x)$ with $\varphi_{V_x}(f(x)) = g|_{V_x}$ for all x . If $x, y \in U$, then $\varphi(f(x))|_{V_x \cap V_y} = g|_{V_x \cap V_y} = \varphi(f(y))|_{V_x \cap V_y}$. Thus, since we have already proved that φ is injective, we get $f(x)|_{V_x \cap V_y} = f(y)|_{V_x \cap V_y}$. The sheaf axiom then yields an $f \in \mathcal{F}(U)$ with $f|_{V_x} = f(x)$. So, $\varphi_U(f) \in \mathcal{G}(U)$ satisfies $\varphi_U(f)|_{V_x} = \varphi_{V_x}(f|_{V_x}) = g|_{V_x}$. Since $\varphi_U(f)$ and g agree on an open cover of U , applying the first sheaf axiom to $\varphi_U(f) - g$ shows that $\varphi_U(f) = g$. Therefore, φ_U is surjective for each U . We have then shown that each φ_U is an isomorphism. We then produce φ^{-1} by $(\varphi^{-1})_U = (\varphi_U)^{-1}$; it is routine to show that φ^{-1} is a sheaf morphism, and that it is the inverse of φ . Therefore, φ is an isomorphism. \square

We use the previous lemma to prove that a sheaf morphism is surjective (resp. injective) if and only if each stalk map is surjective (resp. injective).

Proposition 18. *Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves. Then φ is injective (resp. surjective) if and only if each stalk map $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x$ is injective (resp. surjective).*

Proof. The proof that φ is injective if and only if each φ_x is injective is just a repetition of the injectivity argument of the lemma. However, surjectivity is more difficult. First suppose that φ is surjective. Then $\text{im}(\varphi) = \mathcal{G}$. Since $\text{im}(\varphi)$ is the sheafification of the presheaf $\mathcal{H}(U) = \text{im}(\varphi_U)$, and since stalks are preserved under the sheafification functor, by Lemma 14, we have $\mathcal{G}_x = \varphi_U(\mathcal{F}(U))_x = \varphi_x(\mathcal{F}_x)$. Thus, φ_x is surjective for each x . Conversely, suppose that each φ_x is surjective. The presheaf \mathcal{H} has a natural inclusion into \mathcal{G} , so there is a unique induced morphism $\text{im}(\varphi) \rightarrow \mathcal{G}$, which is injective. The stalk at x of $\text{im}(\varphi)$ is $\varphi_x(\mathcal{F}_x) = \mathcal{G}_x$ since φ_x is surjective. Furthermore, the stalk map $\text{im}(\varphi)_x \rightarrow \mathcal{G}_x$ is injective since the map $\text{im}(\varphi) \rightarrow \mathcal{G}$ is injective; this is a consequence of the injectivity part of the proposition that we have already proved. Therefore, the stalk maps of $\text{im}(\varphi) \rightarrow \mathcal{G}$ are all isomorphisms. Thus, by the lemma, we see that $\text{im}(\varphi) \rightarrow \mathcal{G}$ is an isomorphism. Therefore, φ is surjective. \square

Example 19. We give an example of a surjective sheaf morphism that is not surjective in the category of presheaves. That is, we will produce a sheaf morphism $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ for which $\varphi_U : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ fails to be surjective for some open U . While there are perhaps easier ad-hoc examples, we give a geometric example. The motivation of this example comes from the notion of Cartier divisor. We leave out the verification of our claims and instead refer the reader to [1, Chapter II.6]. Let X be an irreducible variety with function field K , let \mathcal{K}^* be the constant sheaf $\mathcal{K}^*(U) = K^*$, and let \mathcal{O}_X^* be the sheaf of units of the sheaf of regular functions \mathcal{O}_X on X . Finally, consider the quotient sheaf $\mathcal{K}^*/\mathcal{O}_X^*$. This is the sheaf associated to the presheaf $U \mapsto K^*/\mathcal{O}_X(U)^*$. That this presheaf is not a sheaf is critical in this example. The natural sheaf morphism $\pi : \mathcal{K}^* \rightarrow \mathcal{K}^*/\mathcal{O}_X^*$ is surjective since it is surjective at all stalks. However, if we consider the map $\pi_X : \Gamma(X, \mathcal{K}^*) \rightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$ on global sections, then the

quotient $\Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)/\pi_X(\Gamma(X, \mathcal{K}^*))$ is isomorphic to the Picard group $\text{Pic}(X)$ of X (under suitable assumptions on X). If we choose $X = \mathbb{P}^1$, then $\text{Pic}(X) \cong \mathbb{Z}$; in particular, the quotient group above is nontrivial, and so $\pi_X : \Gamma(X, \mathcal{K}^*) \rightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}_X^*)$ is not surjective.

We now describe two functors associated to a continuous map of topological spaces whose study will help us understand the global section functor. Suppose that $f : X \rightarrow Y$ is a continuous map between topological spaces X and Y . If \mathcal{F} is a sheaf on X , then the *direct image sheaf* $f_*(\mathcal{F})$ is the sheaf on Y defined by $f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$. Also, if \mathcal{G} is a sheaf on Y , then the *inverse image sheaf* $f^{-1}(\mathcal{G})$ is the sheafification of the presheaf on X defined by $U \mapsto \varinjlim \mathcal{G}(V)$, where the direct limit is taken over all open sets V containing $f(U)$. A trivial exercise shows that $f_*(\mathcal{F})$ is indeed a sheaf. We have actually seen the direct image sheaf construction in the previous example. If $i : U \rightarrow X$ is the inclusion map, and if \mathcal{F} is a sheaf on U , then $i_*(\mathcal{F})$ is the sheaf on X given by

$$i_*(\mathcal{F})(V) = \mathcal{F}(i^{-1}(V)) = \mathcal{F}(U \cap V),$$

which is the formula for the sheaf \mathcal{G} of that example.

The basic relationship between the functors f^{-1} and f_* is that they are adjoint to each other, as we show in the following lemma.

Lemma 20. *Let $f : X \rightarrow Y$ be a continuous map. Then the functors f^{-1} and f_* are adjoint to each other. That is, $\text{hom}_X(f^{-1}(\mathcal{G}), \mathcal{F}) \cong \text{hom}_Y(\mathcal{G}, f_*(\mathcal{F}))$ for any pair of sheaves \mathcal{F} and \mathcal{G} on X and Y , respectively.*

Proof. This is purely formal; we just have to keep track of definitions. We need to prove that there are functions $\sigma : \text{hom}_X(f^{-1}(\mathcal{G}), \mathcal{F}) \rightarrow \text{hom}_Y(\mathcal{G}, f_*(\mathcal{F}))$ and $\tau : \text{hom}_Y(\mathcal{G}, f_*(\mathcal{F})) \rightarrow \text{hom}_X(f^{-1}(\mathcal{G}), \mathcal{F})$ with σ and τ inverses to each other. To define σ , suppose that $\varphi : f^{-1}(\mathcal{G}) \rightarrow \mathcal{F}$ is a sheaf morphism. Then, for each open set $U \subseteq X$, there is a map $\varphi_U : \varinjlim \mathcal{G}(V) \rightarrow \mathcal{F}(U)$, where the limit is over all open $V \subseteq Y$ with $V \supseteq f(U)$. If $U = f^{-1}(W)$ for some W , then we note that $\varinjlim \mathcal{G}(V) = \mathcal{G}(W)$. Therefore, for any open $W \subseteq Y$, we have the map $\varphi_{f^{-1}(W)} \circ \pi_W : \mathcal{G}(W) \rightarrow \varinjlim \mathcal{G}(V) \rightarrow \mathcal{F}(f^{-1}(W)) = f_*(\mathcal{F})(W)$. A short calculation shows that this is a sheaf morphism; we call this $\sigma(\varphi)$. Therefore, σ is a map $\text{hom}_X(f^{-1}(\mathcal{G}), \mathcal{F}) \rightarrow \text{hom}_Y(\mathcal{G}, f_*(\mathcal{F}))$. Conversely, to define τ , let $\psi : \mathcal{G} \rightarrow f_*(\mathcal{F})$ be a sheaf morphism. Then, for every open $V \subseteq Y$, we have maps $\psi_V : \mathcal{G}(V) \rightarrow \mathcal{F}(f^{-1}(V))$. Let $U \subseteq X$ be open. If $V \subseteq Y$ is open with $f(U) \subseteq V$, then $U \subseteq f^{-1}(V)$. Therefore, we have maps $\mathcal{G}(V) \rightarrow \mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{F}(U)$; the latter map is the restriction map. It is routine to check that these maps have the appropriate compatibility to give a unique map $\varinjlim \mathcal{G}(V) \rightarrow \mathcal{F}(U)$ for every U . Moreover, the uniqueness and the mapping property of sheafification will imply that we then have a sheaf map $f^{-1}(\mathcal{G}) \rightarrow \mathcal{F}$, which we call $\tau(\psi)$. Finally, a routine but somewhat tedious check of definitions shows that $\sigma \circ \tau = \text{id}$ and $\tau \circ \sigma = \text{id}$. Therefore, we have finished the (outline of the) proof of the lemma. \square

One of the main purposes of this note is to show that the global section functor Γ is left exact. If \mathcal{F} is a sheaf on X , then Γ is defined by $\Gamma(\mathcal{F}) = \mathcal{F}(X)$. This is a functor

from the category of sheaves to the category of Abelian groups. While it is easy enough to give a direct proof of this (see Exercise 1.8 in Chapter II of [1]), we invoke category theory machinery as an illustration of such methods. Recall ([2, Theorem 2.6.1]) that a right adjoint of a functor is left exact. We will show that Γ is left exact by proving that it is a right adjoint by identifying it as f_* for an appropriate map f and then quoting Lemma 20. Let X be a topological space, and let $\{*\}$ be a one point space. There is a unique (continuous) map $f : X \rightarrow \{*\}$. Note that, since $\{*\}$ has only one point, a sheaf on $\{*\}$ is nothing more than an Abelian group. While we do not need this, we show after the following proposition that Γ is right adjoint to the constant sheaf functor, that sends an Abelian group A to the sheafification of the presheaf $U \mapsto A$.

Proposition 21. *For the trivial map $f : X \rightarrow \{*\}$ described above, if \mathcal{F} is a sheaf on X , then $f_*(\mathcal{F})$ is the global section functor Γ . Therefore, Γ is left exact.*

Proof. To see this, we have $f_*(\mathcal{F})(*) = \mathcal{F}(f^{-1}(*)) = \mathcal{F}(X)$, which proves that $f_*(\mathcal{F}) = \Gamma$. By Lemma 20 and [2, Theorem 2.6.1], $f_* = \Gamma$ is left exact. \square

We point out that for f above, f^{-1} is the constant sheaves functor. For, if G is an Abelian group and \mathcal{G} is the corresponding sheaf on $\{*\}$, then $f^{-1}(\mathcal{G})$ is the sheaf associated to the presheaf $U \mapsto \mathcal{G}(*) = G$ for any nonempty open set U since $\{*\}$ is the only open set of $\{*\}$ that contains $f(U)$. Thus, $f^{-1}(\mathcal{G})$ is the constant sheaf on X corresponding to G .

We now show that the category $\mathfrak{Ab}(X)$ of sheaves on X has enough injectives. For each $x \in X$, let $i_x : \{x\} \rightarrow X$ be the inclusion map. It is a continuous map. Since $\{x\}$ is a one point space, Example 4 above shows that a (pre)sheaf on $\{x\}$ is the same thing as an Abelian group.

Lemma 22. *With the notation above, we have $\text{hom}(\mathcal{F}_x, A) \cong \text{hom}_X(\mathcal{F}, i_{x*}(A))$ for any Abelian group A . Therefore, i_{x*} is right exact to the stalk functor $\mathcal{F} \rightarrow \mathcal{F}_x$. In particular, i_{x*} is right exact. Thus, if I is an injective Abelian group, then $i_{x*}(I)$ is an injective sheaf.*

Proof. For ease of notation, we write i for i_x . By definition, $i_*(A)$ is the sheaf with $i_*(A)(U) = A$ if $x \in U$ and $i_*(A)(U) = 0$ otherwise. Let $\sigma : \mathcal{F}_x \rightarrow A$ be a group homomorphism. If $x \in U$, we have a group homomorphism $\mathcal{F}(U) \rightarrow \mathcal{F}_x \rightarrow A = i_*(A)(U)$ by composing σ with the natural homomorphism $\mathcal{F}(U) \rightarrow \mathcal{F}_x$. If $x \notin U$, then we have a unique map $\mathcal{F}(U) \rightarrow 0 = i_*(A)(U)$. A short check shows that this gives a sheaf morphism $\mathcal{F} \rightarrow i_*(A)$. Conversely, if $\tau : \mathcal{F} \rightarrow i_*(A)$ is a sheaf morphism, then we have a group homomorphism $\mathcal{F}(U) \rightarrow A$ for every open U containing x . These maps have the appropriate compatibility to yield a group homomorphism $\mathcal{F}_x \rightarrow A$. We have thus produced functions in both directions. A short calculation shows that they are group homomorphisms and are inverses to each other. Therefore, $\text{hom}(\mathcal{F}_x, A) \cong \text{hom}_X(\mathcal{F}, i_*(A))$.

This isomorphism shows that i_* is right adjoint to the stalk functor. By Proposition 2.3.10 of [2] and Lemma 20, if I is an injective Abelian group, then $i_*(I)$ is an injective sheaf. \square

Recall that a category \mathcal{A} is said to have enough injectives if for every object A there is an injective object I and a monic $i : A \rightarrow I$.

Corollary 23. *The category $\mathfrak{Ab}(X)$ of sheaves on X has enough injectives.*

Proof. Let \mathcal{F} be a sheaf on X . For every $x \in X$, there is an injective Abelian group I_x with an injective homomorphism $\mathcal{F}_x \rightarrow I_x$. The sheaf $i_{x*}(I_x)$ is injective by the previous lemma, and so the sheaf $\mathcal{J} = \prod_x i_{x*}(I_x)$ is also injective. By the lemma, the identity map $\mathcal{F}_x \rightarrow \mathcal{F}_x$ yields a sheaf morphism $\mathcal{F} \rightarrow i_{x*}(\mathcal{F}_x)$. Since i_{x*} is a functor, the group homomorphism $\mathcal{F}_x \rightarrow I_x$ gives a sheaf morphism $i_{x*}(\mathcal{F}_x) \rightarrow i_{x*}(I_x)$. Composing these gives a morphism $\mathcal{F} \rightarrow i_{x*}(I_x)$ for each x . This then induces a unique sheaf morphism $\mathcal{F} \rightarrow \mathcal{J}$ by the definition of a product. Note that $\mathcal{J}_x = I_x$ since $(i_{y*}(I_y))_x = 0$ if $y \neq x$. The corresponding map on stalks is $\mathcal{F}_x \rightarrow I_x$, which is injective. Therefore, the morphism $\mathcal{F} \rightarrow \mathcal{J}$ is injective. We have thus produced an injection from any sheaf into an injective sheaf. This proves the corollary. \square

Since $\mathfrak{Ab}(X)$ has enough injectives, we can produce injective resolutions for any sheaf. Therefore, we can construct the right derived functors $R^n(\Gamma)$ of the left exact functor Γ , and so we can define the cohomology groups

$$H^n(X, \mathcal{F}) = R^n(\Gamma)(\mathcal{F})$$

for any sheaf \mathcal{F} on X .

Let $0 \rightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\sigma} \mathcal{H} \rightarrow 0$ be an exact sequence of sheaves. The long exact sequence in cohomology then applies to sheaf cohomology to yield an exact sequence

$$0 \rightarrow \mathcal{F}(X) \xrightarrow{\varphi_X} \mathcal{G}(X) \xrightarrow{\sigma_X} \mathcal{H}(X) \xrightarrow{\delta} H^1(X, \mathcal{F}) \rightarrow H^1(X, \mathcal{G}) \rightarrow \dots$$

From this we see that failure of surjectivity of $\sigma_X : \mathcal{G}(X) \rightarrow \mathcal{H}(X)$ is explained by the map δ into $H^1(X, \mathcal{F})$. In particular, if $H^1(X, \mathcal{F}) = 0$, then σ_X is surjective. More generally, the sequence above shows that σ_X is surjective exactly when δ is the zero map.

References

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