Artin's Construction of an Algebraic Closure

Patrick Morandi

October 8, 2004

In this note we give a construction of an algebraic closure of an arbitrary field. This construction is due to Emil Artin. Zorn's lemma is not invoked in this proof, unlike the one given in the text. We do indirectly use Zorn's lemma since we require the existence of maximal ideals inside arbitrary commutative rings with identity, which does require Zorn's lemma. However, we avoid the use of cardinal arithmetic.

Lemma 1. *Let K/F be an algebraic extension such that every irreducible polynomial in $F[x]$ has a root in K. Then K is algebraically closed, and so is an algebraic closure of F.*

Proof. Let S be the separable closure of F in K. We first claim that S is separably closed. To see this, let $p(x) \in S[x]$ be a separable polynomial over S. If $\alpha_1, \dots, \alpha_n$ are the roots of p in a splitting field M/K of p, then each α_i is also separable over F. Let $T \subseteq M$ be the splitting field of $\min(F, \alpha_1)$; note that each α_i is a root of this polynomial. Thus, T/F is Galois, and $T = F(\beta)$ for some $\beta \in T$. Let $f(x) = \min(F, \beta)$. By hypothesis, there is $\gamma \in K$ with $f(\gamma) = 0$. Note that $\gamma \in F(\beta)$ since $F(\beta)/F$ is Galois. Moreover, $[F(\gamma) : F] = \deg(f) = [F(\beta) : F]$, so $F(\gamma) = F(\beta)$. Therefore, each $\alpha_i \in F(\beta) = F(\gamma) \subseteq K$ since $\gamma \in K$. Thus, as each α_i is separable over F, each $\alpha_i \in S$. This proves that S is separably closed, and so S is a separable closure of F. If $\operatorname{char}(F) = 0$, then $K = S$, and so K is algebraically closed. On the other hand, suppose that $\operatorname{char}(F) = p > 0$. The extension K/S is purely inseparable. Since S is a separable closure of F, if K is not algebraically closed, then there is $a \in K$ with $x^p - a \in K[x]$ irreducible over K. There is an n with $a^{p^n} = s \in S$. Let $s_1, \dots, s_m$ be the roots of $\min(F, s)$. Then $(x - s_1) \cdots (x - s_m) \in F[x]$, and so $f(x) = (x^{p^{n+1}} - s_1) \cdots (x^{p^{n+1}} - s_m) \in F[x]$. Since S/F is Galois, there are F-automorphisms σ_i with $\sigma_i(s) = s_i$. These extend to automorphisms of K by the isomorphism extension theorem, since K/S is normal, as it is purely inseparable. We denote extensions also by σ_i. By hypothesis, there is a $\beta \in K$ with $f(\beta) = 0$. Then $\beta^{p^{n+1}} = s_i$ for some i. Applying σ_i^{-1}, we get $a^{p^n} = s = \sigma_i^{-1}(\beta)^{p^{n+1}}$. Therefore, $a = \sigma_i^{-1}(\beta)^p \in K^p$. This contradicts the irreducibility of $x^p - a$. Therefore, K is indeed algebraically closed. □

Theorem 2. *Let F be a field. Then there exists an algebraic closure of F.*

Proof. By the lemma, we need to construct a field K for which K contains a root of each irreducible polynomial in $F[x]$. To do this, let $\mathcal{S}$ be the set of all monic irreducible polynomials in $F[x]$, and let $R = F\left[\{x_f : f \in \mathcal{S}\}\right]$ be a polynomial ring over F with one variable for each element of $\mathcal{S}$. Let $I = (\{f(x_f) : f \in \mathcal{S}\})$. We claim that I is a proper ideal, and prove this in the next lemma. Given this for now, let M be a maximal ideal of R containing I. Then $R/M := K$ is a field, and F embeds in K. Moreover, $x_f + M$ is a root in K of $f \in \mathcal{S}$. Furthermore, as K is generated over F by the various $x_f + M$, we see that K/F is algebraic. Thus, by the lemma, K is an algebraic closure of F.

The last step in the proof is to demonstrate that the ideal I defined above is not R. We do this in the following lemma. $\square$

Lemma 3. *Let F be a field and let $F[\{x_i\}]$ be the polynomial ring in the variables x_i, where i ranges over $\mathcal{I}$. Suppose for each i that $f_i(x)$ is a monic irreducible polynomial over F. Then the ideal I of $F[x_i]$ generated by $f_i(x_i)$ for each i is a proper ideal.*

Proof. Suppose $I = F[\{x_i\}]$. Then there is an n and polynomials $g_1, \ldots, g_n$ such that $1 = f_1(x_{i_1})g_1(x_{i_1}, \ldots, x_{i_n}) + \cdots + f_n(x_{i_n})g_n(x_{i_1}, \ldots, x_{i_n})$. For simplicity we shall write x_m in place of x_{i_m} for each m. We can assume that all the g_m involve only the variables $x_1, \ldots, x_n$ by increasing the number of f_m if necessary in an equation of this type. Suppose n is chosen to be minimal such that we have such an expression involving n of the x_i. If $S = F[x_1, \ldots, x_n]$ then $(f_1(x_1), \ldots, f_n(x_n)) = S$. Let $R = F[x_1, \ldots, x_{n-1}]$. By minimality of n we have $(f_1(x_1), \ldots, f_{n-1}(x_{n-1})) \neq R$. Let us view the above equation as taking place in $S = R[x_n]$. If $c_m = f_m(x_m) \in R$ we have $J = (c_1, \ldots, c_{n-1}, f_n(x_n)) = S$. Now set $I_0 = (c_1, \ldots, c_{n-1}) \subseteq R$. So $J = (I, f_n(x_n))$. There are ring homomorphisms

$$R[x_n] \longrightarrow (R/I_0)[x_n] \longrightarrow \frac{(R/I_0)[x_n]}{(\overline{f_n(x_n)})}$$

where $\overline{f_n(x_n)}$ is the image of $f_n(x_n)$ in $(R/I_0)[x]$. Since R/I_0 is a nonzero ring and $\overline{f_n(x_n)}$ is not a unit (as f_n is monic of degree at least 1) we see that this last ring is nonzero. Hence the kernel of the composite homomorphism is a proper ideal of S. But J lies in this kernel, so $J \neq S$. This contradiction shows our original I is a proper ideal of $F[\{x_i\}]$, proving the lemma. $\square$