Central Simple Algebras

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September 20, 1998

If F is a field, then a *central simple F-algebra* is a finite dimensional F-algebra S that is simple as a ring, and for which the center $Z(S)$ is equal to F. In this note we prove some fundamental theorems about central simple algebras. We assume the structure theorems for simple Artinian rings. For example, if S is a central simple F-algebra, then Wedderburn's theorem says that $S \cong M_n(D)$, the ring of $n \times n$ matrices over a division ring D. It then follows that $D \cong \mathrm{End}_S(V)$, that $\dim_F(D) < \infty$, and $Z(D) = F$. So, D is a central simple F-division algebra. Alternatively, we may view $S \cong \mathrm{End}_D(V)$, where V is a right D-vector space of dimension n. The Abelian group V is a left S-module via the multiplication $s \cdot v = s(v)$. Moreover, V is the unique up to isomorphism simple left V-module, and every left S-module is isomorphic to a direct sum of (possibly infinitely many) copies of V. For proofs of these theorems, see [2, Sec. 4.4] or [1, Chap. IX].

If A and B are F-algebras, then we may put an F-algebra structure on the F-vector space $A \otimes_F B$, induced by the formula $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. By an argument involving the universal mapping property of tensor products, one can see that this is well defined, and that $A \otimes_F B$ is an F-algebra with this multiplication. One property of tensor products of vector spaces that we will use below is that if $b_1, \dots, b_n$ is an F-basis for B, then every element of $A \otimes_F B$ can be written in the form $\sum_i a_i \otimes b_i$, and the a_i are uniquely determined. In particular, a consequence of this fact is that if $a \in A$ and $b \in B$, then $a \otimes b = 0$ implies that $a = 0$ or $b = 0$; this follows since b is part of an F-basis of B if $b \neq 0$.

Theorem 1 *Let A be a central simple F-algebra and let B be a simple F-algebra with center $Z(B) \supseteq F$. Then $A \otimes_F B$ is a central simple $Z(B)$-algebra.*

Proof. We first prove that $Z(A \otimes_F B) = F \otimes_F Z(B) \cong Z(B)$. Let $\{b_i\}$ be an F-basis for B with $b_1 = 1$. Then we may write every element of $A \otimes_F B$ in the form $\sum_i a_i \otimes b_i$, and the a_i are uniquely determined. Suppose that $x = \sum_i a_i \otimes b_i \in Z(A \otimes_F B)$. For any $a \in A$, we have $(a \otimes 1)x = x(a \otimes 1)$. Thus, $\sum_i aa_i \otimes b_i = \sum a_i a \otimes b_i$. Therefore, since the b_i form a basis for B, we obtain $aa_i = a_i a$ for each i. Since this is true for each $a \in A$, we have $a_i \in Z(A) = F$. So,

$$x = \sum_i a_i \otimes b_i = \sum_i 1 \otimes a_i b_i = 1 \otimes \sum_i a_i b_i.$$

We set $b = \sum_i a_i b_i$. For any $c \in B$, we then have $x(1 \otimes c) = (1 \otimes c)x$, or $1 \otimes bc = 1 \otimes cb$. Since $1 \otimes (bc - cb)$ is then 0, this forces $bc = cb$, so $b \in Z(B)$. Therefore, $x = 1 \otimes b \in F \otimes_F Z(B)$, as desired.

We now prove simplicity. Suppose that I is a nonzero ideal of $A \otimes_F B$. There is a nonzero element $x = \sum_{i=1}^n a_i \otimes b_i \in I$ with n minimal. Then $a_1 \neq 0$. Since A is simple, we have $Aa_1A = A$, so we may write $1 = \sum_{j=1}^m c_j a_1 d_j$ for some $c_j, d_j \in A$. Since I is an ideal, we have

$$\sum_j (c_j \otimes 1)\, x\, (d_j \otimes 1) \in I.$$

In other words, $\sum_i \left(\left(\sum_j c_j a_i d_j \right) \otimes b_i \right) \in I$. Therefore, we have a new element $y = 1 \otimes b_1 + \sum_{i=2}^n a_i' \otimes b_i \in I$. By the minimality of n, we have each $a_i' \neq 0$. For $a \in A$, consider the element $(a \otimes 1)y - y(a \otimes 1) \in I$. This is equal to $\sum_{i=2}^n (aa_i' - a_i'a) \otimes b_i$. This a contradiction to the minimality of n unless $aa_i' = a_i'a$ for each i. Thus, each $a_i' \in Z(A) = F$. Then $y = 1 \otimes \sum(a_i' b_i)$. However, if $b = \sum(a_i' b_i)$, then $BbB = B$ since B is simple. Writing $\sum_j e_j b f_j = 1$ with $e_j, f_j \in B$, we get $1 = \sum_j (1 \otimes e_j) x (1 \otimes f_j) \in I$. This proves that $I = A \otimes_F B$. Therefore, $A \otimes_F B$ is simple. ∎

If R is a ring, then we can make a new ring R^{op}, the *opposite ring* to R, where $(R^{\text{op}}, +) = (R, +)$ as additive groups, and multiplication is defined by $a \cdot b = ba$. To keep confusion down, we will write a^{op} for a viewed as an element of R^{op}. So, the operations in R^{op} are given by $a^{\text{op}} + b^{\text{op}} = (a + b)^{\text{op}}$ and $a^{\text{op}} b^{\text{op}} = (ba)^{\text{op}}$.

Proposition 2 *Let S be a central simple F-algebra. Then $S \otimes_F S^{\text{op}} \cong \text{End}_F(S)$ as F-algebras.*

Proof. We define a map $S \times S^{\text{op}} \to \text{End}_F(S)$ by $(s, t^{\text{op}}) \mapsto \varphi_{(s,t)}$, where $\varphi_{(s,t)}(x) = sxt$. A short calculation shows that $\varphi_{(s,t)}$ is an F-linear endomorphism of S, and that the map $(s, t^{\text{op}}) \mapsto \varphi_{(s,t)}$ is F-bilinear. So, this induces an F-linear map $\varphi : S \otimes_F S^{\text{op}} \to \text{End}_F(S)$ satisfying $\varphi(s \otimes t^{\text{op}}) = \varphi_{(s,t^{\text{op}})}$. The map φ is a ring homomorphism since

$$\begin{aligned}
\varphi\left((a \otimes b^{\text{op}})(s \otimes t^{\text{op}})\right)(x) &= \varphi(as \otimes (tb)^{\text{op}})(x) \\
&= asxtb \\
&= \varphi(a \otimes b^{\text{op}})(sxt) = \varphi(a \otimes b^{\text{op}})\varphi(s \otimes t^{\text{op}})(x)
\end{aligned}$$

for all $x \in S$. By Theorem 1, $S \otimes_F S^{\text{op}}$ is a simple ring, so $\ker \varphi = 0$. Both $S \otimes_F S^{\text{op}}$ and $\text{End}_F(S)$ have F-dimension $\dim_F(S)^2$, so it follows that φ is an isomorphism. ∎

Lemma 3 *Let A and B be central simple F-algebras. Then $C_{A \otimes_F B}(F \otimes_F B) = A \otimes_F F$.*

Proof. Let $\{a_i\}$ be an F-basis for A with $a_1 = 1$. We can write every element of $A \otimes_F B$ uniquely in the form $\sum_i a_i \otimes b_i$. Suppose that $\sum_i a_i \otimes b_i \in C_{A \otimes_F B}(F \otimes_F B)$. Then $\left(\sum_i a_i \otimes b_i\right)(1 \otimes b) = (1 \otimes b)\left(\sum_i a_i \otimes b_i\right)$ for all $b \in B$. Thus, $\sum_i a_i \otimes b_i b = \sum_i a_i \otimes b b_i$. By the uniqueness of the representation, we see that $b_i b = b b_i$, so each $b_i \in Z(B) = F$. Therefore, $\sum_i a_i \otimes b_i \in A \otimes_F F$. The reverse inclusion is clear, so $C_{A\otimes_F B}(F \otimes_F B) = A \otimes_F F$. ■

Theorem 4 (Noether-Skolem) *Let S be a central simple F-algebra. If B is a simple F-subalgebra of S with an F-algebra homomorphism $\varphi : B \to S$, then there is an $x \in S^*$ such that $\varphi(a) = xax^{-1}$.*

Proof. We first consider the case $S = \mathrm{End}_F(V) \cong M_n(F)$, where $n = \dim_F(V)$. We can make V into a B-module in two ways. First, V is a B-module by virtue of the fact that B is a subalgebra of S. Next, we can make V into a B-module via φ by defining $b \cdot v = \varphi(b)(v)$. We refer to V with this B-module structure as V_φ. Note that the F-vector space structure is the same for V and V_φ. Now, let M be the unique (up to isomorphism) simple B-module; M exists since B is simple. Then both V and V_φ are direct sums of copies of M. Say $V \cong M^r$ and $V_\varphi \cong M^s$. Then

$$r \dim_F(M) = \dim_F(V) = \dim_F(V_\varphi) = s \dim_F(M),$$

so $r = s$. This shows that V and V_φ are isomorphic as B-modules. Let $x : V \to V_\varphi$ be a B-module isomorphism. Note that $x \in S^*$ since x is an F-vector space isomorphism. We have, for all $v \in V$ and $b \in B$, that

$$x(bv) = b \cdot xv = \varphi(b)xv.$$

So, as linear transformations, $xb = \varphi(b)x$, or $\varphi(b) = xbx^{-1}$, as desired.

For the general case, consider the map $\varphi \otimes \mathrm{id} : B \otimes_F S^{\mathrm{op}} \to S \otimes_F S^{\mathrm{op}}$. We know that $S \otimes_F S^{\mathrm{op}} \cong M_m(F)$ for some m by Proposition 2. Therefore, by the previous case, there is an $x \in S \otimes_F S^{\mathrm{op}}$ such that

$$(\varphi \otimes \mathrm{id})\,(b \otimes s^{\mathrm{op}}) = y(b \otimes s^{\mathrm{op}})y^{-1}.$$

By setting $b = 1$, we see that y commutes with $F \otimes_F S^{\mathrm{op}}$. By the lemma, the centralizer of $F \otimes_F S^{\mathrm{op}}$ is $S \otimes_F F$. So, we can write $y = x \otimes 1$ for some $x \in S^*$. We then see, by setting $s = 1$, that $xbx^{-1} = \varphi(b)$. ■

Let S be a central simple F-algebra. If B is an F-subalgebra of S, we define the *centralizer* $C_S(B)$ to be the set $C_S(B) = \{s \in S : sb = bs \text{ for all } b \in B\}$. It is easy to see that $C_S(B)$ is an F-subalgebra of S. Moreover, $B \subseteq C_S(B)$ if and only if B is commutative.

Lemma 5 *Let S be an F-algebra, and let B be an F-subalgebra of S. We can view S as a left $S \otimes_F B^{\mathrm{op}}$-module via $(a \otimes b^{\mathrm{op}})c = acb$. Then the left regular representation induces an isomorphism $C_S(B) \to \mathrm{End}_{S\otimes_F B^{\mathrm{op}}}(S)$.*

Proof. Recall that S is a left $S \otimes_F S^{\mathrm{op}}$-module, and so S is also a left $S \otimes_F B^{\mathrm{op}}$-module. We consider, for $c \in S$, the left multiplication map $L_c : S \to S$. Suppose that $c \in C_B(S)$.

Then for $b \in B$ and $a, x \in S$, we have

$$\begin{aligned} L_c((b \otimes a^{\mathrm{op}})\, x) &= L_c(bxa) = cbxa = bcxa \\ &= (b \otimes a^{\mathrm{op}})\, L_c(a). \end{aligned}$$

Therefore, L_c is a $B \otimes_F S^{\mathrm{op}}$-module homomorphism. There is then a well defined map $L : C_S(B) \to \mathrm{End}_{S \otimes_F B^{\mathrm{op}}}(S)$, given by $L(c) = L_c$. It is clear that this is an F-algebra homomorphism. It is also injective, since if $L_c = 0$, then $0 = L_c(1) = c$. Finally, to see that L is surjective, suppose that $\varphi \in \mathrm{End}_{S \otimes_F B^{\mathrm{op}}}(S)$, and set $\varphi(1) = c$. For any $b \in B$, we have $b = (b \otimes 1^{\mathrm{op}})\, 1 = (1 \otimes b^{\mathrm{op}})\, 1$, so

$$\varphi(b) = \varphi((b \otimes 1^{\mathrm{op}})\, 1) = (b \otimes 1^{\mathrm{op}})\, \varphi(1) = bc$$

and

$$\varphi(b) = \varphi((1 \otimes b^{\mathrm{op}})\, 1) = (1 \otimes b^{\mathrm{op}})\, \varphi(1) = cb.$$

Therefore, $c \in C_S(B)$. Thus, $C_S(B) \cong \mathrm{End}_{S \otimes_F B^{\mathrm{op}}}(S)$. ■

Theorem 6 (Double Centralizer) *Let S be a central simple F-algebra and let B be a simple F-subalgebra of S. Then*

1. *$C_S(B)$ is a simple F-algebra of S;*
2. *$\dim_F B \cdot \dim_F C_S(B) = \dim_F S$;*
3. *$C_S(C_S(B)) = B$;*
4. *If $Z(B) = F$, then $S \cong B \otimes_F C_S(B)$.*

Proof. We know that $S \otimes_F B^{\mathrm{op}}$ is simple by Theorem 1. Let M be the unique simple left $S \otimes_F B^{\mathrm{op}}$-module. Let $D = \mathrm{End}_{S \otimes_F B^{\mathrm{op}}}(M)$, a division algebra. Then $S \otimes_F B^{\mathrm{op}} \cong M_n(D)$, where $n = \dim_D(M)$ (and $S \otimes_F B^{\mathrm{op}} \cong M^n$ as $S \otimes_F B^{\mathrm{op}}$-modules). Since S is a left $S \otimes_F B^{\mathrm{op}}$-module, we may write $S \cong M^r$ for some r. By the previous lemma, $C_S(B) \cong \mathrm{End}_{S \otimes_F B^{\mathrm{op}}}(S) \cong M_r(D)$, so $C_S(B)$ is a simple ring. This proves the first statement. Counting dimensions, we have

$$\dim_F(B) \dim_F(S) = n^2 \dim_F(D)$$

and

$$\dim_F(C_S(B)) = r^2 \dim_F(D).$$

Moreover, $\dim_F(S) = r\dim_F(M) = rn\dim_F(D)$. Substituting this into the first equation shows that $\dim_F(B) = n/r$. So,

$$\dim_F(B)\dim_F(C_S(B)) = n/r(r^2\dim_F(D)) = rn\dim_F(D) = \dim_F(S).$$

This proves the second statement.

For the third statement, the definition of centralizer shows that $B \subseteq C_S(C_S(B))$. By the previous statement, we have

$$\dim_F(S) = \dim_F(B)\dim_F(C_S(B))$$

and

$$\dim_F(S) = \dim_F(C_S(B))\dim_F(C_S(C_S(B))).$$

Therefore, $\dim_F(B) = \dim_F(C_S(C_S(B)))$, so $B = C_S(C_S(B))$. Finally, suppose that $Z(B) = F$. Consider the map $B \otimes_F C_S(B) \to S$ given by $b \otimes c \mapsto bc$. This is a well defined F-linear map, and it is a ring homomorphism since any $b \in B$ and $c \in C_S(C)$ commute. Moreover, this map is injective since $B \otimes_F C_S(B)$ is simple; this follows from Theorem 1 and the hypothesis that $Z(B) = F$. Finally, dimension count using the third statement shows that this map is surjective. Thus, $B \otimes_F C_S(B) \cong S$. ■

References

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[2] N. Jacobson, *Basic algebra II*, W. H. Freeman, New York, 1980.