Algebraic Groups, Grassmannians, and Flag Varieties

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In this note we investigate the relation between flag varieties and algebraic groups. We give a detailed description of Grassmannian varieties, partly because they are interesting in their own right, and partly because flag varieties are defined as subvarieties of products of Grassmannians. After discussing Grassmannians, we show how flag varieties arise as quotients of algebraic groups. The standard flag varieties are quotients of $\mathrm{Gl}(n, k)$ for some n, and for other classical algebraic groups we get other forms of flag varieties. In particular, we look at orthogonal groups of even dimensional quadratic spaces, the algebraic groups of type D_n. Throughout this note we assume that k is a fixed algebraically closed field of characteristic not 2. References for algebraic groups include Borel [1] and Humphreys [4]. Brief descriptions of flag varieties and Grassmannians can be found in [1, Sec. 10.3] and [4, Sec. 1.8]. A more complete description of Grassmannian varieties is in [2, Lecture 6]. A reference for algebraic geometry is [3]. Finally, Lam's book [5] is a good reference for quadratic forms.

1 A Small Amount of Algebraic Geometry

In this note we will use ideas of algebraic geometry. If k is an algebraically closed field, then *affine n-space* $\mathbf{A}^n$ is, as a set, the space of all n-tuples of elements of k. We put a topology, the *Zariski topology,* on $\mathbf{A}^n$, by defining a set to be closed if it the set of common zeros of a collection of polynomial equations. In other words, a set C is closed if there is a set S of polynomials in $k[x_1, \dots, x_n]$ with $C = \{P \in \mathbf{A}^n : f(P) = 0 \text{ for all } f \in S\}$. This set is called the zero set of S, and is written $Z(S)$. We call a subset $X \subseteq \mathbf{A}^n$ an *affine variety* if $X = Z(S)$ for some $S \subseteq k[x_1, \dots, x_n]$. If U is an open subset of an affine variety, we will call U a *quasi-affine variety.*

We also will use projective varieties. We define *projective n-space* $\mathbf{P}^n$ as the set of equivalence classes of $(n+1)$-tuples of elements in k (other than $(0, \dots, 0)$) under the equivalence relation $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$ if there is an $\alpha \in k^*$ with $b_i = \alpha a_i$ for all i. We will write $(a_0 : \dots : a_n)$ for the equivalence class of $(a_0, \dots, a_n)$. A more intrinsic definition is that $\mathbf{P}^n$ is the set of all lines passing through the origin in the vector space k^{n+1}. Thinking in this way, if V is a k-vector space, we define $\mathbf{P}(V)$ to be the set of all one dimensional subspaces of V. Any such space can be written in the form kv for some nonzero vector v. By choosing a basis for V, we can identify $\mathbf{P}(V)$ and $\mathbf{P}^{\dim(V)-1}$. The Zariski topology on $\mathbf{P}^n$ defined by setting a subset C of $\mathbf{P}^n$ to be closed if there is a set $S \subseteq k[x_0, \ldots, x_n]$ of *homogeneous* polynomials such that

$$C = \{(a_0 : \ldots : a_n) : f(a_0, \ldots, a_n) = 0 \text{ for all } f \in S\}.$$

While the evaluation $f(a_0, \ldots, a_n)$ is not well defined, since f is homogeneous, the condition $f(a_0, \ldots, a_n) = 0$ is independent of the choice of representation of a point in projective space. A *projective variety* is a closed subset of $\mathbf{P}^n$ for some n. If U is an open subset of a projective variety, we call U a *quasi-projective variety.* We will call X a variety if X is any type of variety described above; i.e., affine, projective, quasi-affine, or quasi-projective variety.

A topological space X is said to be *irreducible* if it is not the union of two proper closed subspaces. An easy calculation shows that if $X \subseteq \mathbf{A}^n$ is an affine variety, and if

$$I(X) = \{f \in k[x_1, \ldots, x_n] : f(P) = 0 \text{ for all } P \in X\},$$

then X is irreducible if and only if $I(X)$ is a prime ideal of $k[x_1, \ldots, x_n]$. A similar definition and result holds for projective varieties. It is not hard to see that a subspace X of a topological space Y is irreducible if and only if the closure $\overline{X}$ of X in Y is irreducible. This allows one to determine when a quasi-affine or a quasi-projective variety is irreducible. For a general variety X, if X is not irreducible, then X can be uniquely written as the union of finitely many closed irreducible subvarieties of X, each maximal with respect to inclusion among the closed irreducible subvarieties of X; these subvarieties are called the irreducible components of X. Note that they need not be disjoint, unlike the connected components of a topological space.

One important property that distinguishes projective and affine varieties is the notion of completeness. A variety X is said to be *complete* if, for all varieties Y, the projection map $X \times Y \to Y$ sends closed sets to closed sets. We list the properties of completeness that we will use below. For references on completeness, see [4, Sec. 6].

Proposition 1.1 *Let X and Y be varieties.*

1. *A variety is complete if and only if it is projective.*
2. *If $\varphi : X \to Y$ is a morphism and X is complete, then $\varphi(X)$ is complete.*
3. *If X is complete and X is a subvariety of Y, then X is closed in Y.*

For example, to see that $\mathbf{A}^n$ is not complete, we look at the projection map $\mathbf{A}^n \times \mathbf{A}^1 \to \mathbf{A}^1$ and the closed subset of $\mathbf{A}^n \times \mathbf{A}^1 = \mathbf{A}^{n+1}$ that is the zero set of $x_1 \cdots x_n y - 1 = 0$; this consists of the points $(a_1, \ldots, a_n, b)$ with $a_1 \cdots a_n b = 1$. The projection into $\mathbf{A}^1$ of this variety is $\mathbf{A}^1 - \{0\}$, which is not a closed subset of $\mathbf{A}^1$.

2 Algebraic Groups

Let G be a group. Suppose that G also has the structure of an affine algebraic variety, and that the multiplication map $G \times G$ and the inversion map $G \to G$ are morphisms of algebraic varieties. Then we call G a *linear algebraic group*.

For example, The group $\mathbf{G}_a$ is the group $(k, +)$. By viewing $k = \mathbf{A}^1$, and recognizing that the formulas for addition and negation are linear polynomials in the components, $\mathbf{G}_a$ is an algebraic group. The multiplicative group $\mathbf{G}_m = (k^*, \cdot)$ is also an algebraic group. While $\mathbf{G}_m$ is an open subset in $\mathbf{A}^1$, we can view it as a closed set in $\mathbf{A}^2$ via the mapping $a \mapsto (a, a^{-1})$; the image of this map is the zero set of the polynomial $xy - 1$. By viewing $\mathbf{G}_m$ then as the set $\{(a, b) : ab = 1\}$ in $\mathbf{A}^2$, we see that multiplication is given by $(a, b)(c, d) = (ac, bd)$ and inversion by $(a, b)^{-1} = (b, a)$. Both these formulas are given by polynomial equations in the components of the pair, so they are morphisms of varieties. This shows that $\mathbf{G}_m$ is indeed an algebraic group.

The special linear group $\mathrm{Sl}(n, k) = \{a \in M_n(k) : \det(a) = 1\}$ can be viewed as a subset of $M_n(k) \cong \mathbf{A}^{n^2}$, and $\mathrm{Sl}(n, k)$ is the zero set of the polynomial $\det - 1$. So, $\mathrm{Sl}(n, k)$ is an affine variety. The formulas for matrix multiplication and inversion show that these operations are given by polynomial equations in the components of a matrix (of determinant 1), so $\mathrm{Sl}(n, k)$ is an algebraic group.

Perhaps the most important example of an algebraic group is the general linear group $\mathrm{Gl}(n, k)$. This is the group of all invertible $n \times n$ matrices. Viewed as a subset of $\mathbf{A}^{n^2}$, it is an open set, consisting of the complement of the zero set of det. However, because it is the complement of the zero set of a single polynomial, it is actually an affine variety (see, for example, [3, Chap. I, Lemma 4.2]). As with $\mathrm{Sl}(n, k)$, the formulas for multiplication and inversion show that $\mathrm{Gl}(n, k)$ is an algebraic group. If we let V be an n-dimensional k-vector space, we sometimes view $\mathrm{Gl}(n, k)$ as the group of invertible linear transformations on V, and we sometimes write $\mathrm{Gl}(V)$. The reason for the name "linear" algebraic group is that a theorem in this subject is that any algebraic group is isomorphic to a subgroup of $\mathrm{Gl}(n, k)$ for some n (see [4, Thm. 8.6]). The center of $\mathrm{Gl}(n, k)$ is the set of scalar matrices, isomorphic to the group k^*. The quotient group $\mathrm{Gl}(n, k)/k^*$ is the group $\mathrm{PGl}(n, k)$, the projective general linear group. One can view it as the complement of the zero set of det inside $\mathbf{P}^{n^2}$. Because it is the complement of a zero set of a single polynomial, it is an affine variety (see [3, Chap. I, Prob. 3.5]).

There are a number of important algebraic subgroups of $\mathrm{Gl}(n, k)$. For example, the group $T(n, k)$ of upper triangular invertible matrices is an algebraic group; it is the zero set inside $\mathrm{Gl}(n, k)$ of the equations $x_{ij} = 0$ for $i > j$. A routine group theory exercise shows that $T(n, k)$ is a solvable group. Less routine is that $T(n, k)$ is irreducible as a variety, although this can be seen from the fact that there is an isomorphism of varieties $T(n, k) \cong \mathbf{G}_m^n \times \mathbf{A}^{n(n-1)/2}$, by viewing elements of $T(n, k)$ as $n(n+1)/2$-tuples of elements of k, the only restriction is that the diagonal elements are nonzero. The group $D(n, k)$ of invertible diagonal matrices is an algebraic group, since it is the zero set inside $\mathrm{Gl}(n, k)$ of the equations $x_{ij} = 0$ for $i \neq j$. As a variety, $D(n,k) \cong \mathbf{G}_m^n$, so $D(n,k)$ is an irreducible variety. Finally, the group $U(n,k)$ of *unipotent* matrices is the set of all upper triangular matrices such that all diagonal entries are 1. This is the zero set inside $\mathrm{Gl}(n,k)$ of the equations $x_{ij} = 0$ for $i > j$ and $x_{ii} - 1 = 0$ for $1 \le i \le n$.

Let (V,q) be a quadratic space; that is, V is a k-vector space and q is a quadratic form on V. The *orthogonal group* $\mathrm{O}(V,q)$ is the group

$$\mathrm{O}(V,q) = \{a \in \mathrm{Gl}(V) : q(a(v)) = q(v)\}.$$

To see that $\mathrm{O}(V,q)$ is an algebraic subgroup of $\mathrm{Gl}(V)$, we need to see that it is a closed subgroup of $\mathrm{Gl}(V)$. Pick a basis $\{v_1,\dots,v_n\}$ of V, and identify $\mathrm{Gl}(V) = \mathrm{Gl}(n,k)$. Also, let B be a symmetric matrix that represents q. Thus, if b is the associated bilinear space to q, then for any $u, v \in V$, viewed as column vectors, we have $b(u,v) = u^T B v$. From this it is quick to see that $\mathrm{O}(V,q) = \left\{a \in \mathrm{Gl}(n,k) : a^T B a = B\right\}$. The condition $a^T B a$, written out in terms of the entries of a, gives n^2 quadratic polynomial equations in the coefficients of a; thus, $\mathrm{O}(V,q)$ is a closed subset of $\mathrm{Gl}(n,k)$. Because k is algebraically closed, if $n = 2m$, then q is hyperbolic. By choosing the basis appropriately, we may represent q by the matrix

$$J_n = \begin{pmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{pmatrix};$$

representing q by this matrix makes certain calculations easier, one of which we will see later. Note that in terms of the bilinear form b, the orthogonal group can be described by $\mathrm{O}(V,q) = \{a : b(a(u), a(v) = b(u,v)\}$.

For one more example, let V be a k-vector space of dimension $2n$, and let h be a skew-symmetric bilinear form on V. Then the *symplectic group* $\mathrm{Sp}(V,h)$ is the group

$$\mathrm{Sp}(V,h) = \{a \in \mathrm{Gl}(V) : h(a(u), a(v)) = h(u,v) \text{ for all } u, v \in V\}.$$

By picking an appropriate basis for V, and using the classification of skew-symmetric forms, we may represent h by the skew-symmetric matrix $H = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$, and the symplectic group is then described by $\mathrm{Sp}(2n,k) = \left\{a \in \mathrm{Gl}(2n,k) : a^T H a = H\right\}$; a similar description to that for $\mathrm{O}(n,k)$. This description together with the argument for $\mathrm{O}(V,q)$ shows that $\mathrm{Sp}(V,h)$ is an algebraic group.

If G is an algebraic group, then G is a union of its irreducible components. One property of algebraic groups is that an algebraic group is irreducible if and only if it is connected. Also, the connected (= irreducible) component G^0 containing the identity of G is a closed subgroup of G; this subgroup is called the identity component of G. Moreover, the other connected components of G are the various cosets of G^0. If $G = \mathrm{O}(V,q)$, then every element of G has determinant ± 1. The set

$$\mathrm{O}^+(V,q) = \{g \in \mathrm{O}(V,q) : \det(g) = 1\}$$

is the identity component of $\mathrm{O}(V,q)$, and the other component consists of the isometries of (V,q) of determinant -1. This group is often denoted $\mathrm{SO}(V,q)$, and called the special orthogonal group.

One important property of algebraic groups is that they often act on varieties. To be more specific, let G be an algebraic group and V a variety, not necessarily affine. Then we say that G *acts* on V as an algebraic group if there is an action $G \times V \to V$ of G (as an abstract group) on the set V, but for which this map is a morphism of algebraic varieties. As with ordinary group actions, for $x \in V$, we can talk about the *orbit* $\mathcal{O}(x)$ of x, the set $\{gx : g \in G\}$, and the stabilizer $G_x = \{g \in G : gx = x\}$, a subgroup of G. Two facts from the theory of algebraic groups are that every orbit is a quasi-projective variety, and the stabilizer of any point is a closed subgroup of G (see [4, Prop. 8.2, Prop. 8.3]).

Example 2.1 Let $G = \mathrm{Gl}(V)$. Let $\dim(V) = n$, so $G \cong \mathrm{Gl}(n,k)$. The group G acts on $\mathbf{P}(V)$ by $g(kv) = k(gv)$. This is an algebraic group action since the mapping $G \times \mathbf{P}(V) \to \mathbf{P}(V)$ is given by linear polynomials. This action is transitive, since if v and w are any two nonzero vectors in V, there is an invertible linear transformation that sends v to w. Let $v = (1,0,\ldots,0)$ and $P = kv$. Then a short calculation shows that the stabilizer G_P is

$$G_P = \{g : g(v) = \alpha v \text{ for some } \alpha \in k^*\}.$$

Thus, for this v, we have

$$G_P = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

This is a closed subgroup of G since it is given by the vanishing of the equations $x_{r1} = 0$ for $2 \le r \le n$. Moreover, there is a 1-1 correspondence $G/G_P \leftrightarrow \mathbf{P}(V)$ via $gG_P \leftrightarrow g(P)$. The set G/G_P then has a structure as a projective variety, by transferring the variety structure of $\mathbf{P}(V)$ to G/G_P via the bijection given above. This variety structure on G/G_P may seem somewhat ad-hoc, but, in fact, it is actually canonical, in an appropriate sense (see [4, Sec. 12])

The action of $\mathrm{Gl}(V)$ on $\mathbf{P}(V)$ given above induces an action of $\mathrm{PGl}(V)$ on $\mathbf{P}(V)$, since the action of k^* on $\mathbf{P}(V)$ is trivial.

Example 2.2 Let V_r be the set of $n \times m$ matrices of rank r. If $r \le \min\{m,n\}$, then this set is nonempty. We view V_r as a subset of $\mathbf{A}^{nm}$ by identifying an $n \times m$ matrix with an nm-tuple. Note that for any s, the set R_s of matrices of rank less than s is a closed subvariety of $\mathbf{A}^{nm}$ since it is the zero set of all the determinants of $s \times s$ subdeterminants. In particular,

R_{r+1} and R_r are closed subvarieties. Since $V_r = R_{r+1} - R_r$, we see that V_r is open in R_{r+1}, so V_r is a quasi-affine variety. The group $\mathrm{Gl}(n,k)$ acts on V_r via left multiplication; this is well defined since rank is preserved under a linear automorphism.

Example 2.3 The group $G = \mathrm{Gl}(n,k)$ acts on affine space $\mathbf{A}^n$. Under this action we have two orbits. One is $\{(0,\dots,0)\}$, and the other is $\mathbf{A}^n - \{(0,\dots,0)\}$. Note that one orbit is closed (the one of minimal dimension), and the other is open. Since there is more than one orbit, G does not act transitively on $\mathbf{A}^n$. If $v = (1,0,\dots,0)$, then the stabilizer of v is the set of matrices of the form

$$\begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix},$$

a closed subgroup of G since it is the zero set of $x_{11} - 1 = x_{21} = \cdots = x_{n1} = 0$.

Example 2.4 Let $G = \mathrm{O}(V,q)$. The group G acts on $\mathbf{P}(V)$ by virtue of the inclusion $G \subseteq \mathrm{Gl}(V)$. Let $v \in V$ with $q(v) = 0$. Then $q(g(v)) = q(v) = 0$ for all $g \in G$. So, the orbit of $P = kv$ under G is contained in the set of isotropic vectors of V. By the theorem of Witt, in fact we see that the orbit is precisely the set of isotropic vectors. Therefore, the orbit of P is $Z(q) \subseteq \mathbf{P}(V)$, a projective variety.

3 Grassmannian Varieties

Let V be a k-vector space of dimension n, and let $\{v_1,\dots,v_n\}$ be a basis for V. For a positive integer $r \le n$, the Grassmannian $\mathrm{Gr}(r,V)$ is defined to be the set of all r-dimensional subspaces of V. We note that $\mathrm{Gr}(1,V)$ is equal to the projective space $\mathbf{P}(V)$. For any r, we wish to show that $\mathrm{Gr}(r,V)$ has a natural structure as a projective variety.

We define the *Plücker embedding* $\mathrm{Gr}(r,V) \to \mathbf{P}(\bigwedge^r(V))$ as follows: let U be a subspace of V of dimension r, and let $\{u_1,\dots,u_r\}$ be a basis for U. We send U to the point $k(u_1 \wedge \cdots \wedge u_r)$ in $\mathbf{P}(\bigwedge^r(V))$. Note that this is well defined, since if $\{u'_1,\dots,u'_r\}$ is another basis for U, then $u'_1 \wedge \cdots \wedge u'_r = \det(M)\,(u_1 \wedge \cdots \wedge u_r)$, where M is the change of basis matrix relating these two bases. So, the resulting point in projective space is uniquely determined. Also, this map is an embedding. To see this, let $W \in \mathrm{Gr}(r,V)$, and let $\{w_1,\dots,w_r\}$ be a basis for W. Set $\omega = w_1 \wedge \cdots \wedge w_r$. We define a map $\varphi_\omega : V \to \bigwedge^{r+1}(V)$ by $\varphi_\omega(v) = v \wedge \omega$. This is a linear map, and it is easy to see that $W \subseteq \ker(\varphi_\omega)$. However, we can see the reverse inclusion: Extend $w_1,\dots,w_r$ to a basis $\{w_1,\dots,w_n\}$ of V. If $v = \sum_i \alpha_i w_i$, then

$$\begin{aligned} v \wedge \omega = \sum_i \alpha_i w_i \wedge \omega &= \sum_{i=r+1}^{n} \alpha_i w_i \wedge w_1 \wedge \cdots \wedge w_r \\ &= \sum_{i=r+1}^{n} (-1)^r \alpha_i w_1 \wedge \cdots \wedge w_r \wedge w_i. \end{aligned}$$

Each of the terms $w_1\wedge\cdots\wedge w_r\wedge w_i$ are basis elements for $\bigwedge^{r+1}(V)$; hence each $\alpha_i = 0$ for $i > r$. Thus, $v = \sum_{i=1}^r \alpha_i w_i \in W$, proving that $W = \ker(\varphi_\omega)$. From this we can see that the Plücker map is an embedding, since if U is another element of $\mathrm{Gr}(r, V)$ with basis $\{u_1, \ldots, u_r\}$, and if $\mu = u_1\wedge\cdots\wedge u_r$, then $k\omega = k\mu$ implies that $\ker(\varphi_\omega) = \ker(\varphi_\mu)$, so $W = U$. We may therefore view $\mathrm{Gr}(r, V)$ as a subset of $\mathbf{P}(\bigwedge^r(V))$. By fixing a basis for V, we get a basis for $\bigwedge^r(V)$, and then we can identify $\mathbf{P}(\bigwedge^r(V))$ with $\mathbf{P}^N$ (with $N = \dim(\bigwedge^r(V)) - 1$); the coordinates of a point $W \in \mathrm{Gr}(r, V)$ in $\mathbf{P}^N$ are called the Plücker coordinates of W. Moreover, if we represent φ_ω as a matrix using a basis for V and the associated basis for $\bigwedge^{r+1}(V)$, we see that the entries of φ_ω consist of zeros and the Plücker coordinates of W up to sign.

Example 3.1 Let V have basis $\{v_1, v_2, v_3, v_4\}$. If W is a subspace of V of dimension 2, then take a basis of W consisting of $w_1 = a_1v_1+a_2v_2+a_3v_3+a_4v_4$ and $w_2 = b_1v_1+b_2v_2+b_3v_3+b_4v_4$. Then

$$\begin{aligned} w_1 \wedge w_2 &= (a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4) \wedge (b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4) \\ &= (a_1b_2 - a_2b_1)v_1 \wedge v_2 + (a_1b_3 - a_3b_1)v_1 \wedge v_3 + (a_1b_4 - a_4b_1)v_1 \wedge v_4 \\ &\quad +(a_2b_3 - a_3b_2)v_2 \wedge v_3 + (a_2b_4 - a_4b_2)v_2 \wedge v_4 + (a_3b_4 - a_4b_3)v_3 \wedge v_4. \end{aligned}$$

Therefore, the Plücker coordinates of $W \in \mathrm{Gr}(2, V)$ are

$$a_1b_2 - a_2b_1,\ a_1b_3 - a_3b_1,\ a_1b_4 - a_4b_1,\ a_2b_3 - a_3b_2,\ a_2b_4 - a_4b_2,\ a_3b_4 - a_4b_3.$$

If $\omega = w_1 \wedge w_2 = \sum_{i<j} x_{ij} v_i \wedge v_j$, then the matrix for φ_ω is

$$\begin{pmatrix} a_2b_3 - a_3b_2 & -(a_1b_3 - a_3b_1) & a_1b_2 - a_2b_1 & 0 \\ a_2b_4 - a_4b_2 & -(a_1b_4 - a_4b_1) & 0 & a_1b_2 - a_2b_1 \\ a_3b_4 - a_4b_3 & 0 & -(a_1b_4 - a_4b_1) & a_1b_3 - a_3b_1 \\ 0 & a_3b_4 - a_4b_3 & -(a_2b_4 - a_4b_2) & a_2b_3 - a_3b_2 \end{pmatrix}.$$

In the special case of $r = 2$ and $\dim(V) = 4$, a calculation shows that $\omega \in \bigwedge^2(V)$ can be written in the form $\omega = w_1 \wedge w_2$ for some $w_i \in V$ if and only if $\omega \wedge \omega = 0$. Writing $\omega = \sum_{i<j\le 4} x_{ij} v_i \wedge v_j$, the condition $\omega\wedge\omega = 0$ gives the equation $2(x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23})v_1 \wedge v_2 \wedge v_3 \wedge v_4 = 0$. Therefore, as $\mathrm{char}(k) \neq 2$, $\mathrm{Gr}(2, V)$ is the zero set of $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$ in $\mathbf{P}\left(\bigwedge^2(V)\right) = \mathbf{P}^5$, and so $\mathrm{Gr}(2, V)$ is a projective variety. It is easy to see that $\mathrm{Gr}(1, V) = \mathbf{P}(V)$ and $\mathrm{Gr}(3, V) = \mathbf{P}\left(\bigwedge^3(V)\right)$, so these grassmannians are also projective varieties.

We now verify, for any r and V, that $\mathrm{Gr}(r, V)$ is a closed subset of $\mathbf{P}(\bigwedge^r(V))$, and so $\mathrm{Gr}(r, V)$ is a projective variety. In order to do this, we need to characterize the *decomposable vectors* $w_1 \wedge \cdots \wedge w_r \in \bigwedge^r(V)$. We will say that $v \in V$ divides $\omega \in \bigwedge^r(V)$ if $\omega = v \wedge \mu$ for some $\mu \in \bigwedge^{r-1}(V)$. We note that v divides ω if and only if $v \wedge \omega = 0$ in $\bigwedge^{r+1}(V)$. To verify this, one direction is clear, so suppose that $v\wedge\omega = 0$. If we choose a basis for V that includes v, then writing ω in the corresponding basis of $\bigwedge^r(V)$, we see that $v \wedge \omega$ is a sum of terms, each either has two v's, or is one of the basis elements of $\bigwedge^{r+1}(V)$. These basis elements only occur once, and so if $v \wedge \omega = 0$, then the only terms in ω that occur each contain a v. Thus, v divides ω. Now, for a given $\omega \in \bigwedge^r(V)$, if $w_1, \dots, w_t$ are linearly independent elements of $\ker(\varphi_\omega)$, then we have $\omega = w_1 \wedge \cdots \wedge w_t \wedge \mu$ for some μ by the preceding line. This forces $t \le r$. Consequently, $\dim(\ker(\varphi_\omega)) \le r$, so $\operatorname{rank}(\varphi_\omega) \ge n - r$. Moreover, if $\omega = w_1 \wedge \cdots \wedge w_r$, then we have $\operatorname{rank}(\varphi_\omega) = n - r$ since the kernel of φ_ω is spanned by the w_i. Thus, $\omega \in \bigwedge^r(V)$ is decomposable if and only if $\operatorname{rank}(\varphi_\omega) \le n - r$. In other words, a point $k\omega \in \mathbf{P}(\bigwedge^r(V))$ lies in $\operatorname{Gr}(r, V)$ if and only if $\operatorname{rank}(\varphi(\omega)) \le n - r$. This is a polynomial condition on the Plücker coordinates of ω, which proves that $\operatorname{Gr}(r, V)$ is a closed subset of $\mathbf{P}(\bigwedge^r(V))$. To be more explicit, given a basis $\{v_1, \dots, v_n\}$ of V, we get the corresponding basis for $\bigwedge^r(V)$, and an isomorphism between $\mathbf{P}(\bigwedge^r(V))$ and $\mathbf{P}^N$ (where $N = \dim(\bigwedge^r(V)) - 1$). With this, if $\omega = \sum \alpha_{i_1 \cdots i_r} v_{i_1} \wedge \cdots \wedge v_{i_r}$, then $\varphi_\omega(v_j)$ is zero or is a multiple of one of the standard basis elements of $\bigwedge^{r+1}(V)$, and the multiple is, up to sign, one of the coefficients of ω. Therefore, the matrix representing φ_ω is a matrix each of whose rows consists of zeros together with coefficients of ω (up to sign). The condition $\operatorname{rank}(\varphi_\omega) \le n - r$ is then a polynomial condition on the coefficients of ω, i.e., a polynomial condition on the coordinates in $\mathbf{P}(\bigwedge^r(V))$ of ω.

Example 3.2 If we continue our example of $r = 2$ and $\dim(V) = 4$ from above, if $\omega = \sum_{i<j} x_{ij} v_i \wedge v_j \in \mathbf{P}\left(\bigwedge^2(V)\right)$, then the matrix representation of φ_ω with respect to the basis $\{v_1 \wedge v_2 \wedge v_3, v_1 \wedge v_2 \wedge v_4, v_1 \wedge v_3 \wedge v_4, v_2 \wedge v_3 \wedge v_4\}$ is

$$\varphi_\omega = \begin{pmatrix} x_{23} & -x_{13} & x_{12} & 0 \\ x_{24} & -x_{14} & 0 & x_{12} \\ x_{34} & 0 & -x_{14} & x_{13} \\ 0 & x_{34} & -x_{24} & x_{23} \end{pmatrix}.$$

Then $\omega \in \operatorname{Gr}(r, V)$ if and only if $\operatorname{rank}(\varphi_\omega) \le 2$, which is true if and only if all 3×3 minors of φ_ω are zero.

Let us now consider a nice open cover of $\operatorname{Gr}(r, V)$. Let W be a subspace of V of dimension $n - r$, and set

$$\mathcal{U}_W = \{U \in \operatorname{Gr}(r, V) : U \oplus W = V\}.$$

Clearly $\operatorname{Gr}(r, V)$ is covered by the various $\mathcal{U}_W$. We wish to show that each $\mathcal{U}_W$ is an open subset of $\operatorname{Gr}(r, V)$ and that $\mathcal{U}_W \cong \mathbf{A}^{r(n-r)}$ as varieties. Let $\omega \in \bigwedge^{n-r}(V)$ correspond to W. We may view ω as a linear map $\bigwedge^r(V) \to \bigwedge^n(V) \cong k$ by $\mu \mapsto \omega \wedge \mu$. Therefore, we can think of ω as a linear form on $\mathbf{P}(\bigwedge^r(V))$. If $U \in \operatorname{Gr}(r, V)$ corresponds to $\mu \in \bigwedge^r(V)$, then $\omega \wedge \mu \ne 0$ if and only if $W \cap U = 0$, if and only if $U \oplus W = V$. In other words, $\mathcal{U}_W$ is the complement of the zero set of ω in $\operatorname{Gr}(r, V)$. So, $\mathcal{U}_W$ is an open subset of $\operatorname{Gr}(r, V)$. To show that $\mathcal{U}_W$ is affine, let U_0 be a fixed complement to W. We give a correspondence between $\mathcal{U}_W$ and $\hom(U_0, W)$ as follows. If $f \in \hom(U_0, W)$, then let U_f be the graph of f. This is a subset of $U_0 \oplus W = V$, and it consists of all elements of the form $(x, f(x))$ for $x \in U_0$. This is clearly a subspace of V. Moreover, $U_f \cap W = 0$ is obvious. Finally, if $\{u_1, \dots, u_r\}$

is a basis for U_0, then $(u_1, f(u_1)), \ldots, (u_r, f(u_r))$ is an independent set in U_f, proving that $\dim U_f = r$. Thus, $U_f \in \mathcal{U}_W$. The map $f \mapsto U_f$ is injective since f is determined from its graph. For the reverse correspondence, let $U \in \mathcal{U}_W$. For $u \in U_0$, write $u = x + y$ with $x \in U$ and $y \in W$. Define f_U by $f_U(u) = -y$. This is well defined, and since $x = u + f(u)$ for all $x \in U$, we have U is the graph of f_U. (Note that f_U is linear since $-f_U$ is an inclusion followed by a projection.) So, our correspondence is a bijection. We will be done by showing that this correspondence is an isomorphism of varieties. Given the basis $\{w_{r+1}, \ldots, w_n\}$ for W, extend it to a basis $\{w_1, \ldots, w_n\}$ of V by adding to it a basis of U_0. Given $U \in \mathrm{Gr}(r, V)$, writing a basis for U in terms of this basis for V, we obtain an $r \times n$ matrix of rank r, whose row space is U. By reducing this matrix, we see that U corresponds to an $r \times n$ matrix of the form $(I_r|A)$, where A is a uniquely determined $r \times (n-r)$ matrix. We can see that this is the case since if $\{u_1, \ldots, u_r\}$ is a basis for U, then we write $u_i = x_i + f(x_i)$, if $f \in \hom(U_0, W)$ is the corresponding map. This equation forces the x_i to be independent. If we write out the x_i (and the $f(x_i)$) in terms of the w_i, then we see that the "left half" of our matrix is invertible, so it can be reduced to I_r. The matrix representation of f is then A. This gives us a new basis for U, which can be written as $u_i = w_i + f(w_i) = w_i + \sum_{j>r} a_{ij} w_j$. The coordinates of $u_1 \wedge \cdots \wedge u_r$ are then polynomials in the a_{ij}, which means the map $\hom(U_0, W) \to \mathcal{U}_W$ is a polynomial map. In fact, the coordinates of $u_1 \wedge \cdots \wedge u_r$ are the $r \times r$ minors of the matrix $(I_r|A)$, as a short calculation demonstrates. For the inverse map, given $u_1 \wedge \cdots \wedge u_r$ written out in its coordinates, the entries of the corresponding matrix A are up to sign various coordinates of $u_1 \wedge \cdots \wedge u_r$. Therefore, the inverse map is a polynomial map, so we get $\mathcal{U}_W \cong \hom_k(U_0, W) \cong \mathbf{A}^{r(n-r)}$ as desired. As a consequence, we see that, as a variety, the dimension of $\mathrm{Gr}(r, V)$ is $r(n-r)$.

Example 3.3 The group $\mathrm{Gl}(V)$ acts on the Grassmannian $\mathrm{Gr}(r, V)$ for any $r \leq \dim(V)$. Note that this action is transitive. If q is a quadratic form on V, then the orthogonal group $\mathrm{O}(V, q)$ acts on $\mathrm{Gr}(r, V)$ also. If W is a totally isotropic subspace of V of dimension r (so $r \leq \frac{1}{2}\dim(V)$), then Witt's theorem shows that the orbit of W under the action of $\mathrm{O}(V, q)$ is the set of all totally isotropic subspaces of V of dimension r. We will write $I_r(V, q)$ for the set of totally isotropic subspaces of (V, q) of dimension r. We prove that $I_r(V, q)$ is a closed subvariety of $\mathrm{Gr}(r, V)$. To do this we use the open cover of $\mathrm{Gr}(r, V)$ and note that a subset of a topological space is closed if its intersection with every set in an open cover is closed in the given open set. So, let W be a subspace of V of dimension $n - r$, and consider the open affine subset $\mathcal{U}_W \cong \hom(U_0, W)$, where U_0 is a fixed complement to W. Let $U \in \mathcal{U}_W$, and associate U with $f \in \hom(U_0, W)$. By viewing elements of V as column matrices, we associate a matrix A to f. With respect to a basis of V, let q be represented by the symmetric matrix B, and let b be the associated symmetric bilinear form on V. Write B in block matrix form

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where the top left is an $r \times r$ block and the bottom right is an $(n-r) \times (n-r)$ block.

We wish to show that the condition U is totally isotropic with respect to B is a polynomial condition on the Plücker coordinates of U. We have $U = \{(z, f(z) : z \in U_0\}$, the graph of f. The condition $b(U,U) = 0$ then, in matrix form, becomes

$$b\left((x, Ax), (y, Ay)\right) = \begin{pmatrix} x^t & (Ax)^t \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} y \\ Ay \end{pmatrix} = 0,$$

where x, y are arbitrary elements of U_0. Multiplying this out, we see that $b(U,U) = 0$ if and only if

$$x^t(B_1 + A^T B_3 + B_2 A + A^T B_4 A)y = 0$$

for all r-tuples x, y. This forces $B_1 + A^T B_3 + B_2 A + A^T B_4 A = 0$, which give polynomial conditions in the coefficients of A. These coefficients are, up to sign, the Plücker coordinates of U. Therefore, $I_r(V, q)$ is the zero set of these polynomial equations, so $I_r(V, q)$ is a closed subset of $\mathrm{Gr}(r, V)$.

4 Flag Varieties

We now consider flag varieties. Let $1 \leq n_1 < n_2 < \cdots < n_r \leq n$ be a sequence of integers. The flag variety $\mathcal{F}(V; n_1, \ldots, n_r)$ is defined by

$$\mathcal{F}(V; n_1, \ldots, n_r) = \{0 \subseteq V_1 \subseteq \cdots \subseteq V_r \subseteq V : \dim(V_i) = n_i\}.$$

There is an obvious embedding $\mathcal{F}(V; n_1, \ldots, n_r) \to \mathrm{Gr}(n_1, V) \times \cdots \times \mathrm{Gr}(n_r, V)$. We claim that the image is a closed subset of this product, which will prove that $\mathcal{F}(V; n_1, \ldots, n_r)$ is a projective variety. We first note to prove that, for $i < j$, if π_{ij} is the projection of $\mathrm{Gr}(n_1, V) \times \cdots \times \mathrm{Gr}(n_r, V)$ onto $\mathrm{Gr}(n_i, V) \times \mathrm{Gr}(n_j, V)$, then $\mathcal{F}(V; n_1, \ldots, n_r)$ is the intersection of all $\pi_{ij}^{-1}(\mathcal{F}(V; n_1, n_j)$. So, it is sufficient to prove that, for $r < s$, the flag variety $\mathcal{F}(V; r, s)$ is closed in $\mathrm{Gr}(r, V) \times \mathrm{Gr}(s, V)$. To verify this, let $\{v_1, \ldots, v_n\}$ be a basis for V and let $(U, W) \in \mathrm{Gr}(r, V) \times \mathrm{Gr}(s, V)$. Let $\{u_1, \ldots, u_r\}$ and $\{w_1, \ldots, w_s\}$ be bases for U and W, respectively. Set $\mu = u_1 \wedge \cdots \wedge u_r$ and $\omega = w_1 \wedge \cdots \wedge w_s$. We have the maps φ_μ and φ_ω, which give the map $\varphi_\mu \oplus \varphi_\omega : V \to \bigwedge^{r+1}(V) \oplus \bigwedge^{s+1}(V)$. It is easy to see that $\ker\left(\varphi_\mu \oplus \varphi_\omega\right) = U \cap W$ since $\ker(\varphi_\mu) = U$ and $\ker(\varphi_\omega) = W$. Consequently,

$$\begin{aligned} \mathrm{rank}(\varphi_\mu \oplus \varphi_\omega) &= \dim(V) - \dim\left(\ker(\varphi_\mu \oplus \varphi_\omega)\right) \\ &= \dim(V) - \dim(U \cap W) \\ &\geq \dim(V) - \dim(U). \end{aligned}$$

So, this tells us that $U \subseteq W$ if and only if $\mathrm{rank}(\varphi_\mu \oplus \varphi_\omega) \leq \dim(V) - \dim(U) = n - r$. By representing $\varphi_\mu \oplus \varphi_\omega$ by a matrix with respect to the basis $\{v_1, \ldots, v_n\}$, we see that the entries of this matrix are, up to sign, the coefficients of μ and ω. Therefore, this rank condition gives polynomial conditions on the coefficients of φ_μ and φ_ω. Thus, $\mathcal{F}(V;r,s)$ is the zero set of these polynomials, so it is a closed subset of $\mathrm{Gr}(r,V)\times\mathrm{Gr}(s,V)$.

Example 4.1 Let $\{v_1,v_2,v_3\}$ be a basis for V. Let $\dim(U)=1$ and $\dim(W)=2$; say $u=c_1v_1+c_2v_2+c_3v_3$ is a basis for U and $w_1=a_1v_1+a_2v_2+a_3v_3$ and $w_2=b_1v_1+b_2v_2+b_3v_3$ is a basis for W. Then

$$w_1\wedge w_2=(a_1b_2-a_2b_1)v_1\wedge v_2+(a_1b_3-a_3b_1)v_1\wedge v_3+(a_2b_3-a_3b_2)v_2\wedge v_3.$$

Moreover, we see that

$$u\wedge w_1\wedge w_2=\left(c_3\left(a_1b_2-a_2b_1\right)-c_2\left(a_1b_3-a_3b_1\right)+c_1\left(a_2b_3-a_3b_2\right)\right)v_1\wedge v_2\wedge v_3.$$

From this we see that $U\subseteq W$ if and only if $c_3\left(a_1b_2-a_2b_1\right)-c_2\left(a_1b_3-a_3b_1\right)+c_1\left(a_2b_3-a_3b_2\right)=0$; since the Plücker coordinates of W are c_1,c_2,c_3 and the Plücker coordinates of W are $a_1b_2-a_2b_1, a_1b_3-a_3b_1, a_2b_3-a_3b_2$, this is a polynomial equation in these coordinates, which shows that, by viewing $\mathrm{Gr}(1,V)\times\mathrm{Gr}(2,V)\subseteq\mathbf{P}(V)\times\mathbf{P}(V)$, the flag variety $\mathcal{F}(V;1,2)$ is the zero set of the polynomial $x_3y_1-x_2y_2+x_1y_3=0$, where we view the coordinates of the first $\mathbf{P}(V)$ as x_1,x_2,x_3 and the second $\mathbf{P}(V)$ as y_1,y_2,y_3.

We now discuss the connection between algebraic groups and grassmannian varieties. We will state and use a number of facts from the theory of algebraic groups, all of whom can be found in [4]. For one such fact, if H is a closed subgroup of G, then there is a variety structure on the set G/H of cosets of H in G such that G/H, together with the canonical projection $\pi:G\to G/H$ (which is then a morphism of varieties) satisfies the following universal mapping property: If $\varphi:G\to V$ is a morphism of algebraic varieties such that each nonempty fibre $\varphi^{-1}(v)$ is a union of cosets of H, then there is a unique morphism $\sigma:G/H\to V$ of varieties such that $\sigma\circ\pi=\varphi$. If G acts on a variety X and H is the stabilizer of some point x, then there is a 1–1 correspondence $G/H\leftrightarrow\mathcal{O}(x)$, and the variety structure of $\mathcal{O}(x)$ then gives rise to a variety structure on G/H so that $G/H\cong\mathcal{O}(x)$ as varieties. In fact, this is the way any quotient structure on G/H arises, because G acts on the variety G/H, and H is the stabilizer of the point eH. The varieties G/H are called *homogeneous spaces* for G.

Recall that if G is an algebraic group, then a *Borel subgroup* B of G is a maximal connected solvable subgroup of G. It is necessarily a closed subgroup of G. All Borel subgroups are conjugate (and all conjugates of a Borel is another Borel), and the quotient variety G/B is a complete (i.e., projective) variety. We can give another characterization of Borel subgroups: a closed connected subgroup H of G is a Borel subgroup of G if and only if H contains no proper subgroups H' with G/H' a projective variety. To see why this is true, we invoke Borel's fixed point theorem [4, Thm. 21.2]: if a connected solvable algebraic group G acts on a nonempty projective variety X, then G has a fixed point in X; that is, there is an $x\in X$ with $gx=x$ for all $g\in G$. To see how this gives our characterization of

Borels, suppose that H' is a closed subgroup of G for which G/H' is projective. If B is a Borel in G, then B acts on G/H' since G acts on G/H'. By Borel's fixed point theorem, there is a $g \in G$ with $b(gH') = gH'$ for all $b \in B$. From this, it follows that $B \subseteq gH'g^{-1}$. Then H' contains the group $g^{-1}Bg$, so, as $g^{-1}Bg$ is a Borel, the hypothesis on H' shows that $g^{-1}Bg = H'$, so H' is Borel. In fact, what we have proven is that if P is a closed subgroup of G for which G/P is a projective variety, then P contains a Borel subgroup of G. Such subgroups of G are called *parabolic*. The converse is also true: If P is a closed subgroup that contains a Borel B, then G/P is a projective variety. For, the canonical map $G/B \to G/P$ is a surjective morphism, and since G/B is complete, its image is complete. This image is G/P, so G/P is complete. Thus, G/P is projective.

We consider the case on $G = \mathrm{Gl}(n,k)$, and we claim that the subgroup $B = T(n,k)$ is a Borel subgroup. We have noted that $T(n,k)$ is closed, solvable, and connected. It is a Borel of $\mathrm{Gl}(n,k)$ by an application of the Lie-Kolchin theorem: If H is a connected solvable subgroup of $\mathrm{Gl}(n,k)$, then H is conjugate to a subgroup of $T(n,k)$. Thus, $T(n,k)$ is maximal among connected solvable subgroups of $\mathrm{Gl}(n,k)$, so $T(n,k)$ a Borel subgroup of $\mathrm{Gl}(n,k)$.

We now relate $T(n,k)$ to (complete) flags. The group $\mathrm{Gl}(V)$ acts on the product $\mathrm{Gr}(1,V) \times \cdots \times \mathrm{Gr}(n,V)$ in the obvious way: $g(V_1,\ldots,V_n) = (g(V_1),\ldots,g(V_n))$. Let $\mathcal{F}(V;1,\ldots,n)$ be the set of complete flags in V. Then $\mathrm{Gl}(V)$ acts transitively on $\mathcal{F}(V;1,\ldots,n)$. To see this, let $G = \{V_i\}$ be a complete flag, and choose a basis $\{v_1,\ldots,v_n\}$ of V such that $\{v_1,\ldots,v_i\}$ is a basis for V_i, for each i. Then if $\{U_i\}$ is another complete flag, choose a basis $\{u_1,\ldots,u_n\}$ of V such that $\{u_1,\ldots,u_i\}$ is a basis of U_i. If we define g by $g(v_i) = u_i$, then $g \in \mathrm{Gl}(V)$ and $g(F) = \{U_i\}$. So, $\mathrm{Gl}(V)$ does act transitively on $\mathcal{F}(V;1,\ldots,n)$. Moreover, by identifying $\mathrm{Gl}(V) = \mathrm{Gl}(n,k)$ via the basis coming from F, it is clear that the stabilizer of G is $T(n,k)$. Consequently, the Borel subgroups of $\mathrm{Gl}(V)$ are the stabilizers of complete flags of V.

We now identify the parabolic subgroups of $\mathrm{Gl}(n,k)$. Let P be a parabolic subgroup of $\mathrm{Gl}(n,k)$. Since P contains a Borel subgroup of $\mathrm{Gl}(n,k)$, by conjugating by an appropriate element, we may suppose that P contains $T(n,k)$. Then a matrix calculation shows that P is in block matrix form

$$P = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{pmatrix}_{n_1,\ldots,n_r},$$

where P is made up of r blocks. Let $\mathcal{F}(V;n_1,\ldots,n_r)$ be the flag variety with subscripts the same as for P. We have seen *HAVE WE?* that $\mathrm{Gl}(V)$ acts transitively on $\mathcal{F}(V;n_1,\ldots,n_r)$. Let $\{e_1,\ldots,e_n\}$ be a basis of V. Set F to be the flag $\{V_i\}$, where V_i is the span of the first n_i of the e's. It is clear that P stabilizes F, and more than that, P is the stabilizer of F. Therefore, P is a stabilizer subgroup of a flag. Conversely, let F be a flag in $\mathcal{F}(V;n_1,\ldots,n_r)$, and let it be a subflag of a complete flag G. Let P be the stabilizer of F, and let B be the stabilizer of G. Then, as we pointed out above, B is a Borel subgroup of $\mathrm{Gl}(n,k)$. Furthermore, we have the obvious inclusion $B \subseteq P$, and since P is closed (it is the stabilizer of a point), so P is a parabolic subgroup of $\mathrm{Gl}(n,k)$. Therefore, the parabolic subgroups of $\mathrm{Gl}(V)$ are precisely the subgroups that stabilize some flag.

We can then use this connection between parabolic subgroups and flags to give another description of $\mathcal{F}(V;n_1,\ldots,n_r)$. Let P be a parabolic subgroup of $\mathrm{Gl}(V)$. Then P is the stabilizer of some flag $F \in \mathcal{F}(V;n_1,\ldots,n_r)$. Moreover, $\mathcal{F}(V;n_1,\ldots,n_r)$ is the orbit of F under $\mathrm{Gl}(V)$, and so $\mathcal{F}(V;n_1,\ldots,n_r) \cong \mathrm{Gl}(V)/P$ as a variety. Therefore, every flag variety is a homogeneous space of $\mathrm{Gl}(n,k)$ for some n.

5 Groups of type D_n

We now consider algebraic groups of type D_n. Let (V,q) be a quadratic space of dimension $2n$ over k, and let $G = \mathrm{O}^+(V,q) = \mathrm{SO}(V,q)$, the connected component of the orthogonal group $\mathrm{O}(V,q)$. This consists of the isometries of q which have determinant 1. Recall that an isometry has determinant ± 1, and that $\mathrm{O}^-(V,q)$ is the set of isometries of determinant -1. If k is algebraically closed, then q is hyperbolic, since n is even. Since $G \subseteq \mathrm{Gl}(V)$, we see that G acts on the flag variety $\mathcal{F}(V;n_1,\ldots,n_r)$. Let F be a flag $\{V_i\}$ consisting of totally isotropic subspaces of V relative to q (so, $n_r \le n$). Recall the theorem of Witt that says any isometry between two subspaces of V extends to an isometry of V. From this we see that any two totally isotropic subspaces of the same dimension differ by an isometry of V, since any vector space isomorphism between them is an isometry. Moreover, we claim that the orbit of F under G is the collection $\mathcal{I}(V;n_1,\ldots,n_r)$ of *isotropic flags*; that is, the collection of flags $\{U_i\}$ such that each U_i is a totally isotropic subspace of V. To prove this, if $F = \{V_1 \subseteq \cdots \subseteq V_r\}$ and $F' = \{V_1' \subseteq \cdots \subseteq V_r'\}$ are two isotropic flags, then we can produce a linear isomorphism g with $g(V_i) = V_i'$. Viewing $g : V_r \to V_r'$, it is an isometry since these are totally isotropic. So, Witt's theorem gives a $g \in \mathrm{O}(V,q)$ with $g(F) = F'$. Conversely, it is clear that any isometry sends totally isotropic subspaces to totally isotropic subspaces, so the orbit of an isotropic flag is contained in $\mathcal{I}(V;n_1,\ldots,n_r)$. Thus, $\mathcal{I}(V;n_1,\ldots,n_r)$ is the orbit of any isotropic flag under the action of $\mathrm{O}(V,q)$. Moreover, we have proven that $I_r(V,q) = \mathcal{I}(V;r)$ is a closed subvariety of $\mathrm{Gr}(r,V)$, and the proof that $\mathcal{F}(V;n_1,\ldots,n_r)$ is a closed subvariety of $\mathrm{Gr}(n_1,V) \times \cdots \times \mathrm{Gr}(n_r,V)$ carries over to prove that $\mathcal{I}(V;n_1,\ldots,n_r)$ is a closed subvariety of $\mathrm{Gr}(n_1,V) \times \cdots \times \mathrm{Gr}(n_r,V)$, so $\mathcal{I}(V;n_1,\ldots,n_r)$ is a projective (i.e., complete) variety.

We now assume that $\dim(V) = 2n$ is even, so $\mathrm{O}(V,q)$ is of "type" D_n. Recall that $I_r(V,q)$ is the set of all r-dimensional totally isotropic subspaces of V relative to q. Then $\mathrm{O}(V,q)$ acts on $I_r(V,q)$ via $W \mapsto g(W)$; this clearly sends totally isotropic subspaces to totally isotropic subspaces since $g \in \mathrm{O}(V,q)$ is an isometry of q. Moreover, $G := \mathrm{O}^+(V,q)$ acts on $I_r(V,q)$ in the same way. First consider the case $r < n$. We claim that G acts transitively on $I_r(V,q)$. To prove this, let U and U' be totally isotropic subspaces of (V,q) of dimension r. We are claiming that there is an element $g \in \mathrm{O}^+(V,q)$ with $g(U) = U'$. We know, by

Witt's theorem, that there is a $g \in \mathrm{O}(V,q)$ with $g(U) = U'$. Suppose that $\det(g) = -1$. Let $u \in U^\perp$ be anisotropic. Such an element exists because if $W \supset U$ is a maximal totally isotropic subspace of V, then $W = W^\perp \subset U^\perp$ and we can take $u \in U^\perp - W^\perp$, since if u is isotropic, then $W + ku$ is a totally isotropic subspace properly containing W, an impossibility. Consider the *reflection* τ_u about u. This is defined by

$$\tau_u(v) = v - \frac{2b(v,u)}{b(u,u)}u.$$

This is an isometry of (V,q) of determinant -1, and $\tau_u|_U = \mathrm{id}$ since $u \in U^\perp$. Therefore $\det(g\tau_u) = 1$ and $g\tau_u(U) = U'$, proving that there is an element of $\mathrm{O}^+(V,q)$ that sends U to U'. Therefore, $\mathrm{O}^+(V,q)$ acts transitively on the projective variety $I_r(V,q)$. However, if $r = n$, then we claim that $I_r(V,q)$ has two $\mathrm{O}(V,q)$-orbits. To see this we first prove that the stabilizer of any $U \in I_n(V,q)$ lies in $\mathrm{O}^+(V,q)$. To see this, let $\{u_1, \dots, u_n\}$ be a basis for U, and let W be a complementary totally isotropic subspace of V; that is, $U \oplus W = V$, and we can choose a basis $\{w_1, \dots, w_n\}$ of W so that $b(u_i, w_j) = \delta_{ij}$ for all i, j. If we represent q with a matrix relative to this basis, then the matrix is $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Any element in the stabilizer of U is of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with A, B, and $D \in M_n(k)$. For this matrix to be in $\mathrm{O}(V,q)$, we must have

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^T \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Multiplying gives the equation

$$\begin{pmatrix} 0 & A^T D \\ D^T A & D^T B + B^T D \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

This forces $D = A^{-T}$, so the determinant of $\begin{pmatrix} A & B \\ 0 & A^{-T} \end{pmatrix}$ is 1, proving that the stabilizer is a subgroup of $\mathrm{O}^+(V,q)$. To see that this gives us two $\mathrm{O}(V,q)$-orbits, we see that if $U \in I_n(V,q)$ and $g \in \mathrm{O}^-(V,q)$, so $\det(g) = -1$, and if $W = g(U)$, then if $W = h(U)$ for some $h \in \mathrm{O}^+(V,q)$, then $U = h^{-1}g(U)$, which means $h^{-1}g \in \mathrm{O}^+(V,q)$, a contradiction. So, if U is any element of $I_n(V,q)$, the two $\mathrm{O}^+(V,q)$-orbits are $\mathcal{O}(U)$ and $\mathcal{O}(g(U))$, where g is any element of $\mathrm{O}^-(V,q)$.

We now relate isotropic flags to algebraic groups of type D_n in an analogous manner to the way we related flags to algebraic groups of type A_n. Let $X = I(V; 1, \dots, n)$ be the projective variety of full isotropic flags. Take $F = (V_1, \dots, V_n)$ be a full isotropic flag, and let $G = (V_1, \dots, V_n, V_{n-1}^\perp, \dots, V_1^\perp)$, a full flag in $\mathcal{F}(V; 1, \dots, 2n)$. If $g \in \mathrm{O}^+(V,q)$ stabilizes F, then $g(V_i) = V_i$, which forces $g(V_i^\perp) = V_i^\perp$ since g is an isometry. So, g also stabilizes G. Therefore, the stabilizer G_F of F in $\mathrm{O}^+(V,q)$ is contained in the stabilizer in $\mathrm{Gl}(V)$ of G.

This latter stabilizer is a conjugate of $T(2n,k)$, a solvable group. Therefore, G_F is a solvable group. By our argument for the $\mathrm{O}^+(V,q)$ action on $I_n(V,q)$ above, we can see that there are two $\mathrm{O}^+(V,q)$-orbits of X, say X_1 and X_2. Each of these orbits has the same dimension, so each are closed in X. Therefore each is a projective variety. We have $\mathrm{O}^+(V,q)/G_F \cong X_i$ as varieties. Therefore, G_F is a parabolic subgroup of $\mathrm{O}^+(V,q)$. By a theorem of Chevalley, G_F is connected since it is a parabolic. It is also closed, being a stabilizer. Since we know that G_F is solvable, we then see that G_F is a Borel subgroup of $\mathrm{O}^+(V,q)$. (Note that G_F contains a Borel since it is parabolic, and this forces G_F to be a Borel since G_F is solvable.) Finally, because any two Borels of an algebraic group are conjugate, and since it is easy to see that $G_{gF} = gG_Fg^{-1}$, we have the conclusion that

$$\left\{ \begin{array}{c} \text{Borel subgroups} \\ \text{of } \mathrm{O}^+(V,q) \end{array} \right\} = \left\{ \begin{array}{c} \text{Stabilizer subgroups} \\ \text{of full isotropic flags} \end{array} \right\}$$

Moreover, the parabolic subgroups of (V,q) are the stabilizers of isotropic flags in V.

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