Isometries of $\mathbb{R}^n$

In this note we investigate isometries, or rigid motions, of $\mathbb{R}^n$. An *isometry* of $\mathbb{R}^n$ is a bijective mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distance. In other words, for any $u, v \in \mathbb{R}^n$, we have $\|f(u) - f(v)\| = \|u - v\|$. We point out that the composition of two isometries is an isometry. For, if f and g are isometries, then

$$\begin{aligned} \|(f \circ g)(u) - (f \circ g)(v)\| &= \|f(g(u)) - f(g(v))\| \\ &= \|g(u) - g(v)\| = \|u - v\| . \end{aligned}$$

Let $E(n)$ be the set of all isometries of $\mathbb{R}^n$. We have just seen that function composition is an operation on $E(n)$. Associativity holds for this operation since it holds for composition of arbitrary functions. The identity function of $\mathbb{R}^n$ is an isometry, so $E(n)$ has an identity. If $f \in E(n)$, then f^{-1} exists since f is a bijection, and f^{-1} is an isometry since

$$\begin{aligned} \left\|f^{-1}(v) - f^{-1}(w)\right\| &= \left\|f(f^{-1}(v)) - f(f^{-1}(w))\right\| \\ &= \|v - w\| \end{aligned}$$

for any $v, w \in \mathbb{R}^n$. Thus, $E(n)$ is a group under composition.

We now describe four classes of isometries.

Translations. Let $b \in \mathbb{R}^n$. Then translation by b is the mapping given by $f(x) = x + b$. We see easily that f is distance preserving since

$$\|f(u) - f(v)\| = \|(u + b) - (v + b)\| = \|u - v\| .$$

Note that $b = f(0)$, so a translation is determined by what it does to the origin.

Rotations. We describe rotations in $\mathbb{R}^2$. Let θ be a fixed angle. We consider the rotation f of the plane centered at the origin that rotates counterclockwise by an angle θ. A geometric argument shows that

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

when $f(x, y)$ is viewed as a column vector. In other words,

$$f(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta).$$

From this formula we can show that f is an isometry. For simplicity, we let $c = \cos\theta$ and $s = \sin\theta$. Let (x, y) and (x', y') be two points in $\mathbb{R}^2$. Then, by writing $a = x - x'$ and $b = y - y'$,

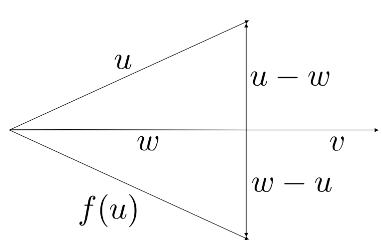
$$\begin{aligned}
\|f(x,y) - f(x',y')\| &= \|(cx + sy, -sx + cy) - (cx' + sy', -sx' + cy')\| \\
&= \|(c(x - x') + s(y - y'), -s(x - x') + c(y - y')\| \\
&= \left((ca + sb)^2 + (-sa + cb)^2\right)^{1/2} \\
&= \left(c^2a^2 + s^2b^2 + 2csab + s^2a^2 + c^2b^2 - 2csab\right)^{1/2} \\
&= \left((c^2 + s^2)a^2 + (c^2 + s^2)b^2\right)^{1/2} \\
&= \left(a^2 + b^2\right)^{1/2} = \|(x, y) - (x', y')\|\,,
\end{aligned}$$

which shows that f is an isometry.

More generally, one can define the rotation centered at a point P of an angle θ. If f is the rotation about the origin of an angle θ and h is translation by P, then one can see that the rotation g centered at b of an angle θ is equal to $h \circ f \circ h^{-1}$. Given this, it then follows that any rotation is an isometry since it is the composition of three isometries.

Reflections. Let L be a line in $\mathbb{R}^n$. Then the reflection f about L is the mapping with the following properties: if $u \in \mathbb{R}^n$, then $f(u)$ is the vector such that (i) $u - f(u)$ is perpendicular to L, and (ii) the distance from $f(u)$ to L is the same as the distance from u to L. You should convince yourself that this is well-defined and that it is an isometry. If $v \in \mathbb{R}^n$ and L is the line through the origin in the direction of v, then the reflection through L is given by the formula

$$f(u) = 2\left(\frac{u \cdot v}{v \cdot v}\right) v - u.$$



This formula is a consequence of the formula for the projection of one vector onto another that one usually sees in a calculus class: we have $w = \alpha v$ for some scalar α, and $(u - w) \cdot v = 0$. This implies that $u \cdot v = w \cdot v = \alpha(v \cdot v)$, so $\alpha = (u \cdot v)/(v \cdot v)$. Finally, $f(u) = w + (w - u) = 2w - u$, which gives the formula above.

For example, if L is the x-axis in $\mathbb{R}^2$, setting $v = (1, 0)$, we get $f(x, y) = (x, -y)$. Also, if L is the y-axis in $\mathbb{R}^2$, then $f(x, y) = (-x, y)$. We show that any reflection f about a line through the origin is an isometry. First, note that $f(u+w) = f(u) + f(w)$ for any $u, w \in \mathbb{R}^n$. Next,

$$\begin{aligned}
\|f(u)\| &= \left\| 2\left(\frac{u \cdot v}{v \cdot v}\right) v - u \right\| \\
&= \left(\left(2\left(\frac{u \cdot v}{v \cdot v}\right) v - u \right) \cdot \left(2\left(\frac{u \cdot v}{v \cdot v}\right) v - u \right) \right)^{1/2} \\
&= \left(4\left(\frac{u \cdot v}{v \cdot v}\right)^2 (v \cdot v) + u \cdot u - 4\left(\frac{u \cdot v}{v \cdot v}\right)(u \cdot v) \right)^{1/2} \\
&= (u \cdot u)^{1/2} = \|u\|^{1/2}.
\end{aligned}$$

Thus,

$$\|f(u) - f(w)\| = \|f(u-w)\| = \|u - w\|$$

from the previous calculation.

Glide Reflections. A glide reflection is the composition of a reflection about a line followed by a translation by a vector on the line. For example, if L is the line $y = x$ in $\mathbb{R}^2$, then the ordinary reflection about L is given by $g(x, y) = (y, x)$. If we let $b = (3, 3)$, a vector on L, then we get a glide reflection by $f(x, y) = g(x, y) + b = (y, x) + (3, 3) = (y + 3, x + 3)$. Any glide reflection is an isometry since it is the composition of two isometries, a reflection and a translation.

We now investigate the structure of isometries. We first consider isometries g with $g(0) = 0$. From the condition $g(0) = 0$ we see for any $u \in \mathbb{R}^n$ that

$$\|g(u)\| = \|g(u) - g(0)\| = \|u - 0\| = \|u\|.$$

In other words, g preserves the length of a vector. Recall that if u and v are vectors, then there is a unique angle θ with $0 \le \theta \le \pi$ such that

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta.$$

This fact is a consequence of the *Cauchy-Schwartz* inequality. A consequence of this is that the dot product is given by $u \cdot v = \|u\|\|v\| \cos\theta$.

Lemma 1. *If g is an isometry of $\mathbb{R}^n$ with $g(0) = 0$, then g is angle preserving. That is, the angle between $g(u)$ and $g(v)$ is the same as the angle between u and v, for all $u, v \in \mathbb{R}^n$.*

Proof. Let g be an isometry. If θ is the angle between two vectors u and v, then $\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta$. If θ' is the angle between $g(u)$ and $g(v)$, then

$$\|g(u) - g(v)\|^2 = \|g(u)\|^2 + \|g(v)\|^2 - 2\|g(u)\|\|g(v)\| \cos\theta'.$$

However, since $\|g(u)\| = \|u\|$ and $\|g(v)\| = \|v\|$, we get

$$\|u - v\|^2 = \|g(u) - g(v)\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta'.$$

This forces $\cos\theta' = \cos\theta$, so $\theta' = \theta$. ∎

Lemma 2. *If g is an isometry g of $\mathbb{R}^n$ with $g(0) = 0$, then g preserves dot products. In other words, $g(u) \cdot g(v) = u \cdot v$ for all $u, v \in \mathbb{R}^n$.*

Proof. If θ is the angle between u and v, then $u \cdot v = \|u\|\|v\|\cos\theta$. Since θ is also the angle between $g(u)$ and $g(v)$ by Lemma 1, we have $g(u) \cdot g(v) = \|g(u)\|\|g(v)\|\cos\theta$. Since $\|g(u)\| = \|u\|$ and $\|g(v)\| = \|v\|$, this yields $g(u) \cdot g(v) = u \cdot v$. ∎

A *linear transformation* of $\mathbb{R}^n$ is a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\varphi(u+v) = \varphi(u)+\varphi(v)$ and $\varphi(\alpha u) = \alpha\varphi(u)$ for all $u, v \in \mathbb{R}^n$ and all scalars $\alpha \in \mathbb{R}$. Recall from linear algebra that, by viewing the elements of $\mathbb{R}^n$ as column vectors, that a linear transformation φ has the form $\varphi(v) = Av$ for some $n \times n$ matrix A.

Proposition 3. *If g is an isometry of $\mathbb{R}^n$ with $g(0) = 0$, then g is a linear transformation.*

Proof. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of $\mathbb{R}^n$. For instance, this can be the standard basis. Let $w_i = g(v_i)$. First of all, we have $\|w_i\| = \|g(v_i)\| = \|v_i\| = 1$ since g is length preserving. Therefore each w_i is a unit vector. Next, since g is angle preserving, the angle between w_i and w_j is equal to the angle between v_i and v_j, which is $\pi/2$. This means that $\{w_1, \ldots, w_n\}$ is also an orthonormal basis of $\mathbb{R}^n$. Recall that if $u = \sum_i \alpha_i v_i$, then the coefficients α_i can be determined by $\alpha_i = u \cdot v_i$. So, we have $\alpha_i = g(u) \cdot g(v_i) = g(u) \cdot w_i$ since g is dot product preserving. This means that $g(u) = \sum_i \alpha_i w_i$ since $\{w_1, \ldots, w_n\}$ is an orthonormal basis. We therefore get that $g(u) = \sum_i (g(u) \cdot v_i)\, w_i$. From this we can see that g is a linear transformation. For, let $u, v \in \mathbb{R}^n$. Then

$$\begin{aligned} g(u+v) &= \sum_i ((u+v) \cdot w_i)\, w_i = \sum_i (u \cdot w_i) w_i + \sum_i (v \cdot w_i) w_i \\ &= g(u) + g(v), \end{aligned}$$

and if γ is any scalar, then

$$\begin{aligned} g(\gamma u) &= \sum_i (\gamma u \cdot w_i) w_i = \sum \gamma (u \cdot w_i) w_i = \gamma \sum_i (u \cdot w_i) w_i \\ &= \gamma g(u). \end{aligned}$$

We have thus proved that g is a linear transformation. ∎

If g is a linear transformation on $\mathbb{R}^n$, viewing the elements of $\mathbb{R}^n$ as column matrices, we can write $g(u) = Au$ for some $n \times n$ matrix A.

We now consider general isometries; that is, we no longer assume that the origin if fixed by the isometry.

Corollary 4. *Let f be an isometry of $\mathbb{R}^n$. Then $f(x) = Ax + b$ for some $n \times n$ matrix A and some $b \in \mathbb{R}^n$.*

Proof. Let $b = f(0)$ and set $g(x) = f(x) - b$. Then g is the composition of f and the translation $x \mapsto x - b$, which is an isometry. So, g is an isometry. Since $g(0) = f(0) - b = b - b = 0$, the previous proposition shows that g is a linear transformation. So, there is a matrix A with $g(x) = Ax$. We then obtain $f(x) = g(x) + b = Ax + b$ as desired. ∎

The matrix A of Corollary 4 is not an arbitrary matrix. We get a restriction on A by knowing that f preserves the dot product. First of all, if $u, v \in \mathbb{R}^n$, then viewing u and v as column matrices, the dot product $u \cdot v$ can be expressed as the matrix product $u \cdot v = u^T v$, where u^T is the transpose of u, and where we identify a 1×1 matrix (a) with the scalar a. From this, we have, for g given by $g(x) = Ax$, that

$$\begin{aligned} g(u) \cdot g(v) &= (Au)^T(Av) = (u^T A^T)(Av) \\ &= u^T A^T A v \\ &= u \cdot v = u^T v. \end{aligned}$$

Since this is true for all $u, v \in \mathbb{R}^n$, a matrix calculation shows that $A^T A = I_n$, the $n \times n$ identity matrix. The set of matrices that satisfy the condition $A^T A = I_n$ is called the *orthogonal group*, and is often denoted $\mathrm{O}_n(\mathbb{R})$. A short argument shows that $\mathrm{O}_n(\mathbb{R})$ is a subgroup of the general linear group $\mathrm{Gl}(n, \mathbb{R})$.

Corollary 5. *If f is an isometry of $\mathbb{R}^n$, then $f(x) = Ax + b$ for some $b \in \mathbb{R}^n$ and some $n \times n$ matrix A with $A^T A = I_n$.*

This corollary shows that any isometry is the composition of a translation with an element of the orthogonal group. A theorem of Dieudonné says that any element of $\mathrm{O}_n(\mathbb{R})$ can be written as a composition of reflections (about lines through the origin). So, any isometry can be obtained from reflections and translations.

Because of the connection between linear transformations and matrices, we get the following connection between $\mathrm{O}_n(\mathbb{R})$ and $E(n)$.

Proposition 6. *Let G be the set of isometries of $\mathbb{R}^n$ that preserve the origin. Then G is a group and $G \cong \mathrm{O}_n(\mathbb{R})$.*

Proof. We leave it to the reader to check that G is a subgroup of $E(n)$. The condition $g(0) = 0$ is necessary to have the identity function in G. We define a map $\sigma : \mathrm{O}_n(\mathbb{R}) \to G$ by $\sigma(A)$ is the isometry $x \mapsto Ax$. In other words, $\sigma(A)(x) = Ax$. We have

$$\begin{aligned} \sigma(AB)(x) &= (AB)x = A(Bx) = \sigma(A)(Bx) \\ &= \sigma(A)(\sigma(B)(x)) \\ &= (\sigma(A)\sigma(B))(x). \end{aligned}$$

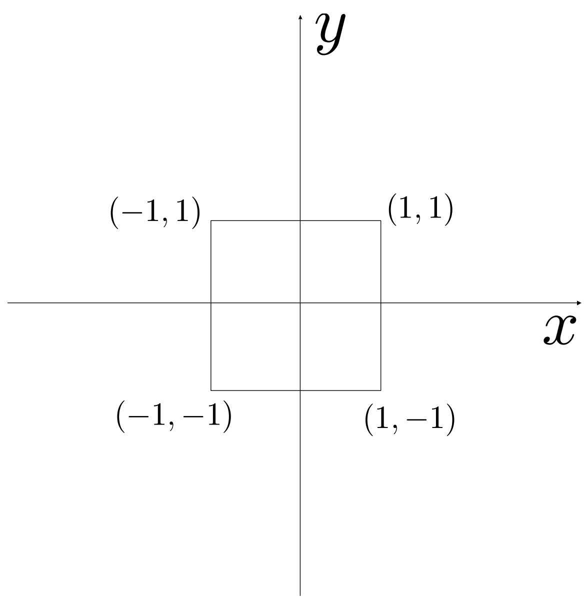
Therefore, $\sigma(AB) = \sigma(A)\sigma(B)$. So, σ is a group homomorphism. If $\sigma(A)$ is the identity function, then $\sigma(A)(x) = x$ for all x. Then $Ax = x$ for all x. But then the matrix A defines the identity linear transformation, so $A = I_n$. So, σ is injective. Finally, if $g \in G$, then $g(x) = Ax$ for some matrix A by Corollary 4; the element $b = 0$ since $b = g(0) = 0$. Now, the argument before the statement of the corollary shows that $A^T A = I_n$, so $A \in \mathrm{O}_n(\mathbb{R})$. This shows that $g = \sigma(A)$, so σ is surjective. Therefore, σ is a group isomorphism. ∎

We can get many interesting groups as subgroups of $E(n)$. The orthogonal group $\mathrm{O}_n(\mathbb{R})$ is one example. For a general class of subgroups, let T be a subset of $\mathbb{R}^n$, and let

$$\mathrm{Sym}(T) = \{f \in E(n) : f(T) = T\}.$$

This set is the *group of symmetries* of T. To explain the notation, $f(T) = \{f(t) : t \in T\}$ is the image of T under f. Thus, the elements of $\mathrm{Sym}(T)$ are those isometries that send T to itself. It is not hard to see that $\mathrm{Sym}(T)$ is a subgroup of $E(n)$, and this will be left to the reader. For example, the *Dihedral group* D_n is the symmetry group of a regular n-gon. By putting the center of the regular n-gon at the origin, we can view D_n as a subgroup of $\mathrm{O}_n(\mathbb{R})$, since the center must be mapped to the center in order for the map to be distance preserving.

Example 7. Consider the group D_4. We think of this as the group of symmetries of the square centered at the origin with vertices $(1,1), (1,-1), (-1,1), (-1,-1)$.



Let σ be the rotation of an angle $\pi/2$ counterclockwise and let τ be the reflection about the x-axis. Viewing σ and τ as matrices, we have

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By multiplying these together in all possible ways, we get that D_4 is isomorphic to the group of matrices

$$\left\{ \begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{array} \right\}.$$

Example 8. Let C be a cube in $\mathbb{R}^3$, and let $\mathrm{Sym}(C)$ be the group of symmetries of C. Considering $\mathrm{Sym}(C)$ as a group of permutations of the eight vertices of C, we can determine the size of $\mathrm{Sym}(C)$. For convenience, we will assume each side of C has length 1. First of all, fixing one vertex v, there are eight choices for the image of v. Given that a choice v' is made, let w be a vertex a distance of 1 from v. There are three vertices a distance of 1 from v'. Then w must map to one of these, so there are three choices for w. So, we have so far a total of $8 \cdot 3 = 24$ possible elements of $\mathrm{Sym}(C)$. We leave it to the reader that an element of $\mathrm{Sym}(C)$ is determined by what it does to v and w, and that all of these 24 possibilities can occur. So, $|\mathrm{Sym}(C)| = 24$. In fact, by considering $\mathrm{Sym}(C)$ as a group of permutations of the four faces of C, one can obtain an isomorphism $\mathrm{Sym}(C) \cong S_4$.