Division Algebras over Local Fields

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Let F be a local field; that is, F is complete with respect to a discrete valuation with finite residue field. In this note we describe all of the F-central division algebras, and in doing so we describe the Brauer group $\mathrm{Br}(F)$. Our approach is purely valuation theoretic and is mostly elementary. In particular, we do not use the completeness assumption except to have Hensel's lemma, except in proving the final result of the note. We recall the statement of Hensel's lemma: if V is a valuation ring of F, then V satisfies Hensel's lemma if for any monic $f(x) \in V[x]$, if $\overline{f(x)}$ factors in $\overline{F}[x]$ as $\overline{f(x)} = \overline{g}(x)\overline{h}(x)$ for some $\overline{g}, \overline{h} \in \overline{F}[x]$ with $\gcd(\overline{g}, \overline{h}) = 1$, then there are $g, h \in V[x]$ with $f = gh$ and $\overline{g(x)} = \overline{g}(x)$ and $\overline{h(x)} = \overline{h}(x)$. If F is complete with respect to a discrete valuation whose valuation ring is V, then it is known that V satisfies Hensel's lemma (see [1, 16.7]). Note that as a consequence of Hensel's lemma, if V is Henselian and $f \in V[x]$ is monic such that $\overline{f}$ has a simple root in $\overline{F}$, then f has a (simple) root in V.

We recall two facts about division algebras over Henselian valued fields: (1) If D is F-central, then D has a valuation extending the (Henselian) valuation on F, and (2) if $\mathcal{L}$ is a subfield of $\overline{D}$ that is a separable extension of $\overline{F}$, then there is a unique up to isomorphism subfield L of D with L/F inertial and $\overline{L} = \mathcal{L}$. The first fact is standard; a proof of a generalization can be found in [4]. We give here a proof of the existence of such inertial lifts. Recall that L/F is *inertial* if $\overline{L}/\overline{F}$ is separable and $[L : F] = [\overline{L} : \overline{F}]$.

Lemma 1 *Let D/F be as above, and let $\mathcal{L} \subseteq \overline{D}$ be a separable extension of $\overline{F}$. Then there is a unique up to isomorphism inertial extension $L \subseteq D$ of F with $\overline{L} = \mathcal{L}$.*

Proof. Since $\mathcal{L}/\overline{F}$ is separable, we may write $\mathcal{L} = \overline{F}(\alpha)$ for some α. Let $p(x) = \min(\overline{F}, \alpha)$ be the minimal polynomial of α over $\overline{F}$. Then p has no repeated roots. We have the factorization $p(x) = (x - \alpha)q(x)$ in $\mathcal{L}[x]$ with $\gcd(x - \alpha, q(x)) = 1$ by the separability of p. Let $a \in D$ be any element with $\overline{a} = \alpha$, and set $K = F(a)$. Then K is Henselian with its unique extension of v, and $\overline{K}$ contains $\overline{F}(\alpha) = \mathcal{L}$. By Hensel's lemma, if $p'(x) \in F[x] \subseteq K[x]$

satisfies $\overline{p'} = p$, there is a root $b \in K$ of p' with $\overline{b} = \alpha$. Let $L = F(b) \subseteq K \subseteq D$. Then

$$[L : F] \leq \deg(p') = \deg(p) = [\mathcal{L} : \overline{F}]$$

and $\alpha = \overline{b} \in \overline{L}$. Thus, $\mathcal{L} \subseteq \overline{L}$. From the fundamental inequality, this forces $\mathcal{L} = \overline{L}$ and $[L : F] = [\overline{L} : \overline{F}]$. Therefore, L/F is an inertial extension with residue field $\mathcal{L}$.

To prove the uniqueness of L, let $L_1$and L_2 be two inertial extensions of F with $\overline{L_1} = \overline{L_2} = \mathcal{L}$. Applying Hensel's lemma to both L_1 and L_2, if $p'(x) \in F[x]$ satisfies $\overline{p'(x)} = p(x)$, there are roots $b_i \in L_i$ of p' with $\overline{b_i} = \alpha$. So, by the dimension count argument of the paragraph above,we get $L_i = F(b_i)$. Moreover, that argument forces p' to be irreducible over F. Thus, $L_1 \cong F[x]/(p') \cong L_2$, so L_1 and L_2 are F-isomorphic. ∎

If $\mathcal{L}/\overline{F}$ is a separable extension, we call an inertial extension L/F with $\overline{L} = \mathcal{L}$ the *inertial lift* of $\mathcal{L}$. We will describe division algebras over F by making heavy use of inertial extensions. For this reason we prove the following fact from commutative valuation theory.

Lemma 2 *Let L/F be an inertial extension. Then L/F is Galois iff $\overline{L}/\overline{F}$ is Galois. When this occurs, the map $\sigma \mapsto \overline{\sigma}$ is a group isomorphism $\mathrm{Gal}(L/F) \cong \mathrm{Gal}(\overline{L}/\overline{F})$, where $\overline{\sigma}$ is defined by $\overline{\sigma}(\overline{u}) = \overline{\sigma(u)}$.*

Proof. Suppose that $\overline{L}/\overline{F}$ is Galois. We may write $\overline{L} = \overline{F}(\alpha)$ for some α. Set $p(x) = \min(\overline{F}, \alpha)$, and let $p'(x) \in F[x]$ be any lift of p. By Hensel's lemma there is a root $a \in L$ of p' with $\overline{a} = \alpha$. Since $\overline{F(a)} \supseteq \overline{F}(\alpha) = \overline{L}$, the fundamental inequality forces $F(a) = L$ since L/F is inertial. Moreover, since p splits over $\overline{L}$ since $\overline{L}/\overline{F}$ is Galois, Hensel's lemma shows that p' splits over L. Thus, L is the splitting field of p' over L, so L/F is Galois. Conversely, suppose that L/F is Galois. If $\overline{L} = \overline{F}(\alpha)$ for some α, let $p = \min(\overline{F}, \alpha)$. Arguing as above, if p' is a lift of p, then p' has a root $a \in L$ with $\overline{a} = \alpha$, and by the fundamental inequality,we get $L = F(a)$. Thus, degree count shows that p' is irreducible over F, so p' splits over L since L/F is Galois. Every root of p' is of the form $\sigma(a)$ for some $\sigma \in \mathrm{Gal}(L/F)$ since L/F is Galois. Since $v \circ \sigma = v$, as both are valuations extending the Henselian valuation $v|_F$, we get $v(\sigma(a)) = 0$, and consequently we see that p splits over $\overline{L}$ since

$$p = \overline{p'} = \overline{\prod(x - \sigma(a))} = \prod \left(x - \overline{\sigma(a)}\right).$$

Thus, $\overline{L}/\overline{F}$ is Galois.

Now assume that L/F is Galois. If $\sigma \in \mathrm{Gal}(L/F)$, then $\overline{\sigma}$ defined as in the statement of the lemma is well defined and is an automorphism of $\overline{L}/\overline{F}$; this uses $v \circ \sigma = v$. So, we get a map $\varphi : \mathrm{Gal}(L/F) \to \mathrm{Gal}(\overline{L}/\overline{F})$ given by $\varphi(\sigma) = \overline{\sigma}$, and it is easy to see that φ is a group homomorphism. Since these groups are the same size, to show that φ is an isomorphism, it suffices to prove that φ is surjective. Take $\tau \in \mathrm{Gal}(\overline{L}/\overline{F})$. Writing $\overline{L} = \overline{F}(\alpha)$ and $L = F(a)$

with $p = \min(\overline{F}, \alpha)$ and $p' = \min(F, a)$, such that $\overline{p'} = p$ as above, we have that $\tau(\alpha)$ is a root of p, so by Hensel's lemma, there is a $b \in L$ with $p'(b) = 0$ and $\overline{b} = \tau(\alpha)$. Since L/F is Galois, there is a $\sigma \in \text{Gal}(L/F)$ with $\sigma(a) = b$. Then $\overline{\sigma}(\alpha) = \overline{\sigma(a)} = \overline{b} = \tau(\alpha)$. Since $\overline{L} = \overline{F}(\alpha)$, this proves that $\overline{\sigma} = \tau$. Hence, φ is surjective. This finishes the proof that φ is an isomorphism. ■

We will use the following notation in this note: F is a local field with valuation v and D is an F-central division algebra. We also denote by v the (unique) extension to D and to any finite extension of F. Let W and V be the valuation rings of D and F, respectively. We denote the value groups of D and F by Γ_D and Γ_F, and the residue rings of D and F by $\overline{D}$ and $\overline{F}$, respectively. We set $q = \left|\overline{F}\right|$, the order of the finite field $\overline{F}$. The *ramification index* of D/F is $e = [\Gamma_D : \Gamma_F]$ and the *residue degree* of D/F is $f = [\overline{D} : \overline{F}]$.

Proposition 3 *The valuation ring W is a free V-module. Therefore, D/F is defectless.*

Proof. Let e and f be the ramification index and residue degree of D/F. If ϵ is the *initial index* of D/F; that is, ϵ is the number of nonnegative elements of Γ_D that are smaller than all positive elements of Γ_F, then it is known that W is a free V-module iff $\epsilon f = [D : F]$. Since $\Gamma_D \cong \mathbb{Z}$, it is easy to see that $\epsilon = e$, so once we prove that W is a free V-module, it will follow that $ef = [D : F]$, so $[D : F]$ is defectless.

Let $x_1, \ldots, x_n$ be elements of W that form an F-basis of D. We may choose the $x_i \in W$ since $WF = D$. Consider the bilinear form $b(x, y) = \text{Trd}(xy)$. This is a nondegenerate form since Trd is a nonzero map. Therefore, there is a dual basis $y_1, \ldots, y_n$ such that $b(x_i, y_j) = \delta_{ij}$, the Kronecker delta. We claim that $W \subseteq \sum V y_j$. To prove this, take $a \in W$, and write $a = \sum \alpha_j y_j$ with $\alpha_j \in F$. Since $x_i \in W$, we have $x_i a = \sum \alpha_j x_i y_j \in W$. Therefore, taking traces, we get $\alpha_i = \text{Trd}(x_i a) \in V$ since W/V is integral. This proves that $W \subseteq \sum V y_j$. Now, V is a Noetherian ring, so W is a Noetherian V-module. Therefore, W is a finitely generated V-module, and W is torsion free. Since V is a PID, this forces W to be free (of rank $[D : F]$). ■

Proposition 4 *With D and F as above, $[\Gamma_D : \Gamma_F] = [\overline{D} : \overline{F}] = \text{ind}(D)$.*

Proof. We wish to show that $e = f = n$, where $n = \text{ind}(D)$. By the previous proposition, $ef = n^2$. We show that $e = f = n$ by showing that $e \leq n$ and $f \leq n$. For f, we note that $\overline{D}$ is a field since it is a finite division ring. Therefore, $\overline{D}$ is a separable field extension of the finite field $\overline{F}$. Therefore, there is an inertial extension L/F with $L \subseteq D$ and $\overline{L} = \overline{D}$. Since

$$f = [\overline{D} : \overline{F}] = [\overline{L} : \overline{F}] = [L : F] \leq n,$$

this forces $f \leq n$.

To prove $e \le n$, note that if $\Gamma_F = \mathbb{Z}$, then $\Gamma_D = e^{-1}\mathbb{Z}$. Let $a \in D$ with $v(a) = e^{-1}$. Let $K = F(a)$, a subfield of D. Then $[K : F] \le n$. However, $\Gamma_K = e^{-1}\mathbb{Z}$ since $e^{-1} \in \Gamma_K$ and $\Gamma_K \subseteq \Gamma_D$. This shows that the ramification index of K/F is e. By the fundamental inequality, $e \le [K : F]$. This forces $e \le n$. Since both e and f are no larger than n, the equation $ef = n^2$ forces $e = f = n$. ■

In the proof of the previous proposition, we see that the inertial extension L that is a lift of $\overline{D}$ is a maximal subfield of D since $[L : F] = [\overline{D} : \overline{F}] = n$. We will describe the structure of D by describing the inertial extensions of F.

Proposition 5 *Let L/F be an inertial extension of degree n. Then L is the splitting field of $x^{q^n} - x$ over F. Moreover, L/F is a cyclic Galois extension.*

Proof. Since $|\overline{F}| = q$ and $[\overline{L} : \overline{F}] = n$, we have $|\overline{L}| = q^n$. Therefore, $\overline{L}$ is the splitting field of $x^{q^n} - x$ over $\overline{F}$ by the theory of finite fields. This polynomial has no repeated roots, so by Hensel's lemma, $x^{q^n} - x$ splits over L. If L_0 is the splitting field inside L of $x^{q^n} - x$ over F, then $\overline{L_0} \supseteq \overline{L}$ since $\overline{L_0}$ contains all of the roots of $x^{q^n} - x$. This forces $\overline{L_0} = \overline{L}$, and the fundamental inequality applied to L_0/F forces $L_0 = L$. This proves that L is the splitting field of $x^{q^n} - x$ over F. So, L/F is Galois. Moreover, $\mathrm{Gal}(L/F) \cong \mathrm{Gal}(\overline{L}/\overline{F})$ by commutative valuation theory. Since $\mathrm{Gal}(\overline{L}/\overline{F})$ is cyclic by finite field theory, $\mathrm{Gal}(L/F)$ is cyclic. ■

Suppose that L/F is an inertial extension of degree n. We can be more specific about a generator of $\mathrm{Gal}(L/F)$. By finite field theory, $\mathrm{Gal}(\overline{L}/\overline{F}) = \langle\sigma\rangle$, where $\sigma(a) = a^q$ for all $a \in \overline{L}$. Let $\sigma_L \in \mathrm{Gal}(L/F)$ be the lift of σ. Thus, $\overline{\sigma_L(x)} = \sigma(\overline{x})$ for all units $x \in L$. The element σ_L is called the *Frobenius automorphism* of L/F. Note that a more proper notation would be $\sigma_{L/F}$, but we omit the F since we always will use F as the base field.

Corollary 6 *Let K and L be inertial extensions of F contained in an algebraic closure of F.*

1. *$K = L$ iff $[K : F] = [L : F]$.*

2. *L is F-isomorphic to a subfield of K iff $[L : F]$ divides $[K : F]$.*

Proof. The first statement follows from the uniqueness of splitting fields and the previous proposition. For the second, by uniqueness of inertial lifts, L is F-isomorphic to a subfield of K iff $\overline{L}$ is $\overline{F}$-isomorphic to a subfield of $\overline{K}$, and this happens iff $[\overline{L} : \overline{F}]$ divides $[\overline{K} : \overline{F}]$ by finite field theory. So, L embeds in K iff $[L : F]$ divides $[K : F]$. ■

We write F_n for the unique up to isomorphism inertial extension of F of degree n, and we denote the Frobenius automorphism of F_n/F by σ_n.

Corollary 7 *D contains a maximal subfield that is cyclic Galois and inertial over F. Therefore, D is the cyclic algebra $(F_n/F, \sigma_n, a)$ for some $a \in F$, where $n = \mathrm{ind}(D)$.*

Proof. As we say in the proof of Proposition 4, there is a subfield L of D which is inertial over F and with $\overline{L} = \overline{D}$. So, $[L : F] = [\overline{D} : \overline{F}] = n$; hence, L is a maximal subfield of D. Moreover, since L/F is inertial, L/F is cyclic Galois, $L \cong F_n$, and σ_n generates $\mathrm{Gal}(F_n/F)$; these facts follow from Proposition 5. Since D has a cyclic maximal subfield F_n, we see that D is a cyclic crossed product $(F_n/F, \sigma_n, a)$ for some $a \in F^*$. ■

From this corollary, we obtain the following structure theorem quite easily.

Theorem 8 *Let A be a central simple F-algebra. If $\deg(A) = m$, then A is a cyclic algebra $(F_m/F, \sigma_m, a)$ for some $a \in F^*$.*

Proof. Write $A = M_t(D)$ for some division algebra D of index n. We already know that F_n is a maximal subfield of D. Since $nt = \deg(A) = m$, and since $[F_m : F_n] = t$, the field F_m embeds in $M_t(F_n) \subseteq M_t(D) = A$. Therefore, F_m is a (strictly) maximal subfield of A that is cyclic Galois over F. Thus, we can write $A = (F_m/F, \sigma, a)$ for some $a \in F^*$. ■

Let π be a uniformizer for F. Then any element $a \in F$ can be written in the form $a = \pi^r u$ with u a unit. In $\mathrm{Br}(F)$ we then have $(F_m/F, \sigma_m, a) \sim (F_m/F, \sigma_m, \pi)^r \otimes_F (F_m/F, \sigma_m, u)$. We get a simple description of $\mathrm{Br}(F)$ by showing that the second term is split.

Proposition 9 *If $u \in F^*$ is a unit, then $(F_m/F, \sigma_m, u)$ is split.*

Proof. Let D be the underlying division algebra of $(F_m/F, \sigma_m, u)$. We wish to show that $D = F$. By Theorem 8, have $D = (F_n/F, \sigma_n, b)$ for some $b \in F^*$. By dimension count, if $A = M_t(D)$, then $m = tn$. By properties of the inflation map of cohomology, we have $D \sim (F_m/F, \sigma_m, b^t)$. Since A is similar to D, this forces $(F_m/F, \sigma_m, b^t) \cong (F_m/F, \sigma_m, u)$. Therefore, $(F_m/F, \sigma_m, b^t u^{-1})$ is split. Thus, $b^t u^{-1} = N_{F_m/F}(c)$ for some $c \in F_m^*$. Taking values gives $tv(b) = mv(c)$, so $v(b) = nv(c)$. However, since F_m/F is an inertial extension, there is an $a \in F$ with $v(a) = v(c)$. So, $b = a^n w$ for some unit $w \in F^*$. This proves that $D \sim (F_n/F, \sigma_n, w)$. Again, by dimension count we get $D = (F_n/F, \sigma_n, w)$. So, there is an $x \in D^*$ with $xdx^{-1} = \sigma_n(d)$ for all $d \in F_n^*$, and $x^n = w$. The condition $x^n = w$ means that $v(x) = 0$, so x is a unit in D. We then get a residue crossed product $(\overline{F_n}/\overline{F}, \overline{\sigma_n}, \overline{w})$ as a subalgebra of $\overline{D}$. However, $[\overline{D} : \overline{F}] = n$ and this crossed product has dimension n^2 over $\overline{F}$. This forces $n = 1$, so $D = F$ as desired. ■

We can now determine $\mathrm{Br}(F)$. Since every finite dimensional F-central division algebra has a maximal inertial subfield, we have $\mathrm{Br}(F) = \bigcup_n \mathrm{Br}(F_n/F)$.

Theorem 10 $\mathrm{Br}(F_n/F) \cong \mathbb{Z}/n\mathbb{Z}$.

Proof. By the previous result and the comment preceding it, we know that every element in $\mathrm{Br}(F_n/F)$ is similar to $(F_n/F, \sigma_n, \pi)^r$ for some r. So, $\mathrm{Br}(F_n/F)$ is the cyclic group generated by the class of $E = (F_n/F, \sigma_n, \pi)$. We will be done by showing that $\exp(E) = n$.

It is clear that $\exp(E)$ divides n. Suppose that $\exp(E) = r$. Then $(F_n/F, \sigma_n, \pi^r)$ is split, so $\pi^r = N_{F_n/F}(c)$ for some $c \in F_n^*$. Taking values, and remembering that F_n/F is inertial, we get $r = nv(c)$. So, $v(c) = r/n$. But, since the value group of F_n is $\mathbb{Z}$, this forces n to divide r. So, $r = n$, so $\exp(E) = n$. ■

Theorem 11 *There is an isomorphism* $\mathrm{Br}(F) \to \mathbb{Q}/\mathbb{Z}$ *given by*

$$k/n + \mathbb{Z} \to [(F_n/F, \sigma_n, \pi^k)],$$

where $n \in \mathbb{N}$ *and* $0 \leq k < n$.

Proof. We define a map $\mathbb{Q} \to \mathrm{Br}(F)$ by $\varphi(k/n) = [(F_n/F, \sigma_n, \pi^k)]$. To see that this is well defined, note that by properties of inflation, $(F_n/F, \sigma_n\pi^k) \sim (F_{nt}/F, \sigma_{nt}, \pi^{kt})$ for any $t > 0$. Let $k/n = k'/n'$. We have $\varphi(k/n) = [(F_n/F, \sigma_n, \pi^k)] = [(F_{nn'}/F, \sigma_{nn'}, \pi^{kn'})]$ and $\varphi(k'/n') = [(F_{n'}/F, \sigma_{n'}, \pi^{k'})] = [(F_{nn'}/F, \sigma_{nn'}, \pi^{k'n})]$. Since $kn' = k'n$, we see that $\varphi(k/n) = \varphi(k'/n')$. It is easy to see that φ is a group homomorphism, since if k/n and k'/n' are in $\mathbb{Q}$, then

$$\begin{aligned}
\varphi(k/n + k'/n') &= \varphi\left((kn' + k'n)/nn'\right) \\
&= [(F_{nn'}/F, \sigma_{nn'}, \pi^{kn'+k'n})] \\
&= [(F_{nn'}/F, \sigma_{nn'}, \pi^{kn'})][(F_{nn'}/F, \sigma_{nn'}, \pi^{k'n})] \\
&= [(F_n/F, \sigma_n, \pi^k)][(F_{n'}/F, \sigma_{n'}, \pi^{k'})] = \varphi(k/n)\varphi(k'/n')
\end{aligned}$$

by properties of inflation again. It is clear that $\mathbb{Z} \subseteq \ker\varphi$ since $(F_n/F, \sigma_n, \pi^{nt})$ is split for all t. Conversely, if $k/n \in \ker\varphi$, then $(F_n/F, \sigma_n, \pi^k)$ is split. Since the exponent of $(F_n/F, \sigma_n, \pi)$ is n by Theorem 10, this forces n to divide k, so $k/n \in \mathbb{Z}$. Finally, φ is surjective by Theorem 8. We then have an induced isomorphism $\mathbb{Q}/\mathbb{Z} \cong \mathrm{Br}(F)$. ■

Let $\mathrm{inv} : \mathrm{Br}(F) \to \mathbb{Q}/\mathbb{Z}$ be the inverse map to the isomorphism in the theorem above. The element $\mathrm{inv}([A])$ is called the invariant of A, and this number completely determines A. The following corollary gives the basic properties of this function.

Corollary 12 *Let A and B be central simple F-algebras. Then*

1. $A \sim B$ *iff* $\mathrm{inv}(A) = \mathrm{inv}(B)$.
2. $A \sim F$ *iff* $\mathrm{inv}(A) = 0$.
3. $\mathrm{ind}(A)$ *is the order of* $\mathrm{inv}(A)$ *in* $\mathbb{Q}/\mathbb{Z}$.
4. $\mathrm{inv}(A \otimes_F B) = \mathrm{inv}(A) + \mathrm{inv}(B)$.

5. $\mathrm{inv}(A^m) = m\,\mathrm{inv}(A)$.

Proof. All of these properties follow immediately from the definition of inv, except perhaps for (3). To prove (3), suppose that $\mathrm{inv}(A) = k/n + \mathbb{Z}$ with $n > 0$ and $\gcd(k, n) = 1$. Then the order of $\mathrm{inv}(A)$ in $\mathbb{Q}/\mathbb{Z}$ is n. However, the index of $D = (F_n/F, \sigma_n, \pi)$ is n, and A is similar to $D^{\otimes k}$, so $\mathrm{ind}(A) = n$ since $\gcd(k, n) = 1$. ■

Corollary 13 *If A is a central simple F-algebra, then* $\mathrm{ind}(A) = \exp(A)$.

Proof. By the previous corollary, $\mathrm{ind}(A)$ is equal to the order of $\mathrm{inv}(A)$ in $\mathbb{Q}/\mathbb{Z}$. By using the isomorphism $\varphi = \mathrm{inv}^{-1}$, this order is equal to the order of $[A]$ in $\mathrm{Br}(F)$. Since this order is $\exp(A)$, we get $\mathrm{ind}(A) = \exp(A)$. ■

The final result of this note describes how the invariant behaves under scalar extension. In this result we need to assume that F is complete; it is not sufficient to assume that F is Henselian with value group $\mathbb{Z}$ and finite residue field for the proof we give. What we need from completeness is that any finite extension of complete fields is defectless. This can be found in [3, Chap. II, Thm. 11].

Proposition 14 *If E/F is a field extension with $m = [E : F]$, and if A is a central simple F-algebra, then* $\mathrm{inv}(A \otimes_F E) = m\,\mathrm{inv}(A)$.

Proof. Let $\mathrm{inv}(A) = k/n + \mathbb{Z}$. By the isomorphism $\mathrm{Br}(F) \to \mathbb{Q}/\mathbb{Z}$, we have $A \sim (F_n/F, \sigma_n, \pi^k)$. Let L be the maximal unramified extension of F in E. We write $K = F_n$ for ease of notation. Let $r = [KE : K]$ and $s = [KE : E]$. By the theorem of natural irrationalities and Proposition 5, we have

$$s = [K : K \cap E] = [K : K \cap L].$$

Also, let $l = [L : K \cap L]$. By Proposition 5, since K and L are inertial over $K \cap L$, we have $\gcd(s, l) = 1$. Let $e = e(E/F)$, the ramification index of E/F. If a is a uniformizer for E, then π and a^e have the same value in L. If $t = [K \cap L : F]$, then $lt = [L : F] = f(E/F) := f$, so

$$\begin{aligned} A \otimes_F E &\sim (K/F, \sigma_K, \pi^k) \otimes_F E \\ &\sim (EK/E, \sigma_{K/F}^{t}, a^{ek}) \\ &\sim (EK/E, \sigma_{K/F}^{rl}, a^{ekl}) \\ &\sim (EK/E, \sigma_{K/F}^{f}, a^{ekl}), \end{aligned}$$

the third similarity follows since $\gcd(s, l) = 1$. Now let us determine the Frobenius automorphism of EK/K. Since EK/E is inertial, $EK = E_s$, the inertial extension of E of degree s.

The Frobenius automorphism τ of EK/E is determined by $\overline{\tau}(\overline{a}) = (\overline{a})^{|\overline{E}|} = \overline{a}^{fq}$. Therefore, $\tau = \sigma_{K/F}^f$. Here is the one place that we invoke completeness: since E is a complete field, E/F is defectless, so $m = [E : F] = ef = elt$, so $ekl = km/t$. Thus,

$$A \otimes_F E \sim (E_s/E, \sigma_E, a^{km/t}).$$

Therefore, since $n = st$,

$$\operatorname{inv}(A \otimes_F E) = \frac{km/t}{s} = \frac{mk}{st} = \frac{mk}{n} = m \operatorname{inv}(A).$$

■

From this result we can determine exactly when a field extension of F splits a central simple F-algebra.

Corollary 15 *A field extension E of F splits A iff* $\operatorname{ind}(A)$ *divides* $[E : F]$.

Proof. Let $\operatorname{inv}(A) = k/n + \mathbb{Z}$ with $n > 0$ and $\gcd(k, n) = 1$. We have seen that $n = \operatorname{ind}(A)$. Since $\operatorname{inv}(A \otimes_F E) = [E : F] \operatorname{inv}(A)$ by the previous corollary, we see that E splits A iff $\operatorname{inv}(A \otimes_F E) = 0 + \mathbb{Z}$ iff $[E : F]k/n \in \mathbb{Z}$. Since $\gcd(k, n) = 1$, this forces n to divide $[E : F]$, as desired. ■

We point out that if F is only a Henselian discretely valued field with finite residue field, then an extension E/F need not be defectless in the case $\operatorname{char}(F) = p > 0$. If E/F is separable, the proof of Proposition 3 can be modified (use $T_{E/F}$ instead of Trd) to prove that E/F is defectless. However, it is possible for E/F to be defective if E/F is inseparable. We give an example of this, and use this to produce an example where the previous corollary does not hold.

Example 16 Let k be a finite field of characteristic p. It is known that $k((x))$ is not algebraic over $k(x)$. Let $y \in k((x))$ be transcendental over k. Then we have the chain of fields $k(x) \subseteq k(x, y^p) \subseteq k(x, y) \subseteq k((x))$. With the x-adic valuation on $k((x))$, this valuation restricts to a valuation on $F_0 := k(x, y^p)$ and $K_0 := k(x, y)$ so that $\Gamma_{K_0} = \Gamma_{F_0} = \mathbb{Z}$ and $\overline{K_0} = \overline{F_0} = k$. If we form the Henselization F of F_0 and set $K = FK_0$, then since F/F_0 is separable, $[K : F] = p$, and K/F is purely inseparable. Moreover, we have

$$\Gamma_K = \Gamma_{K_0} = \Gamma_{F_0} = \Gamma_F$$

and

$$\overline{K} = \overline{K_0} = \overline{F_0} = \overline{F}.$$

Thus, K/F is defective.

Example 17 Let K/F be as in the previous example, and let D be a division algebra of invariant $1/p + \mathbb{Z}$. For example, we could take $D = (F_n/F, \sigma_n, \pi)$. We have $\Gamma_D \cap \Gamma_K = \Gamma_F$ and $\overline{D} \otimes_{\overline{F}} \overline{K} = \overline{D}$. So, by [2, Thm. 1], $D \otimes_F K$ is a division algebra. Therefore, K does not split D even though $\operatorname{ind}(D)$ divides $[K : F]$. Thus, the previous corollary is false for F. Moreover, $\operatorname{inv}(D \otimes_F K) = [K : F] \operatorname{inv}(D)$ is also false since $[K : F] \operatorname{inv}(D) = 0$ while $\operatorname{inv}(D \otimes_F K)$ is nonzero.

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