Examples of Localization

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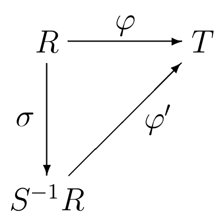
In this note we give a number of examples of localization. While we do not give the construction of localization, we do recall the universal mapping property: Let R be a commutative ring and let S be a multiplicatively closed subset of R. Denote by σ the canonical homomorphism $\sigma : R \to S^{-1}R$ (defined by $\sigma(r) = r/1$). Suppose that there is a ring homomorphism $\varphi : R \to T$ with $\varphi(S) \subseteq T^*$. Then there is a unique ring homomorphism $\varphi' : S^{-1}R \to T$ such that $\varphi' \circ \sigma = \varphi$; that is, the following diagram commutes:



Recall that the universal mapping property has the following consequence. If R' is a ring together with a ring homomorphism $\sigma' : R \to R'$ such that (R', σ') satisfies the same mapping property as $(S^{-1}R, \sigma)$, then $S^{-1}R \cong R'$.

Example 1 Let R be an integral domain and let $S = R - \{0\}$. Then S is multiplicatively closed, and $S^{-1}R$ is the quotient field of R. The universal mapping property for $S^{-1}R$ can be phrased in the following way: Let F be a field such that there is an injective ring homomorphism $\varphi : R \to F$. Then there is an injective ring homomorphism $\varphi' : S^{-1}R \to F$ such that $\varphi' \circ \sigma = \varphi$. So, if F contains (an isomorphic copy of) R, then F also contains $S^{-1}R$. So, the quotient field of R is the smallest field that contains R.

Example 2 Let $f \in R$ be a non-nilpotent element. If $S = \{f^n : n \geq 0\}$, then S is multiplicatively closed, and S does not contain 0 since f is not nilpotent. The ring $S^{-1}R := R_f$ can be written as $R_f = \{a/f^n : a \in R, n \geq 0\}$. Recalling the description of ideals in a localization, we see that the ideals of R_f are of the form $S^{-1}I$ with I an ideal of R that does not contain any power of f. From this fact we can prove the following result: let P be an ideal of R maximal (with respect to inclusion) among the ideals of R that are disjoint from

S. Then P is a prime ideal. Note that a Zorn's lemma argument gives the existence of P; the condition that f is not nilpotent is required to make $\{I : I \cap S = \varnothing\}$ a nonempty set. To prove this, given P, we have $S^{-1}P \neq S^{-1}R$ since $P \cap S = \varnothing$. Let M be a maximal ideal of $S^{-1}R$ that contains $S^{-1}P$. Then $\sigma^{-1}(M)$ is an ideal of R that contains P. Moreover, $\sigma^{-1}(M) \cap S = \varnothing$ since M is generated by the image of $\sigma^{-1}(M)$. By maximality, we get $P = \sigma^{-1}(M)$. However, since M is a prime ideal of $S^{-1}R$, we see that $P = \sigma^{-1}(M)$ is a prime ideal of R.

Example 3 Let P be a prime ideal of R, and let $S = R - P$. That S is multiplicatively closed is equivalent to the condition that P is prime. The localization $S^{-1}R$ is usually denoted by R_P. The prime ideals of R_P are in 1–1 correspondence with the prime ideals Q of R with $Q \cap S = \varnothing$. The condition $Q \cap S = \varnothing$ is equivalent to the condition $Q \subseteq P$. Thus, the ideal $S^{-1}P$ is the unique maximal ideal of R_P. Therefore, R_P is a *local ring*; in other words, R_P has a unique maximal ideal. Local rings have a number of nice properties; we will see some throughout the semester. Note that if T is a local ring with unique maximal ideal M, then $T^* = T - M$.

Example 4 Let $P_1, \dots, P_n$ be prime ideals of R, and set $S = R - \bigcup_{i=1}^n P_i$. Since $S = \bigcap_{i=1}^n (R - P_i)$, we see that S is multiplicatively closed. The prime ideals of $S^{-1}R$ are in 1–1 correspondence with the prime ideals Q of R with $Q \cap S = \varnothing$; that is, with the prime ideals Q of R that are contained in $\bigcup_{i=1}^n P_i$. However, by the prime avoidance theorem, any such prime ideal is contained in one of the P_i. Therefore, $\{S^{-1}P_i : 1 \le i \le n\}$ contains the maximal ideals of $S^{-1}R$. If the P_i are pairwise incomparable ($P_i \not\subseteq P_j$ for all $i \neq j$), then this is exactly the set of maximal ideals of $S^{-1}R$. In any case, $S^{-1}R$ is a *semilocal ring*; a ring with only finitely many maximal ideals.

Example 5 Let $S = R^*$, the group of units of R. Then $S^{-1}R = R$, which can be seen from the universal mapping property: the identity map $\mathrm{id} : R \to R$ satisfies $\mathrm{id}(S) \subseteq R^*$, so there is a unique group homomorphism $\varphi : S^{-1}R \to R$ such that $\varphi \circ \sigma = \mathrm{id}$. So, σ is injective. However, it is easy to see that σ is surjective, since if $r \in R$ and $s \in S = R^*$, then $r/s = (rs^{-1})/1 \in \mathrm{im}(\sigma)$. Thus, σ is an isomorphism, so $S^{-1}R \cong R$.

Example 6 Let R be a ring, and let $S = \{\text{non-zero divisors of } R\}$. Then S is multiplicatively closed, and the canonical map $\sigma : R \to S^{-1}R$ is injective. Moreover, $S^{-1}R$ is the largest localization of R for which the canonical map is injective. For, if T is a multiplicatively closed subset of R such that $\tau : R \to T^{-1}R$ is injective, then T cannot contain a zero divisor, since if $t \in T$ is a zero divisor, then for any nonzero $r \in R$ with $rt = 0$, we have $r/1 = 0/1$ in $T^{-1}R$ since $rt = 0$. This proves that $T \subseteq S$. Therefore, by the universal mapping property for $T^{-1}R$, as the map $\sigma : R \to S^{-1}R$ satisfies $\sigma(T) \subseteq (S^{-1}R)^*$, we see that there is a ring homomorphism $\varphi : T^{-1}R \to S^{-1}R$ that satisfies $\varphi(r/t) = r/t \in S^{-1}R$. Moreover, φ is injective, since $r/t = 0/1$ in $S^{-1}R$ means that there is an $s \in S$ with $rs = 0$, but this forces $r = 0$ since s is not a zero divisor. So, $\varphi : T^{-1}R \to S^{-1}R$ is injective, so $S^{-1}R$ contains an isomorphic copy of $T^{-1}R$. It is in this sense that $S^{-1}R$ is the largest localization such that the canonical map is injective.

The ring $S^{-1}R$ is called the *total quotient ring* of R. This is a generalization of the notion of the quotient field of an integral domain.

Example 7 Let R be an integral domain with quotient field F, and let $S = R - \{0\} \subseteq R[x]$. Then $S^{-1}(R[x]) = F[x]$, which we can see from the universal mapping property. Since the identity homomorphism $\mathrm{id} : R[x] \to F[x]$ satisfies $\mathrm{id}(S) \subseteq F[x]^*$, we get a ring homomorphism $\varphi : S^{-1}R[x] \to F[x]$ defined by $f(x)/s = s^{-1}f(x)$. This map is surjective, since any polynomial in $F[x]$ can be written in the form $(a_0/\alpha + \cdots + (a_n/\alpha)x^n$ with $a_i \in R$, $\alpha \in R - \{0\}$ by using common denominators, so this polynomial is the image of $(a_0 + \cdots + a_n x^n)/\alpha$. Also, φ is injective, since $\mathrm{id} : R[x] \to F[x]$ is injective. So, $S^{-1}R[x] \cong F[x]$.

Example 8 Let X be an open connected subset of $\mathbb{C}$, and let R be the set of functions analytic on X and let F be the set of functions analytic on X except perhaps for having poles at finitely many points. Then R is a commutative ring under pointwise addition and multiplication. Moreover, we can define pointwise operations on F, by defining $(f+g)(P) = f(P) + g(P)$ and $(fg)(P) = f(P)g(P)$ whenever f and g are both defined at P (you should think some about this). With this definition, F is a ring containing R. Let $P \in X$, and let $R_P \subseteq F$ be the set of functions that are analytic at P. So, any $f \in R_P$ is analytic on an open neighborhood of P. We clearly have $R \subseteq R_P \subseteq F$, and R_P is a subring of F. Moreover, let $M_P = \{f \in R : f(P) = 0\}$. Then M_P is an ideal of R, as is easily checked. Also, since $fg(P) = f(P)g(P)$, we see that $fg \in M_P$ forces $f \in M_P$ or $g \in M_P$. This means that M_P is a prime ideal of R. Now, let $f \in R_P$. Suppose that the poles of f in X are at $Q_1, \dots, Q_t$, and that Q_i is a pole of order n_i of f. Then $g(x) := f(x)(x - Q_1)^{n_1} \cdots (x - Q_t)^{n_t}$ is analytic on X. Moreover, $s(x) := (x - Q_1)^{n_1} \cdots (x - Q_t)^{n_t}$ is analytic on X, and $s(P) \neq 0$. So, $s \in R - M_P$, and $f(x) = g(x)/s(x)$. So, R_P is the localization R_{M_P}.

Example 9 Here is an example from algebraic geometry, and it is an algebraic analogue of the previous example. Let $f(x, y) = y^2 - (x^3 - x) \in \mathbb{C}[x, y]$, and let $X = \{(a, b) : f(a, b) = 0\}$. Let $\mathbb{C}[X] = \mathbb{C}[x, y]/(f)$. We can think of $\mathbb{C}[X]$ as the ring of polynomial functions on X since two polynomials give the same function on X iff they differ by a multiple of f (this uses the fact that (f) is a prime ideal of $\mathbb{C}[x, y]$). Also, let $\mathbb{C}(X)$ be the quotient field of $\mathbb{C}[X]$. We will think of $\mathbb{C}(X)$ as the field of rational functions on X. If $P \in X$, let R be the subring of $\mathbb{C}(X)$ consisting of the rational functions that are defined at P. One can show that if $M_P = \{f \in \mathbb{C}[X] : f(P) = 0\}$, then M_P is a prime ideal of $\mathbb{C}[X]$ and that $\mathbb{C}[X]_{M_P} = R$. The name "localization" comes from this type of example; by passing from $\mathbb{C}[X]$ to R we are focusing our attention on the functions defined in a neighborhood of P. In other words, we are localizing our attention at P. Note that we could have started with any irreducible polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ and repeated this construction. Furthermore, we could have started with any prime ideal I of $\mathbb{C}[x_1, \ldots, x_n]$ and repeated this with $X = \{P \in \mathbb{C}^n : f(P) = 0 \text{ for all } f \in I\}$ and $\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]/I$.

Example 10 Let R_1 and R_2 be commutative rings, and let $R = R_1 \oplus R_2$. Let P_1 be a prime ideal of R_1, and set $P = P_1 \oplus R_2$. Then P is a prime ideal of R. Moreover, we claim that $R_P \cong (R_1)_{P_1}$. To prove this, let σ and σ_1 be the canonical maps $\sigma : R \to R_P$ and $\sigma_1 : R_1 \to (R_1)_{P_1}$. Let $\pi : R \to R_1$ be the standard projection. Then $\pi(P) = P_1$, and so $\pi(S - P) \subseteq (R_1)^*_{P_1}$. So, by the universal mapping property, there is a ring homomorphism $\pi' : R_P \to (R_1)_{P_1}$, and this map satisfies $\pi'((r_1, r_2)/(s_1, t_2)) = r_1/s_1$. Conversely, there is a ring homomorphism $\tau : R_1 \to R_P$ given by $\tau(r) = (r, 0)/(1, 1)$. Note that since $(1, 0)/(1, 1) = (1, 1)/(1, 1)$ in R_P, the map τ does send 1 to 1, so τ is a ring homomorphism. The map τ sends $R_1 - P_1$ into the units of R_P, so the universal mapping property gives us a map $\tau' : (R_1)_{P_1} \to R_P$. This map satisfies $\tau'(r/s) = (r, 0)/(s, 0)$. From the definition of localization, $(r, r_2)/(s, t_2) = (r, 0)/(s, 0)$ for any $r \in R_1, s \in R_1 - P_1$ and $r_2, t_2 \in R_2$. This proves that $\pi' \circ \sigma' = \text{id}$, and the equality $\sigma' \circ \tau' = \text{id}$ is simple. So, $R_P \cong (R_1)_{P_1}$.