Non Abelian Cohomology

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Let G be a group and let A be a G-module. In this note we will define $H^0(G, A)$ and $H^1(G, A)$, and we will show that they have a natural structure as pointed sets. Moreover, given a short exact sequence $1 \to A \xrightarrow{p} B \xrightarrow{q} C \to 1$ of G-modules, we will obtain a long exact sequence

$$1 \to H^0(G,A) \xrightarrow{p_0^*} H^0(G,B) \xrightarrow{q_0^*} H^0(G,C) \xrightarrow{\delta} H^1(G,A) \xrightarrow{p_1^*} H^1(G,B) \xrightarrow{q_1^*} H^1(G,C). \qquad (1)$$

Furthermore, if $A \subseteq Z(B)$, we show that this long exact sequence can be extended one more term to

$$1 \to H^0(G,A) \xrightarrow{p_0^*} H^0(G,B) \xrightarrow{q_0^*} H^0(G,C) \xrightarrow{\delta} H^1(G,A) \xrightarrow{p_1^*} H^1(G,B) \xrightarrow{q_1^*} H^1(G,C) \xrightarrow{\delta} H^2(G,A). \qquad (2)$$

1 Definition of $H^i(G, A)$.

As in the abelian case, we define $H^0(G, A) = A^G$, the set of G-fixed elements of A. For $H^1(G, A)$, we first define a function $f : G \to A$ to be a cocycle if for all $\sigma, \tau \in G$, we have

$$f(\sigma)\sigma(f(\tau)) = f(\sigma\tau).$$

Next, we say that two cocycles f, g are cohomologous if there is an $a \in A$ with

$$g(\sigma) = a^{-1}f(\sigma)\sigma(a)$$

for each $\sigma \in G$. It is easy to see that the relation of cohomology is an equivalence relation, so we define $H^1(G, A)$ to be the set of equivalence classes of cocycles from G to A. While we do not get a group structure on $H^1(G, A)$, we have a distinguished element, the trivial cocycle 1 defined by $1(\sigma) = 1$. We will view $H^1(G, A)$ as a pointed set with distinguished element 1. Note that $H^0(G, A)$ is a group, so it is also a pointed set with distinguished element 1.

2 Functorial Properties.

Given a map $p : A \to B$ of G-modules, we define maps $p_i^* : H^i(G, A) \to H^i(G, B)$. For $i = 0$ this is easy. We let $p_0^* = p|_{A^G}$. This gives a map from $H^0(G, A)$ to $H^0(G, B)$ since p is a map of G-modules, so if $a \in A^G$, then $\sigma(p(a)) = p(\sigma(a)) = p(a)$ for each $\sigma \in G$, so $p(a) \in B^G$. For H^1, we define p_1^* as follows. If $f \in H^1(G, A)$, then $p_1^*(f) = p \circ f$. We can see that this is a well defined map, since if $\sigma, \tau \in G$, we have

$$\begin{aligned} p_1^*(f)(\sigma\tau) &= p(f(\sigma\tau) = p(f(\sigma)\sigma(f(\tau)) \\ &= p(f(\sigma) \cdot p(\sigma(f(\tau)) \\ &= p_1^*(f)(\sigma) \cdot \sigma(p_1^*(f(\tau))), \end{aligned}$$

so $p_1^*(f)$ is a cocycle in $H^1(G, B)$. Moreover, if $f, g \in H^1(G, A)$ are cohomologous, then there is an $a \in A$ with $g(\sigma) = a^{-1}f(\sigma)\sigma(a)$, so $p(g(\sigma)) = p(a)^{-1}p(f(\sigma))\sigma(p(a))$, so $p_1^*(g)$ is cohomologous to $p_1^*(f)$.

The functoriality of p_i^* is clear. If $A \xrightarrow{p} B \xrightarrow{q} C$ is an exact sequence of G-modules, then $(q \circ p)_1^* = q_1^* \circ p_1^*$ is immediate from the definition, as is $(q \circ p)_0^* = q_0^* \circ p_0^*$.

We now define a boundary map $\delta : H^0(G, C) \to H^1(G, A)$. If $c \in C^G$, let $b \in B$ with $q(b) = c$, and define $\delta(c)$ by $\delta(c)(\sigma) = b^{-1}\sigma(b)$. Note that $b^{-1}\sigma(b) \in \ker(q) = \mathrm{im}(p)$. We identify A with $p(A)$ here for simplicity. We first show that $\delta(c)$ is a cocycle, and then we show that different choices of b yield cohomologous cocycles, so that the map δ is well defined. To show that $\delta(c)$ is a cocycle, if $\sigma, \tau \in G$, then

$$\begin{aligned} \delta(c)(\sigma)\sigma(\delta(c)(\tau)) &= b^{-1}\sigma(b)\sigma(b^{-1}\tau(b)) \\ &= b^{-1}\sigma\tau(b) = \delta(c)(\sigma\tau). \end{aligned}$$

Therefore, $\delta(c)$ is a cocycle. If $b' \in B$ with $q(b') = c$, then $b' = ba$ for some $a \in A$. Then

$$\begin{aligned} (b')^{-1}\sigma(b') &= a^{-1}b^{-1}\sigma(b)\sigma(a) \\ &= a^{-1}\delta(c)\sigma(a), \end{aligned}$$

so the cocycle obtained from b' is cohomologous in A to the one obtained from b. Thus, δ is well defined.

Suppose that $A \subseteq Z(B)$. We construct a boundary map $H^1(G, C) \to H^2(G, A)$ in this case. Let $f \in H^1(G, C)$, and for each $\sigma \in G$, let $b_\sigma \in B$ with $q(b_\sigma) = f(\sigma)$. We define $\delta(f)$ by

$$\delta(f)(\sigma, \tau) = b_\sigma\sigma(b_\tau)b_{\sigma\tau}^{-1}.$$

We need to show that $\delta(f)$ is a 2-cocycle with values in A, that δ is compatible with the cohomology relation on $H^1(G, C)$, and that $\delta(f)$ does not depend on the choice of the b_σ.

If we apply q to $\delta(f)(\sigma,\tau))$,we get $f(\sigma)\sigma(f(\tau))f(\sigma\tau)^{-1} = 1$ since f is a cocycle. Thus, $\delta(f)$ has values in A. Next, we show that $\delta(f)$ is independent of the choice of the b_σ. Suppose that $q(b'_\sigma) = f(\sigma) = q(b_\sigma)$. Then $b'_\sigma = b_\sigma a_\sigma$ for some $a_\sigma \in A$. Then

$$\begin{aligned} b'_\sigma \sigma(b'_\tau)(b'_{\sigma\tau})^{-1} &= b_\sigma a_\sigma \sigma(b_\tau)\sigma(a_\tau) b_{\sigma\tau}^{-1} a_{\sigma\tau}^{-1} \\ &= a_\sigma \sigma(a_\tau) a_{\sigma\tau}^{-1} \cdot b_\sigma \sigma(b_\tau) b_{\sigma\tau}^{-1} \\ &= a_\sigma \sigma(a_\tau) a_{\sigma\tau}^{-1} \delta(f)(\sigma,\tau), \end{aligned}$$

In this calculation we used the assumption that $A \subseteq Z(B)$. Therefore, the cohomology class in $H^2(G,A)$ does not depend on the choice of the b_σ. Finally, suppose that f, g are cohomologous in $H^1(G,C)$. Then there is a $c \in C$ with $g(\sigma) = c^{-1} f(\sigma)\sigma(c)$. Choose b_σ with $q(b_\sigma) = f(\sigma)$, and let $b \in B$ with $q(b) = c$. Then $g(\sigma) = q(b^{-1} b_\sigma \sigma(b))$. If we use $b'_\sigma = b^{-1} b_\sigma \sigma(b)$ to define $\delta(g)$, we have

$$\begin{aligned} \delta(g)(\sigma,\tau) &= b^{-1} b_\sigma \sigma(b) \cdot \sigma(b^{-1} b_\tau \tau(b)) \cdot (b^{-1} b_{\sigma\tau} \sigma\tau(b))^{-1} \\ &= b^{-1} b_\sigma \sigma(b) \sigma(b)^{-1} \sigma(b_\tau) \sigma\tau(b) \sigma\tau(b)^{-1} b_{\sigma\tau}^{-1} b \\ &= b^{-1} b_\sigma \sigma(b_\tau) b_{\sigma\tau}^{-1} b = b^{-1} \delta(f)(\sigma,\tau) b \\ &= \delta(f)(\sigma,\tau). \end{aligned}$$

The final equality is true since the values of $\delta(f)$ lie in A, which is a central subgroup of B. Thus, the boundary map δ is well defined.

3 The Long Exact Sequence.

We now prove that the sequence 1 is an exact sequence of pointed sets, and that the sequence 2 is exact if $A \subseteq Z(B)$. We recall what this means. If $h : X \to Y$ is a map of pointed sets, then h is a homomorphism if h sends the distinguished element of X to the distinguished element of Y. The kernel of a homomorphism $h : X \to Y$ is the set of all elements of X that map to the distinguished element of Y. A sequence $X \xrightarrow{h} Y \xrightarrow{h'} Z$ of homomorphisms of pointed sets is then exact if $\ker(h') = \operatorname{im}(h)$.

To show the sequence 1 is exact, we verify this one term at a time.

At $H^0(G,A)$: This is clear since $p_0^* = p|_{A^G}$ and p is injective.

At $H^0(G,B)$: By the functoriality of the maps, $g_0^* \circ p_0^* = (g \circ p)_0^* = 0$ since $g \circ p$ is the zero map. Consequently, $\ker(g_0^*)$ contains $\operatorname{im}(p_0^*)$. For the reverse inclusion, if $b \in \ker(g_0^*)$, then $g(b) = 1$, and $b \in B^G$. There is an $a \in A$ with $p(a) = b$. Moreover, $p(\sigma(a)) = \sigma(p(a)) = p(a)$, so $\sigma(a) = a$ since p is injective. Thus, $a \in A^G$, so $b \in \operatorname{im}(p_0^*)$.

At $H^0(G,C)$: Take $c \in \operatorname{im}(q_0^*)$, say $c = q(b)$ with $b \in B^G$. Then $\delta(c) = b^{-1}\sigma(b) = 1$, so $\delta(c) = 1$. Conversely, if $\delta(c)$ is trivial, then there is an $a \in A$ with $\delta(c)(\sigma) = a^{-1}\sigma(a)$. If $c = q(b)$, then $b^{-1}\sigma(b) = a^{-1}\sigma(a)$, so $ba^{-1} \in B^G$. Since $q_0^*(ba^{-1}) = c$, the element c lies in the image of q_0^*.

At $H^1(G, A)$: Let $c \in C^G$, and consider $\delta(c)$. This cocycle sends σ to $b^{-1}\sigma(b)$, where $q(b) = c$. Applying p_1^*, we get the cocycle $b^{-1}\sigma(b) \in H^1(G, B)$, which is clearly trivial. Conversely, let $f \in \ker(p_1^*)$. Then there is a $b \in B$ with $f(\sigma) = b^{-1}\sigma(b)$. However, $f(\sigma) \in A$, so $q(b^{-1}\sigma(b)) = 1$. Thus, if $c = q(b)$, then $c^{-1}\sigma(c) = 1$, so $c \in C^G$. It is now clear that $\delta(c) = f$.

At $H^1(G, B)$: If $f \in H^1(G, A)$, then $q_1^*(p_1^*(f)) = (q \circ p)_1^*(f)$ is trivial since $q \circ p$ is the zero map. Thus, $\mathrm{im}(p_1^*) \subseteq \ker(q_1^*)$. Conversely, let $g \in \ker(q_1^*)$. Then there is a $c \in C$ with $q(g(\sigma)) = c^{-1}\sigma(c)$ for all $\sigma \in G$. Let $b \in B$ with $q(b) = c$. Then $q(g(\sigma)) = q(b^{-1}\sigma(b))$, so $g(\sigma) = b^{-1}a_\sigma\sigma(b)$ for some $a_\sigma \in A$. Note that we can write $g(\sigma)$ this way since A is normal in B. Then $a_\sigma = bg(\sigma)\sigma(b)^{-1}$, so the map a that sends σ to a_σ is a cocycle with values in A; that is, $a \in H^1(G, A)$. The map a is sent to g by p_1^*, so we get the reverse inclusion.

At $H^1(G, C)$: Here we assume $A \subseteq Z(B)$ in order to have a map $\delta : H^1(G, C) \to H^2(G, A)$. Suppose $f \in H^1(G, B)$. We can use $f(\sigma)$ to represent $q_1^*(f)$ in the definition of δ. Then $\delta(q_1^*(f))(\sigma, \tau) = f(\sigma)\sigma(f(\tau))f(\sigma\tau)^{-1} = 1$, so $\mathrm{im}(q_1^*) \subseteq \ker(\delta)$. For the converse, suppose $\delta(g) = 1$. Then there are $a_\sigma \in A$ with $\delta(g)(\sigma, \tau) = a_\sigma\sigma(a_\tau)a_{\sigma\tau}^{-1}$. If $q(b_\sigma) = g(\sigma)$, then $b_\sigma\sigma(b_\tau)b_{\sigma\tau}^{-1} = a_\sigma\sigma(a_\tau)a_{\sigma\tau}^{-1}$. Using that $A \subseteq Z(B)$, we see that if $b'_\sigma = b_\sigma a_\sigma^{-1}$, then $b'_\sigma\sigma(b'_\tau)(b'_{\sigma\tau})^{-1} = 1$. Also, $q(b'_\sigma) = g(\sigma)$, so we can represent $\delta(g)$ using the b'_σ. Thus, the function f defined by $f(\sigma) = b'_\sigma$ is a cocycle in $H^1(G, B)$, and it maps to g.