

# Projective Modules and $S^2$

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In this note we discuss a non-free projective module that is associated to the 2-sphere  $S^2$ . The proof that this module is projective needs only basic properties of projective modules, but the proof that the module is not free requires a result of algebraic topology.

Let  $A = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$  and, for simplicity of notation, write  $x, y, z$  for the images of  $x, y, z$  in  $A$ . The ring  $A$  can be thought of as the ring of polynomial functions on the sphere  $S^2 = \{(a, b, c) \in \mathbb{R}^3 \mid a^2 + b^2 + c^2 = 1\}$ . For, any polynomial  $f \in \mathbb{R}[x, y, z]$  is a function  $f : S^2 \rightarrow \mathbb{R}$ , and two polynomials  $f, g$  agree as functions on  $S^2$  iff they differ by an element in the ideal  $(x^2 + y^2 + z^2 - 1)$ . Let

$$P = \{(a, b, c) \in A^3 \mid ax + by + cz = 0\}$$

There is an  $A$ -module homomorphism  $T : A^3 \rightarrow A$  given by  $(a, b, c) \mapsto ax + by + cz$ , and  $P$  is the kernel. The map  $T$  is surjective, since  $T(x, y, z) = x^2 + y^2 + z^2 = 1$ , hence  $T(rx, ry, rz) = r$  for any  $r \in A$ . Thus, there is an exact sequence

$$0 \rightarrow P \rightarrow A^3 \rightarrow A \rightarrow 0.$$

Moreover, since  $A$  is a free  $A$ -module, hence projective, this sequence splits, so  $A^3 \cong P \oplus A$ . This shows that  $P$  is a projective module. We can be more specific by noting that this sequence splits because the map  $\varphi : A \rightarrow A^3$  given by  $\varphi(r) = r(x, y, z) = (rx, ry, rz)$  is a 1-sided inverse of  $T$  (that is,  $T \circ \varphi = \text{id}|_A$ ). So,  $A^3$  is the internal direct sum of  $P$  and  $A(x, y, z)$ .

The module  $P$  is an example of a *stably free* module; a module  $M$  is stably free if  $M \oplus A^n \cong A^m$  for some  $n, m$ . However, with some topology and differential geometry, we can show that  $P$  is not a free module. Suppose  $P$  is free. Then the rank of  $P$  is two, since  $P \oplus A(x, y, z) = A^3$ . If  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{v} = (v_1, v_2, v_3)$  are basis elements for  $P$ , then

$(x, y, z), \vec{u}, \vec{v}$  form a basis for  $A^3$ . Let

$$B = \begin{bmatrix} x & y & z \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \in M_3(A)$$

and let  $f = \det(B) \in A$ . If  $F$  is the quotient field of  $A$ , then we know that the elements  $(x, y, z), \vec{u}, \vec{v}$  form a basis for  $F^3$  iff the matrix  $B$  is invertible as a matrix over  $F$ , iff  $f \neq 0$ . However, from the formula for inverses in terms of determinants, the inverse  $B^{-1}$  has entries in  $A$  iff  $f \in A^*$ ; i.e., if  $f$  is a unit in  $A$ . Some of the standard arguments for matrices over a field can be checked to see that they prove that the three vectors  $(x, y, z), \vec{u}, \vec{v}$  form a basis for  $A^3$  iff  $B^{-1} \in M_3(A)$ , iff  $f$  is a unit in  $A$ . Note that the units of  $A$  are precisely the constants.

We will use this fact together with some topology to show that  $P$  is not free. For each point  $\vec{a} = (a_1, a_2, a_3)$  in  $S^2$ , we have a ring homomorphism  $\varphi_{\vec{a}} : A \rightarrow \mathbb{R}$  given by evaluation at  $\vec{a}$ . That is, if  $g(x, y, z) \in A$  is a polynomial function on  $S^2$ , then  $\varphi_{\vec{a}}(g) = g(a_1, a_2, a_3)$ . The function  $\varphi_{\vec{a}}$  induces a function  $\varphi_{\vec{a}} : A^3 \rightarrow \mathbb{R}^3$  by

$$\varphi_{\vec{a}}(g_1, g_2, g_3) = (\varphi_{\vec{a}}(g_1), \varphi_{\vec{a}}(g_2), \varphi_{\vec{a}}(g_3)) = (g_1(\vec{a}), g_2(\vec{a}), g_3(\vec{a})).$$

Notice that if  $(g_1, g_2, g_3) \in P$ , then the image of this triple lands in the tangent space of  $S^2$  at the point  $\vec{a}$ . This is because the tangent space  $T_{\vec{a}}(S^2)$  of  $S^2$  at  $\vec{a}$  is the vector space

$$T_{\vec{a}}(S^2) = \{(x, y, z) \in \mathbb{R}^3 \mid a_1x + a_2y + a_3z = 0\}.$$

This is the subspace of  $\mathbb{R}^3$  that is parallel to the usual tangent plane at  $\vec{a}$ . If we assume that  $f$  is a unit, then  $\varphi_{\vec{a}}(f)$  is a nonzero real number for all  $\vec{a}$ , since if  $fg = 1$  in  $A$ , then  $1 = \varphi_{\vec{a}}(1) = \varphi_{\vec{a}}(f)\varphi_{\vec{a}}(g)$ . Consequently, because  $\varphi_{\vec{a}}$  is a ring homomorphism, the formula for determinant shows that

$$\varphi_{\vec{a}}(f) = \det \begin{bmatrix} \varphi_{\vec{a}}(x) & \varphi_{\vec{a}}(y) & \varphi_{\vec{a}}(z) \\ \varphi_{\vec{a}}(u_1) & \varphi_{\vec{a}}(u_2) & \varphi_{\vec{a}}(u_3) \\ \varphi_{\vec{a}}(v_1) & \varphi_{\vec{a}}(v_2) & \varphi_{\vec{a}}(v_3) \end{bmatrix}.$$

Since this is nonzero for all  $\vec{a}$ , each row is nonzero for all  $\vec{a}$ . In particular, for the middle row, we see that this means the vector  $\varphi_{\vec{a}}(\vec{u})$  is a nonzero tangent vector to  $S^2$  at  $\vec{a}$ . By noting that all maps are continuous, this means the function  $\vec{a} \mapsto \varphi_{\vec{a}}(\vec{u})$  is a tangent vector field on  $S^2$  that never vanishes. Such a vector field cannot exist, however, from topology. (See Theorem 10.4 in Chapter 8 of Munkres book *Topology A First Course*, or the current 542

text). Therefore,  $P$  is not a free module.

This example also gives an example of a (locally trivial) vector bundle, the tangent bundle on  $S^2$ , that is not a trivial bundle.