On Severi-Brauer Varieties

Patrick J. Morandi

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In this note we give an argument to show that the Severi-Brauer variety of $A = M_n(F)$ is a closed subset of $\mathbb{P}(\bigwedge^n A)$ and that it is isomorphic to $\mathbb{P}^{n-1}$. In doing so we utilize both the Segre embedding and the n-uple embedding of projective space!

Let V be an n-dimensional F-vector space and let $A = \mathrm{End}_F(V)$. First consider the map $\mathbb{P}(V) \times \mathbb{P}(V^*) \to \mathbb{P}(A)$ given by $\varphi(Fv, F\sigma) = F(v \otimes \sigma)$. We are using the canonical vector space isomorphism $V \otimes_F V^* \to \mathrm{End}_F(V)$ given by the map $(v \otimes \sigma)(w) = \sigma(w)v$. On coordinates we may view this map from $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}(A)$ by $\varphi(v, w) = vw^t$, or, more explicitly, by $\varphi((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = (a_i b_j)$. In other words, this is the Segre embedding. We then map $\mathbb{P}(A) \to \mathrm{SB}(A)$ by $Fc \mapsto cA$. We note that this map is well defined by the first of the following facts:

1. The map φ is an isomorphism of varieties from $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ to the variety X_1 of rank one matrices in A;
2. If $c = \varphi(v, w)$ is a rank 1 matrix, then the column space of c is Fv;
3. If $c, d \in X_1$, then $cA = dA$ iff the column spaces of c and d are equal;
4. If $d \in cA$ is nonzero, then $dA = cA$;
5. If $c = \varphi(v, w)$, then $cA = \{\varphi(v, u) : u \in \mathbb{P}^{n-1}\}$.

We leave the proofs of these statements until later; each is quite simple. We recall that we view $\mathrm{SB}(A)$ as a subset of the grassmanian $\mathrm{Gr}(n, A)$ of n-dimensional subspaces of A, and this embeds in $\mathbb{P}(\bigwedge^n A)$ via a subspace of A with basis $u_1, \ldots, u_n$ goes to $F(u_1 \wedge \cdots \wedge u_n)$. If $c = \varphi(v, w)$, then $cA = \{\varphi(v, u) : u \in \mathbb{P}^{n-1}\}$, so if F^n has the standard basis $e_1, \ldots, e_n$, we see that cA contains $\varphi(v, e_i)$ for each i. The matrix $\varphi(v, e_i)$ has column i equal to v and all other columns equal to 0. Thus, these matrices are linearly independent, hence they form a basis for cA. Thus, as a point in $\mathbb{P}(\bigwedge^n A)$, cA corresponds to $F(\varphi(v, e_1) \wedge \cdots \wedge \varphi(v, e_n))$. To see what this is explicitly, say $v = (a_1, \ldots, a_n)$. Then the wedge of these n matrices is

$$\begin{aligned} &\left(\sum a_i e_{i1}\right) \wedge \left(\sum a_i e_{i2}\right) \wedge \cdots \wedge \left(\sum a_i e_{in}\right) \\ = &\sum_{i_1, i_2, \ldots, i_n} a_{i_1} \cdots a_{i_n} \cdot e_{i_1 1} \wedge \cdots \wedge e_{i_n n}, \end{aligned}$$

where $\{e_{ij}\}$ is the set of standard matrix units for A. Each of the elements $e_{i_1 1} \wedge \cdots \wedge e_{i_n n}$ is a distinct basis vector of $\bigwedge^n A$ when we use the standard basis of $\bigwedge^n A$ relative to the basis $\{e_{ij}\}$ of A. Thus, identifying $\mathbb{P}(\bigwedge^n A)$ with $\mathbb{P}^{\binom{n^2}{n}}$ in the obvious way, if C is the closed subset of $\mathbb{P}^{\binom{n^2}{n}}$ given by the vanishing of all coordinates not associated to the basis vectors of $\bigwedge^n A$, then we have a map $\varphi \to \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to C \subseteq \mathbb{P}^{\binom{n^2}{n}}$ given by $\varphi((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \{a_{i_1} \cdots a_{i_n}\}$. Viewing $C = \mathbb{P}^{\binom{n^2}{n}-1}$, this induces a map $f : \mathbb{P}^{n-1} \to C$, given by $(a_1, \ldots, a_n) \mapsto \{a_{i_1} \cdots a_{i_n}\}$, which is just the n-uple embedding. Thus, f is a variety isomorphism onto $\operatorname{im}(f)$, but $\operatorname{im}(f) = \mathrm{SB}(A)$. Since $\mathbb{P}^{n-1}$ is complete, $\operatorname{im}(f)$ is closed in C, hence in $\mathbb{P}(\bigwedge^n A)$, and $\mathrm{SB}(A) = \operatorname{im}(f) \cong \mathbb{P}^{n-1}$.

We now give the simple proofs of the five statements above. First, for (1), note that if $w = (b_1, \ldots, b_n)$, then the i-th column of vw^t is $b_i v$. So, if v and w are nonzero (as must be for Fv, Fw to be points in $\mathbb{P}(V)$), then the column space of vw^t is Fv, a one dimensional space, so $\operatorname{rank}(vw^t) = 1$. Notice that we have proven (2). Thus, $\operatorname{im} \varphi \subseteq X_1$. To see that φ is injective, if $\varphi(v, w) = \varphi(v', w')$, then comparing column spaces, we have $Fv = Fv'$, i.e., v and v' are the same points in projective space. By similar arguments, the row space of $\varphi(v, w)$ is Fw, so we get $Fw = Fw'$. Thus, φ is injective. For surjectivity, let $c \in X_1$. If the column space of c is generated by v, the i-th column of c is equal to $b_i v$ for some b_i. Then $w := (b_1, \ldots, b_n)$ is nonzero otherwise $c = 0$. So, $c = \varphi(v, w)$. Viewing $\varphi : \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \to \mathbb{P}(A)$, on coordinates we have seen that φ is the Segre embedding into $\mathbb{P}^{n^2-1}$. So, φ is an isomorphism onto its image. This finishes the proof of (1).

For the proof of (4), if c has rank 1 and $d \in cA$ is nonzero, then $dA \subseteq cA$, so by minimality of cA we see that $dA = cA$. Thus (4) is true. For (5), if the columns of an $a \in A$ are $a_1, \ldots, a_n$, then for $c = \varphi(v, w)$, we have $ca = vw^t a = v \cdot (w^t a_1, w^t a_2, \ldots, w^t a_n)$, so the i-th column of ca is $(w^t a_i)v$. From this we see that $cA \subseteq \{\varphi(v, u) : u \in \mathbb{P}^{n-1}\}$. For the converse, since F^n is a simple A-module, there is an $a \in A$ with $u^t = w^t a$ given that $w \neq 0$. So, $\varphi(v, u) = vu^t = vw^t a = ca \in cA$. This proves (5). Since we have already proven (2), we only have to prove (3). Let $c, d \in X_1$. If $c = \varphi(v, w)$ and $d = \varphi(v', w')$, then Fv is the column space of c and Fv' is the column space of d. If $Fv = Fv'$, then $c \in \{\varphi(v', x) : x \in \mathbb{P}^{n-1}\} = dA$ by (5). Similarly $d \in cA$, so $cA = dA$. Conversely, if $cA = dA$, then $\{\varphi(v, x) : x \in \mathbb{P}^{n-1}\} = \{\varphi(v', y) : y \in \mathbb{P}^{n-1}\}$ by (5) . Since φ is injective (on the level of projective space), this forces $Fv = Fv'$, so the column space of c and d are equal.