I will present an example of an interaction between (non-classical) propositional logics and (classical) model theory which was made possible due to categorical logic.

I will investigate the existence of model-completions for equational theories arising from propositional logics (such as the theory of Heyting algebras and various kinds of theories related to propositional modal logic). The existence of model-completions turns out to be related to proof-theoretic facts concerning interpretability of second order propositional logic into ordinary propositional logic through the so-called ‘Pitts’ quantifiers’ or ‘bisimulation quantifiers’.

For an equational theory $T$ (satisfying a certain assumption which is rather strong in general, but which is often satisfied in varieties of algebras arising from logic), we have a characterization of the existence of the model completion:

$T$ admits a model completion iff $T$ is an r-Heyting category,

where $T$ is the opposite of the category $Alg(T)_{fp}$ of finitely presented $T$-algebras. In other words $T$ admits a model completion iff the category $T$ derived from $T$ has some nice categorical structure.

The notion of r-Heyting category is obtained from the notion of Heyting category by replacing ‘subobject’ by ‘regular subobject’ everywhere in the definition.

Next I will use this characterization to two kinds of varieties of algebras: Heyting algebras and modal algebras. Usually it is not easy to decide directly whether $T$ is an r-Heyting category. But, as this is a purely categorical
property, we can study it in any category equivalent to $T$. The strategy we adopt for an equational theory $T$ can be summarized in the following four steps:

1. **Embedding.** Find an r-Heyting category $E$ and an embedding 
   \[ \Phi_T : T \rightarrow E \]
   which is conservative, preserves finite limits and all the other r-Heyting category structure that exists in $T$.
   Conservativity ensures that the operations that can be performed in $T$ and are preserved by $\Phi_T$ satisfy automatically any exactness properties that these operations satisfy in $E$. In particular, the operations of left ($\exists_f$) and right ($\forall_f$) adjoint to the pullback functors $f^*$ (operating on regular subobjects) in $T$, if they exist, they automatically satisfy the Beck-Chevalley condition.
   In the applications the category $E$ is (equivalent to) the category of sheaves on the opposite of the category of finite $T$-algebras with the canonical topology.

2. **Duality.** Identify the image of $\Phi_T$ in $E$, i.e. describe in a convenient way a subcategory $M_T$ of $E$ so that we have a factorization of $\Phi_T$

   \[ T \xrightarrow{\Phi_T} E \xrightarrow{\Psi_T} M_T \]

   with the first component being an equivalence of categories and $\Psi_T$ being an inclusion.
   In the applications this point is slightly reversed. It is usually more natural to define a 'duality' functor in the opposite direction, i.e. $M_T \rightarrow T$.

3. **Combinatorial condition for existence of adjoints.** Now the existence of the adjoints is reduced to the verification whether the existing adjoints
in \( \mathcal{E} \) when applied to objects coming from \( T \) give objects coming from \( T \), as well.

In applications, with the help of an appropriate description of \( M_T \), this can be reduced to an equivalent condition of a combinatorial nature, expressed in terms of Ehrenfeucht-Fraissé games on finite Kripke models.

4. **Verification of combinatorial conditions.** Last, but not least, the combinatorial conditions should be verified to establish whether the adjoints do exist, if they do \( T \) is an r-Heyting category.

Using the above method I will describe the few equational theories of Heyting and modal \( S_4 \)-algebras that admit model completions.

**References**