Math 581 Assignment 2
due Friday 7 September

If $V$ is an $F$-vector space, we will denote by $\text{Aut}_F(V)$ the group of all $F$-vector space automorphisms of $V$. It is a subgroup of $\text{Aut}(V)$, the group of all group automorphisms of $(V,+)$. If $R$ is a ring, we'll denote by $R^*$ the group of invertible elements of $R$, under multiplication.

**Instructions.** Do the first three problems and one of Problems 4 and 5.

1. Let $\varphi : G \to H$ be a homomorphism.
   
   (a) Prove that $\varphi(e_G) = e_H$ and, for each $a \in G$, that $\varphi(a^{-1}) = \varphi(a)^{-1}$.
   
   (b) If $\varphi$ is surjective and $G$ is Abelian, prove that $H$ is Abelian.

2. Let $G, H$ be groups. Prove that $G \times H \cong H \times G$.

3. Let $C = \langle a \rangle$ be a cyclic group of order $n < \infty$.
   
   (a) Define $\sigma_i : C \to C$ by $\sigma_i(x) = x^i$. Prove that $\sigma_i$ is a homomorphism, and is an automorphism if and only if $\gcd(i,n) = 1$.
   
   (b) Prove that $\sigma_i = \sigma_j$ if and only if $i \equiv j \pmod{n}$.
   
   (c) Prove that $\text{Aut}(C) = \{\sigma_i : i \in \mathbb{Z}, \gcd(i,n) = 1\}$.
   
   (d) Prove that $\sigma_i \circ \sigma_j = \sigma_{ij}$. Conclude that $\text{Aut}(C) \cong (\mathbb{Z}_n)^*$.

4. Let $S$ be a set with $|S| \geq 3$. Prove that $Z(\text{Perm}(S)) = \{e\}$.

5. Prove that $\mathbb{R}^*$ is not isomorphic to $\mathbb{C}^*$.

**Optional Problems**

1. Prove that $\text{Aut}_\mathbb{R}(\mathbb{R}) = \{\sigma \in \text{Aut}(\mathbb{R}) : \sigma \text{ is continuous}\}$ and that $\text{Aut}_\mathbb{R}(\mathbb{R}) \neq \text{Aut}(\mathbb{R})$.

2. For $a \in \mathbb{Q}$ with $a \neq 0$, define $L_a : \mathbb{Q} \to \mathbb{Q}$ by $L_a(x) = ax$. Prove that $\text{Aut}(\mathbb{Q}) = \{L_a : a \in \mathbb{Q}^*\}$ and that $\text{Aut}(\mathbb{Q}) \cong \mathbb{Q}^*$.

3. Prove that $(\mathbb{R}, +) \cong (\mathbb{C}, +)$.
   
   (Hint: Vector space argument.)