Math 581 Assignment 8  
due Friday 2 November

Recall that $K/F$ is a normal extension if $K$ is the splitting field over $F$ of some polynomial in $F[x]$. This terminology is used in Problem 2.

1. Let $K/F$ be a field extension and let $f(x), g(x) \in F[x]$. Prove that the gcd of $f(x), g(x)$ in $F[x]$ is the same as the gcd of $f(x), g(x)$ in $K[x]$. Conclude that the gcd of $f(x), g(x)$ is not 1 if and only if $f(x), g(x)$ have a common root in some extension field of $F$.

2. If $F \subseteq L \subseteq K$ is a tower of fields with $K/F$ a normal extension, prove that $K/L$ is a normal extension. Give an example of $K/F$ normal with $L$ a field with $F \subseteq L \subseteq K$ and $L/F$ not normal.

3. Let $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$, where $\omega = e^{2\pi i/3}$. Recall that $K$ is the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Prove that there is an automorphism $\sigma$ of $K$ with $\sigma|_{\mathbb{Q}(\sqrt[3]{2})} = \text{id}$ and $\sigma(\sqrt[3]{2}) = \sqrt[3]{2} \omega$. Also prove that there is an automorphism $\tau$ of $K$ with $\tau|_{\mathbb{Q}(\sqrt[3]{2})} = \text{id}$ and $\tau(\omega) = \omega^2$. (Hint: Recall that $[K : \mathbb{Q}] = 6$. Use this to find the minimal polynomial of $\omega$ over $\mathbb{Q}(\sqrt[3]{2})$ and the minimal polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}(\omega)$.)

4. Let $K$ be a splitting field over $F$ of some polynomial $f(x) \in F[x]$. If $p(x) \in F[x]$ is irreducible and has a root in $K$, prove that $p(x)$ splits over $K$. (Hint: Use the Isomorphism Extension Theorem.)

Optional Problems

1. Let $F$ be a field of characteristic $p$. Let $F^p = \{a^p : a \in F\}$.
   
   (a) Prove that $F^p$ is a subfield of $F$.
   
   (b) If $F$ is a finite field, prove that $F^p = F$.
   
   (c) Prove that each finite extension of $F$ is separable if and only if $F^p = F$.

2. Let $F$ be a field of characteristic $p > 0$, and suppose $K = F(a, b)$ be a field extension of $F$ with $a^p, b^p \in F$. Prove that if $c \in K$, then $c^p \in F$, and conclude that $[F(c) : F] \leq p$.

3. Let $K = \mathbb{Z}_p(x, y)$, the rational function field in two variables, and let $F = \mathbb{Z}_p(x^p, y^p)$. Prove that $[K : F] = p^2$ and that $K \neq F(c)$ for any $c \in K$. 

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