In many ways, abstract algebra began with the work of Abel and Galois on the solvability of polynomial equations by radicals. The key idea Galois had was to transform questions about fields and polynomials into questions about finite groups. For the proof that it is not always possible to express the roots of a polynomial equation in terms of the coefficients of the polynomial using arithmetic expressions and taking roots of elements, the appropriate group theoretic property that arises is the idea of solvability.

**Definition 1.** A group $G$ is solvable if there is a chain of subgroups

$$\langle e \rangle = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that, for each $i$, the subgroup $H_i$ is normal in $H_{i+1}$ and the quotient group $H_{i+1}/H_i$ is Abelian.

An Abelian group $G$ is solvable; the chain of subgroups $\langle e \rangle \subset G$ satisfies the definition. Also, the symmetric groups $S_3$ and $S_4$ are solvable by considering the chains $\langle e \rangle \subset A_3 \subset S_3$ and $\langle e \rangle \subset H \subset A_4 \subset S_4$, respectively, where

$$H = \{e, (12)(34), (13)(24), (14)(23)\}.$$

We shall show below that $S_n$ is not solvable if $n \geq 5$. This is the group theoretic result we need to show that the roots of the general polynomial of degree $n$ (over a field of characteristic 0) cannot be written in terms of the coefficients of the polynomial by using algebraic operations and extraction of roots.

We now begin to work toward showing that the symmetric group $S_n$ is not solvable if $n \geq 5$. If $G$ is a group, let $G'$ be the **commutator subgroup** of $G$; that is, $G'$ is the subgroup of $G$ generated by all $ghg^{-1}h^{-1}$ with $g, h \in G$. It is a fairly straightforward exercise to show that $G'$ is a normal subgroup of $G$ and that $G/G'$ is Abelian. In fact, if $N$ is a normal subgroup of $G$, then $G/N$ is Abelian if and only if $G' \subseteq N$. We define $G^{(i)}$ by recursion by setting by $G^{(1)} = G'$ and $G^{(i+1)} = (G^{(i)})'$. We then obtain a chain

$$G \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots \supseteq G^{(n)} \supseteq \cdots$$

such that $G^{(m+1)}$ is normal in $G^{(m)}$ and $G^{(m)}/G^{(m+1)}$ is Abelian for all $m$. 

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**Solvable Groups**

Mathematics 581, Fall 2012

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Lemma 2. \( G \) is solvable if and only if \( G^{(n)} = \langle e \rangle \) for some \( n \).

Proof. Suppose that \( G^{(n)} = \langle e \rangle \) for some \( n \). Then the chain

\[
G \supseteq G^{(1)} \supseteq \cdots \supseteq G^{(n)} = \langle e \rangle
\]

shows that \( G \) is solvable. Conversely, suppose that \( G \) is solvable, and let

\[
\langle e \rangle = H_n \subset H_{n-1} \subset \cdots \subset H_0 = G
\]

be a chain of subgroups such that \( H_{m+1} \) normal in \( H_m \) and \( H_m/H_{m+1} \) is Abelian for all \( m \). Then \( G/H_1 \) is Abelian, so \( G' = G^{(1)} \subseteq H_1 \). Thus, \( (G^{(1)})' \subseteq H_1' \). Because \( H_1/H_2 \) is Abelian, \( H_1' \subseteq H_2 \). Therefore, \( G^{(2)} = (G^{(1)})' \subseteq H_2 \). Continuing this process shows that \( G^{(n)} \subseteq H_n = \langle e \rangle \), so \( G^{(n)} = \langle e \rangle \).

Proposition 3. Let \( G \) be a group, and let \( N \) be a normal subgroup of \( G \). Then \( G \) is solvable if and only if \( N \) and \( G/N \) are solvable.

Proof. We have \( N^{(m)} \subseteq G^{(m)} \) and \( (G/N)^{(m)} = (G^{(m)})N/N \) for all \( m \). Thus, if \( G \) is solvable, there is an \( n \) with \( G^{(n)} = \langle e \rangle \). Therefore, \( N^{(n)} = \langle e \rangle \) and \( (G/N)^{(n)} = \langle e \rangle \), so both \( N \) and \( G/N \) are solvable. Conversely, suppose that \( N \) and \( G/N \) are solvable. Then there is an \( m \) with \( (G/N)^{(m)} = \langle e \rangle \), so \( G^{(m)} \subseteq N \). There is an \( n \) with \( N^{(n)} = \langle e \rangle \), so \( G^{(n+m)} = (G^{(m)})^{(n)} \subseteq N^{(n)} = \langle e \rangle \). Therefore, \( G^{(n+m)} = \langle e \rangle \), so \( G \) is solvable.

Lemma 4. If \( n \geq 5 \), then \( A_n \) is a simple group.

We saw this in class; a proof is on the class website.

Corollary 5. If \( n \geq 5 \), then \( S_n \) is not solvable.

Proof. Since \( A_n \) is simple and non-Abelian, \( A'_n = A_n \). Thus, we see for all \( m \) that \( A_n^{(m)} = A_n \neq \langle e \rangle \), so \( A_n \) is not solvable. By the proposition above, \( S_n \) is also not solvable.