Math 582 Assignment 9
due Friday May 3

Instructions. Throughout this assignment $A$ will be a Dedekind domain. Work Problems 1, 2 and two others.

1. If $I$ and $J$ are nonzero ideals of $A$, we say $I$ divides $J$ if there is an ideal $K$ with $J = IK$. Prove that $I$ divides $J$ if and only if $J \subseteq I$.

2. Let $P_1, \ldots, P_n$ be maximal ideals of $A$, and let $I = P_1^{e_1} \cdots P_n^{e_n}$ and $J = P_1^{f_1} \cdots P_n^{f_n}$ with $0 \leq e_i, f_i$ for each $i$. Prove that
   \begin{enumerate}[(a)]   
   
   \item $I + J = P_1^{g_1} \cdots P_n^{g_n}$, where $g_i = \min\{e_i, f_i\}$ for each $i$.
   
   \item $I \cap J = P_1^{h_1} \cdots P_n^{h_n}$, where $h_i = \max\{e_i, f_i\}$ for each $i$.
   \end{enumerate}

3. (a) Let $R$ be a commutative ring. If $M, N$ are distinct maximal ideals of $R$ and if $n, m \in \mathbb{N}$, prove that $M^n + N^m = R$.
   
   (b) Let $M_1, \ldots, M_n$ be maximal ideals of $A$, and let $f_i \in \mathbb{N}$. Pick $x_i \in M_i^{f_i} - M_i^{f_i+1}$. Show that the Chinese Remainder Theorem implies there is $a \in A$ with $a + M_i^{f_i+1} = x_i + M_i^{f_i+1}$ for each $i$. Furthermore, prove that $(a) = M_1^{f_1} \cdots M_n^{f_n} K$ for some ideal $K$ whose prime factorization does not involve any of the $M_i$.

4. If $I$ is a nonzero ideal of $A$, show that every ideal of $A/I$ is principal. Conclude that any ideal in a Dedekind domain can be generated by two elements.

5. If $S \subseteq A$ is multiplicatively closed with $0 \notin S$, prove that $A_S$ is a Dedekind domain, provided that $A_S$ is not equal to the quotient field of $A$.

6. If $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in A[x]$, the content of $f$ is the ideal $(a_0, \ldots, a_n)$. Prove the following generalization of Gauss’ lemma, that $c(fg) = c(f)c(g)$.

7. Prove that $A$ is a UFD if and only if $A$ is a PID.

8. Let $A = \mathbb{Z}[\sqrt{-5}]$. By a previous HW problem $A$ is the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-5})$, so it is a Dedekind domain. Note that $2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ in $A$. Let $P = (2, 1 + \sqrt{-5})$, $Q_1 = (3, 1 + \sqrt{-5})$, and $Q_2 = (3, 1 - \sqrt{-5})$.
   
   (a) Prove that $P, Q_1, Q_2$ are prime ideals of $A$.
   
   (b) Prove that $(2) = P^2$.
   
   (c) Prove that $(3) = Q_1Q_2$.
   
   (d) Prove that $(1 + \sqrt{-5}) = PQ_1$ and $(1 - \sqrt{-5}) = PQ_2$. 

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