Math 582 Exam 1
due Friday 8 March

Instructions. You may not consult anybody other than me on this exam.

1. Let $I$ be a nonzero ideal of a commutative ring $R$. Prove that $I$ is a free $R$-module if and only if $I = Ra$ for some $a \in I$ which is not a zero divisor.

2. Let $V$ be a finite-dimensional vector space over a field $F$. If $v, w \in V$ are nonzero, prove that $v \otimes w = w \otimes v$ in $V \otimes_F V$ if and only if $w = \alpha v$ for some $\alpha \in F$.

3. Let

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

be a split exact sequence of $R$-modules. If $M$ is any $R$-module, prove that the sequence

$$0 \rightarrow \text{hom}_R(M, A) \xrightarrow{f^*} \text{hom}_R(M, B) \xrightarrow{g^*} \text{hom}_R(M, C) \rightarrow 0$$

is exact.

4. Let $R$ be an integral domain with quotient field $F$. An $R$-submodule $I$ of $F$ is called a fractional ideal if there is $r \in R$ nonzero with $rI \subseteq R$.

(a) If $I$ is a finitely generated $R$-submodule of $F$, show that $I$ is a fractional ideal.

(b) Let $I$ be a fractional ideal of $R$, and suppose there is a fractional ideal $J$ with $IJ = R$, where $IJ$ is the set of all sums of elements of the form $xy$ with $x \in I$ and $y \in J$. Prove that $I$ is projective as an $R$-module.

(Hint for (b): If you write $1 = \sum_{i=1}^n a_i b_i$ with $a_i \in I$, show that $I = \sum_{i=1}^n Ra_i$. Use that to produce a homomorphism $R^n \rightarrow I$, and show that the corresponding short exact sequence splits by using the $b_i$ to produce a splitting map $I \rightarrow R^n$.)