**Math 582 Final Exam**  
8 May 2013

**Instructions.** Recall that if \( \varphi : A \to B \) is a ring homomorphism and \( P \) is an \( B \)-module, then \( P \) is an \( A \)-module via \( r \cdot p = \varphi(r)p \). If \( M \) is an \( A \)-module, then \( M \otimes_A B \) is an \( B \)-module via \( s \cdot (m \otimes t) = m \otimes st \). More generally, if \( M \) is an \( A \)-module and \( P \) is an \( B \)-module, then \( M \otimes_A P \) is an \( B \)-module via \( s \cdot (m \otimes p) = m \otimes sp \).

1. Let \( A \) be a Dedekind domain with quotient field \( F \).
   
   (a) If \( J_1 \supseteq J_2 \) are ideals of \( A \), prove that \( (A : J_1) \subseteq (A : J_2) \).
   
   (b) If \( J \) is a fractional ideal, prove that \( A \) satisfies the ACC on fractional ideals contained in \( J \). That is, if \( I_1 \subseteq I_2 \subseteq \cdots \) is a chain of fractional ideals with \( I_n \subseteq J \) for each \( n \), prove there is an \( m \) with \( I_m = I_{m+1} = \cdots \).

   (c) If \( I \) is a nonzero ideal of \( A \), prove that \( A/I \) is an Artinian ring.

2. Let \( \varphi : A \to B \) be a ring homomorphism between commutative rings. If \( M \) is a projective \( A \)-module, prove that \( M \otimes_A B \) is projective as an \( B \)-module.

3. Let \( A \) be a commutative ring and let \( a_1, \ldots, a_n \in A \).
   
   (a) Suppose that \( (a_1, \ldots, a_n) = A \). Prove that \( (a_1^r, \ldots, a_n^r) = A \) for each \( r \geq 1 \).

   (b) Suppose that \( (a_1, \ldots, a_n) = A \). Prove that \( A \) is a Noetherian ring if and only if each \( A_{a_i} \) is a Noetherian ring, where \( A_{a_i} \) is the localization of \( A \) at \( \{a_i^n : n \geq 0\} \).

4. Let \( A \subseteq B \) be commutative rings. Let \( M \) be an \( A \)-module and let \( N, P \) be \( B \)-modules. You may use without proof that \( (M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P) \) as \( B \)-modules.
   
   (a) If \( M \) is an \( A \)-module and \( P \) is an \( B \)-module, prove that \( M \otimes_A P \cong (M \otimes_A B) \otimes_B P \) as \( B \)-modules.

   (b) If \( M \) is flat as an \( A \)-module, prove that \( M \otimes_A B \) is flat as a \( B \)-module.