The Math in Good Will Hunting

Today we will watch a bit of the movie Good Will Hunting, staring Matt Damon and Robin Williams, and talk about the math that shows up. Here is a bit of information about the movie from IMDB.com.

Though Will Hunting (Matt Damon) has genius-level intelligence (such as a talent for memorizing facts and an intuitive ability to prove sophisticated mathematical theorems), he works as a janitor at MIT and lives alone in a sparsely furnished apartment in an impoverished South Boston neighborhood. An abused foster child, he subconsciously blames himself for his unhappy upbringing and turns this self-loathing into a form of self-sabotage in both his professional and emotional lives. Hence, he is unable to maintain either a steady job or a steady romantic relationship.

The first week of classes, Will solves a difficult graduate-level math problem that Professor Gerald Lambeau (Stellan Skarsgård), a Fields Medalist and combinatorialist, left on a chalkboard as a challenge to his students, hoping someone might solve it by the semester's end. Everyone wonders who solved it, and Lambeau puts another problem on the board – one that took him and his colleagues two years to prove. Will is discovered in the act of solving it, and Lambeau initially believes that Will is vandalizing the board and chases him away. When Will turns out to have solved it correctly, Lambeau tries to track Will down.

Clicker Question

Do you think somebody who is untrained in mathematics could solve complicated math problems?

A Yes

B No

C I don't know

Answer

This has happened. Ramanujan, who will be mentioned in the movie, had no formal education in mathematics yet ended up as one of the most influential mathematicians of the first half of the 20th century in spite of dying when he was 33 years old.

We'll watch some bits of the movie now.

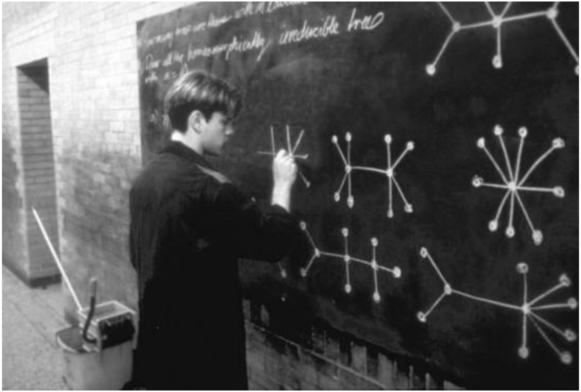
Clicker Question

Do you think Matt Damon is convincing as an untrained mathematical genius?

A Yes

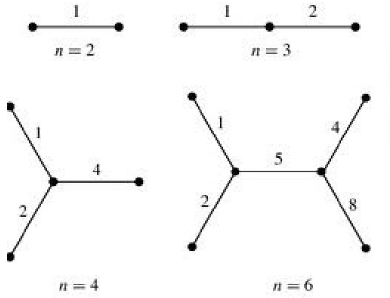
B No

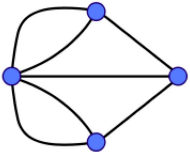
Here is an image of Will solving the problem. Notice that the pictures involved dots and lines connecting them. These are graphs; the mathematical objects we studied a couple weeks ago.



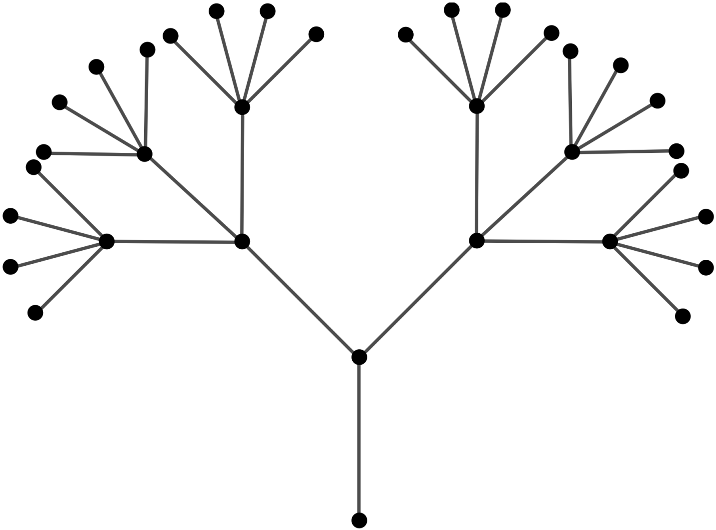
Trees

The problem Will solves is one of graph theory. A tree is a special type of graph. It is a graph for which there are no cycles. The graphs on the left are all examples of trees. The graph on the right (from the 7 bridges problem) is not a tree. Vertices of a tree which are connected to only one other vertex are called leaves.





Here is an example of a tree which indicates more why these kinds of graphs are called trees.



Homeomorphically Irreducible Trees

THE NUMBER OF HOMEOMORPHICALLY IRREDUCIBLE TREES, AND OTHER SPECIES

BY

FRANK HARARY and GEERT PRINS

Princeton [1] *and Amsterdam* [1]

Table of Contents

1. Introduction

Our object is to augment the already rich literature on the enumeration of trees by the addition of several previously uncounted species. Interest is moreover derived from the fact that we use variations of one general method in each of these cases; a method which is also applicable to numerous counting problems not treated in this paper (e.g., see Riordan

[1] The work of the first author was supported by grants from the National Science Foundation to the Institute for Advanced Study and from the Office of Naval Research to Princeton University, while on leave from the University of Michigan; the work of the second author was done at the University of Michigan and in Amsterdam.

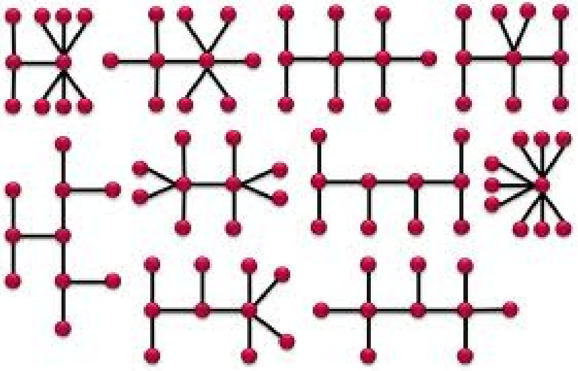
The problem Will solves is to determine all "homeomorphically irreducible trees with 10 vertices".

Let's look at a short video talking about what this means. But, we mention one point in a little more detail than in the video. The degree of a vertex on a graph is the number of edges going into it. This notion came up while solving the 7 bridges of Königsberg problem. A tree is irreducible if the degree of a vertex is never 2.

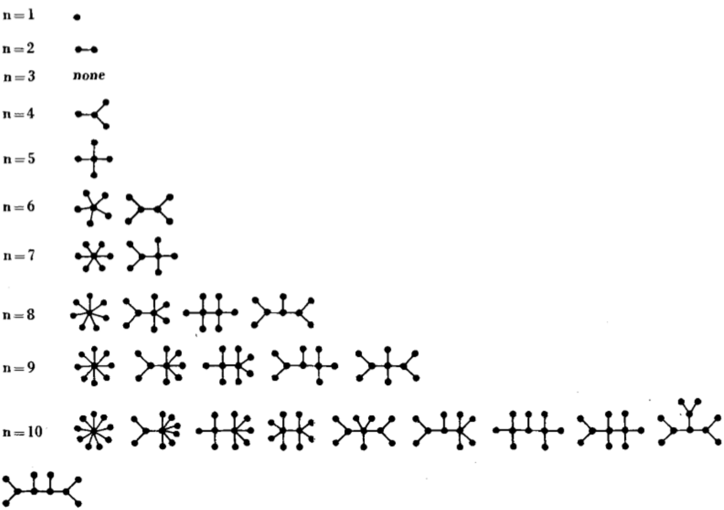
The problem in Good Will Hunting

NPR story on the math of the movie

The 10 Homeomorphically Irreducible Trees with 10 Vertices



Diagrams of all Homeomorphically Irreducible Trees with $n \leqslant 12$ Points



Determination of all the Trees

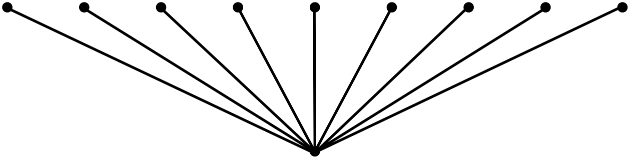
It is correct that the problem Will Hunting solved is much easier than indicated in the movie. Here is one approach to solving it. The problem is to find all 10 vertex trees which are irreducible. So, we can't have any vertex of degree 2.

We approach the problem by focusing on a vertex of largest possible degree. Since a vertex can be connected to at most every other vertex, the largest possible degree is 9.

Perhaps what is most important is to find a systematic approach to listing all possible trees. Drawing some trees satisfying the requirement is not terribly hard. Knowing when you have them all is the harder part of the problem.

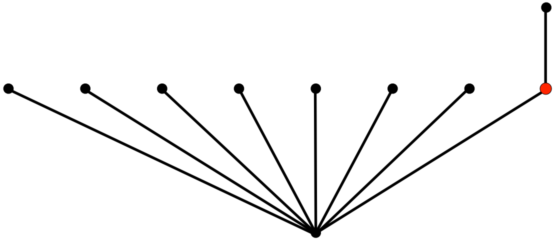
A Vertex of Degree 9

If one vertex has degree 9, then it must be connected to every other vertex. There can be no other connections else there is a cycle. Here is a drawing of such a tree.



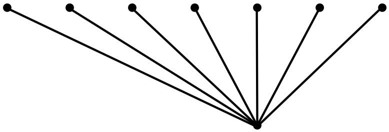
A Vertex of Degree 8?

If there is a vertex of degree 8, then it is connected to all but one other vertices, as in the following picture. The 10th vertex must be connected to another vertex, but that will make such a vertex have degree 2. This is then not allowed, so this possibility cannot happen.

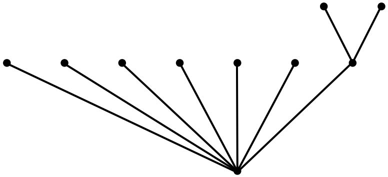


A Vertex of Degree 7

If there is a vertex of degree 7, there aren't too many options. The following picture shows one vertex connected to 7 others.

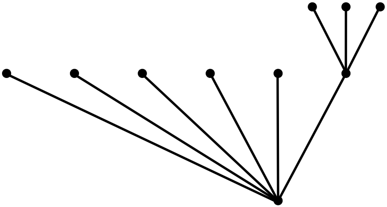


We have to add two more vertices. We cannot connect them to the base and keep it having degree 7. We can't connect one each to two of the other vertices else these will have degree 2. So, both must be connected to just one of the top vertices.

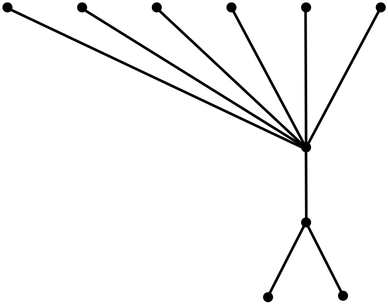


A Vertex of Degree 6

If there is a vertex of degree 6 then, as with the case of degree 7, there is only one possibility. The reasoning is very similar to that case. The following picture shows the only possibility.



There are different ways to draw these options. For example, here is another way to draw the degree 7 option.



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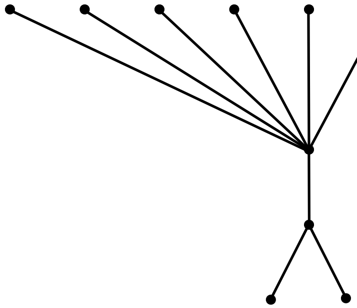

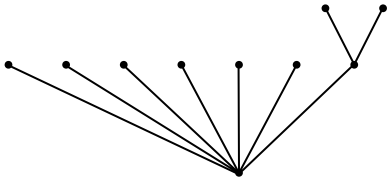

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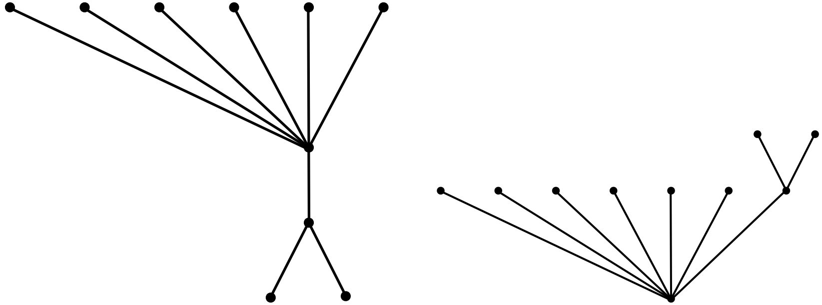
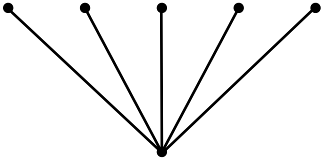


These two represent the same graph; we can just drag the three vertices on the top right along with edges connecting them to the bottom or the picture.

A Vertex of Degree 5

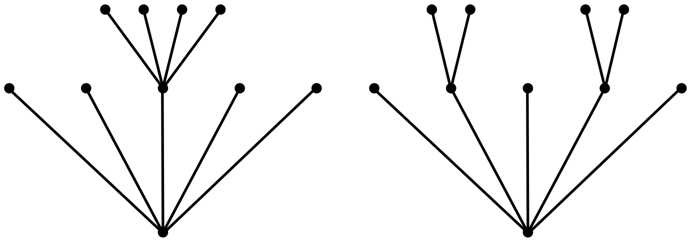
It turns out that having a vertex of degree 5 is perhaps the most complicated case.

If we have one vertex connected to five others, then we have this as part of the tree.

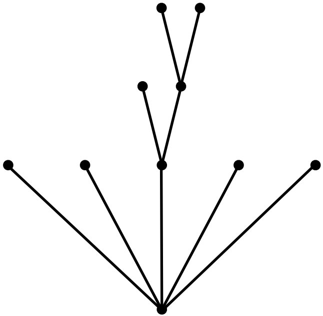


We need to add four more vertices. We have to avoid vertices of degree 2. So, if we connect vertices to one of the top vertices, we have to connect more than 1.

One option is to connect 4 vertices to one of them. Another is to connect 2 vertices each to two of them.

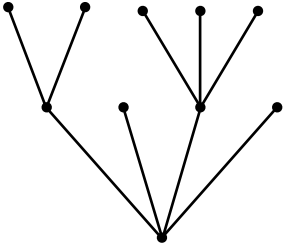


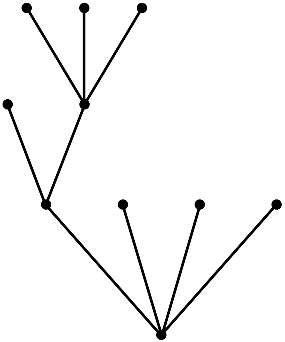
The final option is to connect two vertices to one of the original five vertices at the top, and then the final two vertices to one of those just added.



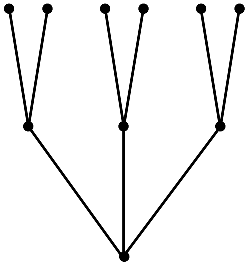
A Vertex of Degree 4

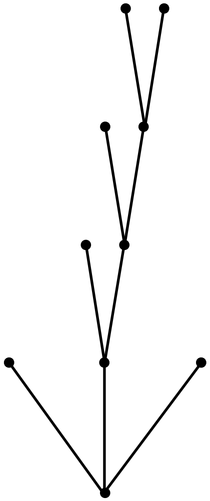
With similar reasoning we can see that there are two trees whose largest degree is 4, and two trees which have no vertex of degree larger than 3. Without going through the details, here are pictures of them.





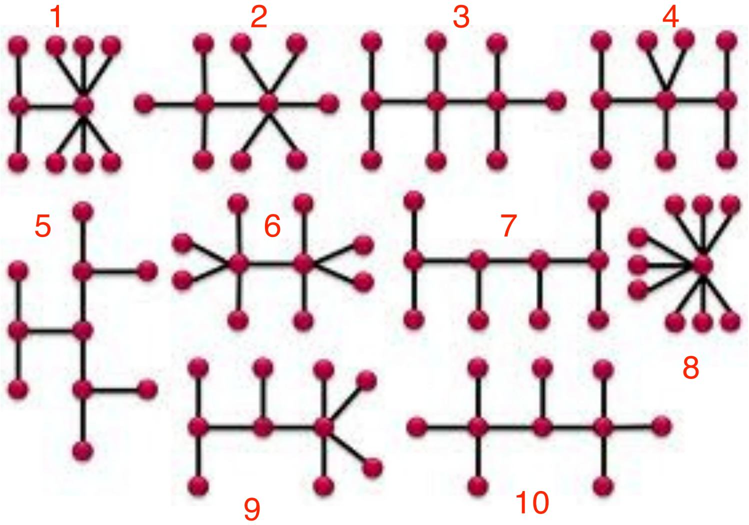
And, finally, the trees with no vertex of degree larger than 3.





All 10 trees

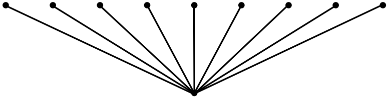
Here is a picture from the web listing all 10 trees.

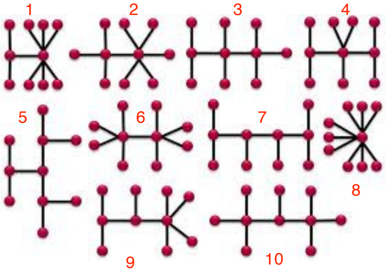


These pictures look different than the ones shown earlier, but they represent the same information. The idea of the word "homeomorphically" is that how a tree is drawn isn't important; if you can shift the vertices around without changing the edges, you have the same graph, essentially.

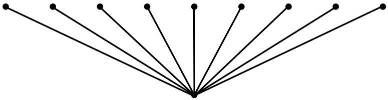
We'll compare the graphs in this latest picture with the ones drawn earlier. The issue will be to try to figure out, given one of the trees drawn before, which one of the 10 in the latest picture has the same information.

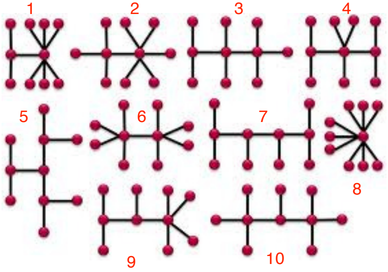
Which of these trees is the same as the tree on the left? Enter the number on your clicker.



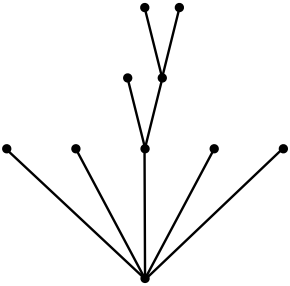


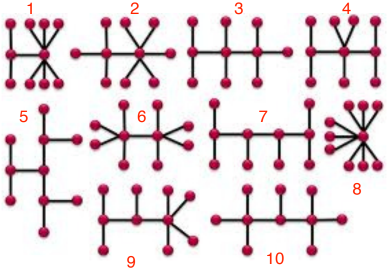
8 It is the tree with a vertex connected to all other vertices.





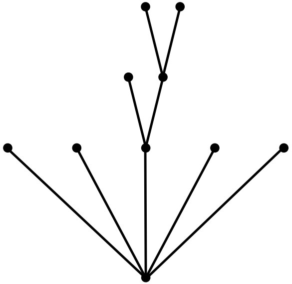
Which of these trees is the same as the tree on the left? Enter the number on your clicker.

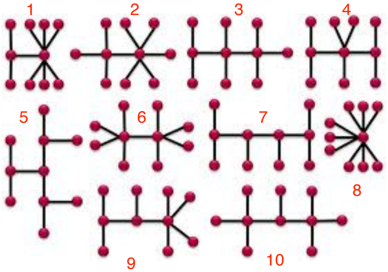




Answer

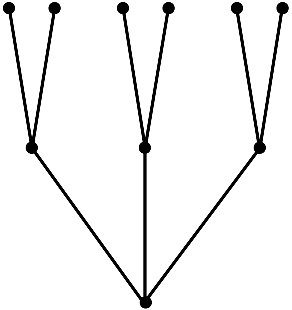
9 It is a tree with one vertex of degree 5. Those are 4, 6, and 9. It has 7 leaves so cannot be 6, which has 8 leaves. If we look at the degree 5 vertex, it is connected to only one vertex of degree 3. The degree 5 vertex in tree 4 is connected to two degree 3 vertices. So, it must be tree 9.

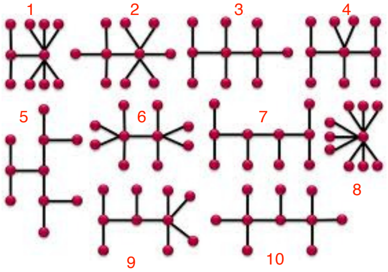




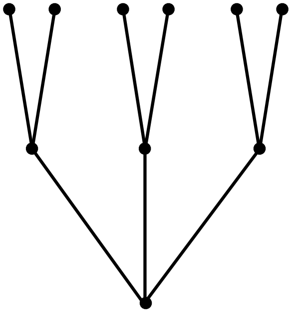
Clicker Question

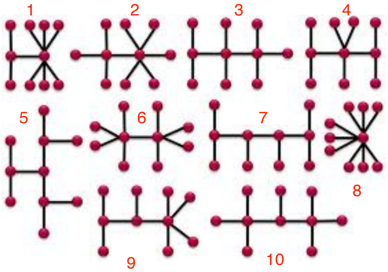
Which of these trees is the same as the tree on the left? Enter the number on your clicker.





5 It is a tree with no vertex of degree larger than 3. There are two such, 5 and 7. The way to distinguish 5 and 7 is that the tree on the left has a vertex of degree 3 that is not connected to any leaf. That happens in 5 but not in 7.





Next Week

Next week we'll discuss interest rates. We'll start by talking about compound interest, the type that is paid in savings accounts and is the basis of all discussion of interest rates. We'll then move on to loans, including home and car loans, and then discuss credit cards and annuities.