January 22

Interest Rates
Compound Interest

Look how your money grows if you save it for 15 years!

The money that you put into a savings account earns interest and grows.
If you put $100 in a bank account at 5% interest per year, you will have $105 after one year. You earn $5 in interest.

How much do you have after two years if you neither deposit or withdraw anything?
You have more than $110.

You have the $105 plus 5% of $105

105 + .05*105 = 110.25

(5% = .05. Divide the percentage by 100 to get the decimal equivalent)
Starting point: 100

after one year: \(100 + 0.05 \times 100 = 100 \times (1.05)\)

after two years: \(105 + 0.05 \times 105 = 105 \times (1.05)\),
which is equal to \(100 \times 1.05 \times 1.05\)

shorter way to write this: \(100 \times (1.05)^2\)

on a calculator, \(100 \times (1.05)^2\)
shorter way to write \(100 \times 1.05 \times 1.05\) is:

\[100 \times (1.05)^2\]

on a calculator, \(100 \times (1.05)^2\)
after three years, you have

\[ 100 \times (1.05)^3 = \$115.76 \]

the point is that you are making interest on interest.

It doesn’t seem like this is significant; you only gained 76¢ over getting $5 per year in interest.
Compound Interest, 5% per year
Questions

- How much will you have after 10 years?
- After 50 years?
- After 200 years?

$5 per year for 200 years is $1000. Not compounding, you’d have $1,100 after 200 years.
Answers

- $100 \times (1.05)^{10} = $162.89$
- $100 \times (1.05)^{50} = $1,146,74$
- $100 \times (1.05)^{200} = $1,729,258!$
General Formula

If you put money in the bank at a given yearly interest rate, convert the rate to a decimal. The formula for how much money you have after so many years is

\[
\text{future amount} = \text{principal} \times (1 + \text{interest rate})^{\text{number of years}}
\]
Or, in symbols,

\[ F = P \times (1 + r)^n \]

where,

\( F \) = future amount

\( P \) = principal (initial investment),

\( r \) = interest rate (converted to decimal)

\( n \) = number of years
Questions

- If you invest $1,000 at 6% per year, how much do you have in 5 years?
- What about after 20 years?
- Same two questions, except that you receive 8% interest per year.
Answers

• $1000 \times 1.06^5 = $1,338.22

• $1000 \times 1.06^{20} = $3,207.14

• $1000 \times 1.08^5 = $1,469.33

• $1000 \times 1.08^{20} = $4,660.96
Compounding does not have to be once per year. Banks generally compound each month.

The formula $F = P \times (1+r)^n$ works with $r$ the interest rate per period and $n$ the number of periods.
If you invest $100 at 5% per year, compounded monthly, how much do you have in 2 years?
2 years = 24 months

5% per year means 5/12 % per month

monthly interest rate is then

\[
\frac{5/12}{100} = \frac{.05}{12}
\]

future value = \(100 \times (1 + \frac{.05}{12})^{24} = \$110.49\)

on a calculator: \(100 \times (1+.05/12)^{24} =\)
Questions

• If you invest $1000 at 6% per year, compounded monthly, how much will you have in 5 years?

• What about after 20 years?

• What if, instead, you get 8% per year?
Answers

• $1000 \times (1 + \frac{.06}{12})^{60} = $1,348.85

• $1000 \times (1 + \frac{.06}{12})^{240} = $3,310.20

• $1000 \times (1 + \frac{.08}{12})^{60} = $1,489.85

• $1000 \times (1 + \frac{.08}{12})^{240} = $4,926.80
Spreadsheets allow one to do multiple calculations more easily. There are also computer programs and webpages which are set up to do interest rate calculations.
Useful Exercise

Do a web search for interest rate calculators. You will probably find many webpages which will do these calculations for you.
After searching google for interest rate calculator, I found:


Can do loans, annuities, etc. at that site
The amount of time it takes for money to double does not depend on how much you invest.

If \( P \times (1.05)^n = 2P \)

then

\((1.05)^n = 2\)

and \(n\) can be found with logarithms or trial and error; \(n\) is about 14 years.
For those who have seen logarithms, if

$$(1+r)^n = 2$$

then taking logs of both sides and simplifying gives

$$n = \frac{\log(2)}{\log(1+r)}$$

This is the exact value of how many periods it takes to double at an interest rate of $r$ per period.
So, if you start with $100 and invest at 5% per year, it doubles to $200 in about 14 years. After another 14 years it doubles again to $400. After another 14 years, it doubles to $800. So, you make $100 interest the first 14 years, but $200 the second 14 years, and $400 the third 14 years.
Doubling Rule of Thumb

Divide the yearly interest rate percentage into 72. The result is about how many years it takes for money to double.

This is only a rough estimate. Its accuracy depends on the interest rate and the period in which money is compounded.

(See wikipedia.org, search for rule of 72)
Inflation

The effect of inflation on the value of money is similar to that of compound interest.
If \( P \) is the present value of money, at an inflation rate of \( r\% \) per year (made into a decimal), the equivalent value \( n \) years later is

\[
F = P \times (1 + r)^n
\]
If you earn $50,000 a year today, what will you have to earn in 20 years in order to have the same income level, if inflation were 2% per year?

What if inflation were 5% per year?
At 2% per year,

$$50000 \times (1 + .02)^{20} = \$74,297$$

At 5% per year,

$$50000 \times (1.05)^{20} = \$132,665$$
The calculation just done is an example of what is called a future value calculation (and can be done with a compound interest rate calculator).

Asking the question in reverse is an a present value calculation.
If you think you will need $500,000 in 30 years, and inflation is 2% per year, how much would this amount be worth now?
Answer: $276,035.44

You can do this on a present value calculator on the web, or use the formula

\[ P = F \left(1 + r\right)^{-n} \]

Where \( P \) is the present value, \( F \) the future value, \( r \) is the interest rate per year, \( n \) is the number of years.
Enter

\[ 500000 \times (1 + 0.02)^{-30} = \]

Make sure to use the minus sign key, not the subtraction key