Graph Theory
Graph theory was invented by a mathematician named Euler in the 18th century. We will see some of the problems which motivated its study. However, it wasn’t studied too systematically until the latter half of the 20th century. Computer Science applications have driven its development, since many of its problems are naturally modeled via graphs.
More specifically, we will focus on three (or four) problems. The first, the historical motivation of the subject, is the 7 bridges of Königsburg problem. The second is how to find your way through a maze. The third is how many colors does it take to color a map?
7 Bridges of Königsburg

The origin of graph theory was the following problem. In the city of Königsburg, in present day Lithuania, there are seven bridges passing over the river and connecting various parts of the city.

The following picture shows the city and its bridges.
The problem, possibly originating from people strolling around the city, is this: Is it possible to cross each bridge exactly once and end up where you started?

Alternatively, is it possible to cross each bridge exactly once, regardless of where you end relative to where you started?
Euler solved this problem by representing the situation as a structure which we now call a graph. This use of the term graph is different than that occurring in algebra.

We will illustrate how Euler solved the 7 bridges problem. We will also address other problems which can be solved by the use of graph theory.
What is a graph?

A graph consists of a bunch of points, usually called vertices. Some of the vertices are connected to each other. When a vertex is connected to another, that connection is called an edge. We can draw edges as straight line segments or curves.

Here are some examples of graphs.
The top two graphs look different, but they represent the same information. Both have the top three vertices connected to each of the three bottom vertices. That the edges in the top left figure sometimes are drawn with straight lines and sometimes with curves does not matter. Nor does it matter where the vertices are positioned.
Euler represented the 7 bridges problem as a graph in the following way. Each land mass was represented as a vertex. Two vertices are connected by an edge if the corresponding land masses are connected by a bridge. The graph representing the problem is shown on the next slide. As we have indicated, the shape of the edges is irrelevant. Only what matters are which vertices are connected.

Since there are 4 land masses, there are 4 vertices. The 7 bridges correspond to 7 edges.
Graph of the 7 Bridges of Königsburg
A **path** on a graph is a journey through various vertices, where you can go from one to another as long as there is an edge connecting them. A **circuit** is a path which returns to the starting point.

This idea comes from the original motivation for graphs. A path in the 7 bridges graph can be though of as a walk across various bridges.
In honor of Euler’s work, we call a path which crosses each edge exactly once an **Euler path**.

If the Euler path starts and ends at the same vertex, then it is called an **Euler circuit**.

In terms of graph theory, the 7 bridges problem is then: is there an Euler circuit (or Euler path) on the graph representing Königsburg?
Euler discovered that, in order to have an Euler circuit, the number of edges connected to each vertex must be even.

He also saw that in order to have an Euler path, every vertex, except for at most two, must have an even number of edges connected to it. If two have an odd number of edges, those could be the start and the end of the path.
The graph to the left has an Euler circuit. The one to the right does not, but it does have an Euler path, when one starts at the top left vertex and finishes at the top right.
Roughly, Euler reasoned that if there was an Euler path or circuit, then for any vertex other than the start or finish, each time you reached the vertex, you need two edges, one to get there and one to get away. The number of edges connected to the vertex is then twice the number of times you cross the vertex, and so is an even number.
Since the 7 bridges graph has 3 vertices with an odd number of edges connected to them, there is no Euler path or Euler circuit.

Thus, it is impossible to walk across each of the 7 bridges exactly once.
Mazes are often difficult to solve because it is hard to distinguish dead ends. Identifying, and then ignoring, dead ends will result in a path through the maze. Mazes appear to be difficult in part because they cause one to make lots of turns, even when you don’t actually have to choose one direction or another.

By representing a maze as a graph, we have a method to be able to ignore the turns which only complicate the look of a maze.
To represent a maze as a graph, we need to focus on what is the important information of a maze. The turns which are forced upon us without requiring us to make a decision are not important. What we need to consider are the junctions where we have a choice of a turn.
We represent a maze as a graph by letting the vertices be the *junctions*; those spots where we have a choice of a direction to turn. We also include the start and finish of the maze.

Two vertices are connected with an edge if you can get from one to another without crossing any other junction. In other words, if you go from one junction to another without having the choice of making a turn, then those two junctions are connected with an edge.
How to draw the graph of a maze
First draw all possible paths. Those are shown in blue in the figure below to the right.
Next, erase the boundaries, leaving only the paths. This is not necessary, but can help to do the next step.
Mark all the junctions, including the start and finish. Recall that the junctions are the vertices of the graph.
Draw the edges by connecting two vertices only if you can get from one to the other without crossing another junction. Drawing the edges as straight lines makes the situation as simple as possible.
Dead ends can be represented as short paths that don’t end at a vertex. Alternatively, they can be ignored, especially if it is clear what are dead ends.
Once you have the graph, find a path from the start to the finish. Comparing the graph and the original drawing of paths will then give you a route through the maze.
The mazes shown in class today were created at the web site

http://hereandabove.com/maze/