Graph Theory, continued
Coloring Maps and the Four Color Problem

Map makers have usually drawn maps in such a way that when two regions border each other, they aren’t given the same color. This makes it easier to read the map.
Over the years, map makers have been able to draw maps with at most four colors without coloring a pair of bordering regions the same. This led people to the following question:

Can every map be colored, so that bordering regions have different colors, with at most four colors?

This is known as the **Four Color Problem**.
Over the years, map makers have been able to draw maps with at most four colors without coloring a pair of bordering regions the same.

In 1852, Francis Guthrie, while coloring a map of counties in England, noticed that he could do this with only four colors. He asked his brother, a mathematician, if that always happened. This led to the following question:

Can every map be colored, so that bordering regions have different colors, with at most four colors?

This is now known as the Four Color Problem.
Several top mathematicians worked on the problem, and there were several incorrect attempts. In 1890, Percey Heawood, while trying to understand a purported proof by Kempe, found a flaw in Kempe's proof, and subsequently proved that all maps can be colored with at most 5 colors.
To understand the problem better, let’s consider some examples.

First, we can associate a graph to a map in the following way. The vertices of the graph are the regions. Two regions are connected by an edge if they are adjacent. This means they share a border other than a corner.
New Mexico is adjacent to Texas, Arizona, Colorado, and Oklahoma, but it is not adjacent to Utah, because they only share a corner.
Graph of the 4 corners region
For another example, consider the piece of the U.S. consisting of California, Nevada, and Arizona. Then this piece of the map is represented by the following graph.
If we color California red, then Nevada must be another color, say blue, since it is adjacent to California. Because Arizona is adjacent to both California and Nevada, it must be another color, say green. We thus need at least 3 colors.
More generally, if the graph of a map contains a piece as below, where there are 3 vertices and each is connected to the other 2, then the map needs at least 3 colors.
For another example, consider the following map of Europe.
More specifically, let’s consider the following piece involving Belgium, France, Luxembourg, and Germany.
The graph associated to this piece of the map is as follows.
The significance of this piece is that it indicates that we need at least four colors. First, suppose we color Luxembourg red. France needs a different color since it is adjacent to it. Say we color France blue. Belgium cannot be red or blue since it is adjacent to both France and Luxembourg. Say we color Belgium green. Germany, since it is adjacent to all three, cannot be any of these three colors, so it must be given a fourth color. Thus, we cannot get by with only three colors.
More generally, if the graph representing a map has a piece consisting of 4 vertices, with each vertex connected to each other, as in the following picture, then the map will require at least 4 colors. This graph looks different from the previous 4 vertex graph, but it has the same information.
A more complete graph representing the map of Europe, which involves Austria, Belgium, the Czech Republic, France, Germany, Italy, Poland, Portugal, Spain, and Switzerland is the following.
We have seen that a 3 vertex graph where each vertex is connected to all others needs 3 colors, and a 4 vertex graph where each vertex is connected to all others needs 4 colors. Does this pattern hold in general? Here is the 5 vertex version:
How many colors does it need? The answer is 5. If we used 4 or fewer colors, since there are 5 vertices, then two vertices would have to have the same color. This is not possible, since all vertices are connected.

Doesn’t this disprove the Four Color Problem?
How many colors does it need? The answer is 5. If we used 4 or fewer colors, since there are 5 vertices, then two vertices would have to have the same color. This is not possible, since all vertices are connected.

Doesn’t this disprove the Four Color Problem? Actually no, since it turns out that this is not the graph of any map. We will say more about this later.
The attempt to solve the Four Color Problem has had many incorrect solutions.

For example, consider the following graph.
This is a map with 10 regions; the “outside” is one of the regions. This coloring makes it appear that we need 5 colors to color it.
However, it can be colored with 4 colors, as the following picture shows.
Through a considerable amount of graph theory, the Four Color Problem was reduced to a finite, but large number of special cases.

Appel and Haken published an article in Scientific American in 1977 which showed that the answer to the problem is yes: you can color any map with at most four colors and not need to color any adjacent regions with the same color.
This was the first widely known mathematical proof which used a computer to check many cases. At the time this use of machines was very controversial, since nobody could check all the details, only the commands in the program used to do the checking.
Planar Graphs

The graphs that arise from a map turn out to have the following property: they can be drawn in such a way that no two edges cross.

The pentagon graph we saw earlier is not a planar graph. This is why it does not represent the graph of a map.
For example, consider the graph

![Graph Image]
We can redraw it by having one of the diagonal edges drawn outside of the square. Thus, it is a planar graph, even though the original drawing does not indicate so.
One of Euler’s other contributions to graph theory was the following result about planar graphs:

If \( V \) is the number of vertices, \( E \) the number of edges, and \( F \) the number of regions formed by the graph, then

\[
V - E + F = 2.
\]
Euler came up with this by considering solids. Regions were viewed as faces of the solid.

Examples of solids include the so-called “Platonic Solids”, drawn below. There are five such solids, including the cube.
Suppose there are 3 utilities and 3 houses. Each house is to be connected to each utility (by, e.g., a pipe, or wire). Is it possible to do this without having the connections crossed?

In the next picture, think about the utilities as the vertices in red and the houses in blue.
This almost works, but not quite. We have two edges which cross. No matter how hard you try, it is not possible to draw this graph without crossing edges.

It turns out that Euler’s formula is actually a result about the shape of space. His formula helps to classify surfaces. Planar graphs mean something different if you draw them on a surface other than an ordinary 2-dimensional surface.
For example, the graph representing the houses and utilities problem is a planar graph if you draw it on a torus, a fancy name for a donut.

The website

http://www.lsus.edu/sc/math/rmabry/live3d/k33-torus.htm

gives an animation of this graph.