Poker: Further Issues in Probability

20 February 2012
How to Succeed at Poker (3 easy steps)

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* Step 3 might take a bit of money!
Deck of Cards

Poker is played with a deck of 52 cards. Each card has a suit and a value.
There are 4 suits and 13 values. The suits are
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- Spades ♠
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- Spades ♠
- Hearts ♥
Suit and Values

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- Spades ♠
- Hearts ♥
- Diamonds ♦
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- Spades ♠️
- Hearts ♥️
- Diamonds ♦️
- Clubs ♣️
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- Spades ♠
- Hearts ♥
- Diamonds ♦
- Clubs ♣

There are 13 values: 2 through 10, Jack, Queen, King, and Ace.

2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
A poker hand is made up of 5 cards. The different poker hands are, from best to worst, are:

Royal Flush, Straight Flush, Four of a Kind, Full House, Flush, Straight, Three of a Kind, Two Pair, One Pair.
Royal Flush
Royal Flush

Straight Flush
Four of a Kind
Four of a Kind

Full House
Flush
Flush

Straight
3 of a kind
3 of a kind

2 pair
1 pair
# Table of Poker Hands

<table>
<thead>
<tr>
<th>Hand</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Royal Flush</td>
<td>$A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit$</td>
</tr>
<tr>
<td>Straight Flush</td>
<td>$J\heartsuit, 10\heartsuit, 9\heartsuit, 8\heartsuit, 7\heartsuit$</td>
</tr>
<tr>
<td>4 of a kind</td>
<td>$8\spadesuit, 8\heartsuit, 8\diamondsuit, 8\clubsuit, 7\heartsuit$</td>
</tr>
<tr>
<td>Full House</td>
<td>$4\heartsuit, 4\diamondsuit, 4\clubsuit, J\spadesuit, J\clubsuit$</td>
</tr>
<tr>
<td>Flush</td>
<td>$K\spadesuit, 10\spadesuit, 8\spadesuit, 7\spadesuit, 3\spadesuit$</td>
</tr>
<tr>
<td>Straight</td>
<td>$Q\heartsuit, J\diamondsuit, 10\spadesuit, 9\clubsuit, 8\heartsuit$</td>
</tr>
<tr>
<td>3 of a kind</td>
<td>$8\heartsuit, 8\diamondsuit, 8\clubsuit, J\heartsuit, 6\spadesuit$</td>
</tr>
<tr>
<td>2 pair</td>
<td>$K\spadesuit, K\heartsuit, 9\spadesuit, 9\diamondsuit, A\spadesuit$</td>
</tr>
<tr>
<td>1 pair</td>
<td>$7\heartsuit, 7\clubsuit, 10\clubsuit, J\heartsuit, 4\spadesuit$</td>
</tr>
</tbody>
</table>
Definition of the Poker Hands

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- **Royal flush**: Ace, King, Queen, Jack, 10 in the same suit.
- **Straight flush**: 5 consecutive cards in the same suit (but not a royal flush).
- **4 of a kind**: 4 cards of the same value.
- **Full house**: 3 cards of the same value and a pair.
- **Flush**: 5 cards in the same suit.
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• 1 pair: 2 cards of the same value.
How many poker hands are there?

The number is how many ways you can choose 5 cards out of 52. But how many is that?
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Counting the number of hands requires a different approach than what we did last week. We could think about dealing 5 cards as a 5-stage situation. However, thinking about first card, second card, etc., will over count, since it doesn’t matter the order you receive cards.
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We’ll look at some smaller examples.
How many ways are there to choose 1 item out of 4?
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There are 4 ways: If the items are numbered 1, 2, 3, 4, then we can select 1, or 2, or 3, or 4.
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It is not a coincidence that the number of ways turned out to be the number of items. In general, if you select 1 item out of a group, the number of ways to do this is the number of items in the group.
How many ways are there to choose 2 items out of 4?
How many ways are there to choose 2 items out of 4?

If the items are labeled 1, 2, 3, 4, we could have

\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},

so there are 6 ways, provided that the order in which we choose them does not matter (like in a poker hand).
If we try to model this with a probability tree, we might draw:
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For example, \( \{1, 2\} \) is listed as \( 1, 2 \) and as \( 2, 1 \). While in some situations these are different, for picking poker hands, the order doesn’t matter. Each of the six outcomes is listed twice.
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With this tree there are 4 choices for the first stage and 3 choices for the second stage, giving 12 total. But, we are double counting each outcome, so there really are \( \frac{4 \cdot 3}{2} = 6 \) outcomes.
How many ways are there to choose 3 items out of 4?
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There are 4: If we label them 1, 2, 3, 4 as before, the ways are

\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}
How many ways are there to choose 3 items out of 4?

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\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}
\]

One way of thinking about this is that choosing 3 of 4 is the same as choosing one not to choose, and there are 4 ways to choose 1 out of 4.
How many ways are there to choose 4 items out of 4?
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There is only 1: We must choose all 4 items.
If we try to use this line of reasoning with poker hands, we could say there are 52 choices for selecting the first card. Then there are 51 choices for selecting the second card, 50 for the third card, 49 for the fourth card, and 48 for the fifth card. However, we are counting each hand multiple times.
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The number of times we are counting a given hand is the number of ways we can rearrange 5 cards.
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The number of times we are counting a given hand is the number of ways we can rearrange 5 cards.

To arrange 5 cards, there are 5 choices for the first, 4 for the second, 3 for the third, 2 for the fourth, and 1 for the fifth. There are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ total ways to rearrange 5 cards. So, we are counting each hand 120 times above.
This means the number of different hands is

\[
\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{311,875,200}{120}
\]

\[
= 2,598,960
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or nearly 2.6 million hands!
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In other words, the number of ways to choose 5 out of 52 cards is 2,598,960.
Choosing \( r \) things out of \( n \)

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This number is often written

$$nC_r$$

or \( \binom{n}{r} \). It is often a combination or a binomial coefficient. More on that next time.

Besides counting poker hands, we’ll need it to compute probabilities of poker hands and of winning the lottery.
The formula for $nC_r$ is

$$nC_r = \frac{n!}{r! \cdot (n - r)!}$$
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The explanation point is spoken *factorial*; that is, 5! is spoken as 5 factorial.
To make sense of this formula, we recall that in counting poker hands, one thing we had to compute was

\[ 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \]

This kind of expression is common and is abbreviated \( 5! \). More generally, if \( n \) is a whole number, then \( n! \) represents multiplying the first \( n \) whole numbers:

\[ n! = 1 \cdot 2 \cdot 3 \cdots (n - 1) \cdot n \]
So, the number of poker hands is

\[ {52 \choose 5} = \frac{52!}{5! \cdot 47!} \]
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\[ 52 \binom{5}{} = \frac{52!}{5! \cdot 47!} \]

If we look at \( \frac{52!}{47!} \) we have

\[ \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdots 2 \cdot 1}{47 \cdots 2 \cdot 1} \]
So, the number of poker hands is

\[ \binom{52}{5} = \frac{52!}{5! \cdot 47!} \]

If we look at \( \frac{52!}{47!} \) we have

\[
\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdots 2 \cdot 1}{47 \cdots 2 \cdot 1}
\]

which is equal to \( 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \), and we had computed the number of hands as

\[
\frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}
\]
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Calculating Combinations

Scientific calculators have a factorial button on them. Some fancier calculators even have a combinations function. If you google combinations calculator you’ll find web apps for computing $nC_r$.

Microsoft Excel and other spreadsheets can compute $nC_r$ and $n!$. 
Next Time

We will figure out the probability of being dealt a given poker hand when given 5 cards. This will tell us why the order of poker hands is determined; the better the hand the smaller the chance it comes up.
A game is played with the spinner drawn above. A person spins the spinner twice. The person wins if they get red on the first spin and blue on the second spin. What is the probability of winning this game?

You are free to use whatever method (probability tree, listing outcomes, etc.) you wish, but you need to give some explanation of how you got your answer.
4! is equal to

A  1
B  4
C  6
D  24
E  120