Graph Theory: The Four Color Theorem

28 March 2012
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The problem of map coloring arose as a topic of mathematical interest. It generated a lot of interest and trying to determine the smallest number of colors any map needs was worked on by many people for many years. It turns out to be quite a difficult problem, and had many incorrect solutions before the result was finally proved in 1977 that 4 colors suffice.
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The result we will discuss is now known as the Four Color Theorem.
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First, we can associate a graph to a map in the following way. The vertices of the graph are the regions. Two regions are connected by an edge if they share a border other than a corner. When two regions share a border other than a corner we’ll call them adjacent.
New Mexico is adjacent to Texas, Arizona, Colorado, and Oklahoma, but it is not adjacent to Utah, because they only share a corner.
Graph of the 4 corners region

UT - CO
AZ - NM
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If we color California red, then Nevada must be another color, say blue, since it is adjacent to California. Because Arizona is adjacent to both California and Nevada, it must be another color, say green. We thus need at least 3 colors.
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More generally, if the graph of a map contains a piece as below, where there are 3 vertices and each is connected to the other 2, then the map needs at least 3 colors.
Clicker Question

What is the fewest number of colors we can color this piece of the U.S. map, with 6 states, and not have two states who share a border colored with the same color? Enter the number on your clicker.
Answer

We need four colors, as the following graph indicates.
If we start coloring the vertex in the middle and then color by going around in a circle, we’ll see that we can’t do this with 3 colors. There are five vertices on the outside. By a similar argument, we can see that if there is an odd number of vertices on the outside, all connected to a central vertex, we need 4 colors.
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This doesn't work with an odd number of vertices.
For another example, consider the following map of Europe.
More specifically, let’s consider the following piece involving Belgium, France, Luxembourg, and Germany.
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The graph associated to this piece of the map is as follows.

Can you color this graph with three colors?

A Yes
B No
No, this piece of the map needs at least four colors. First, suppose we color Luxembourg red. France needs a different color since it is adjacent to it.
Say we color France blue. Belgium cannot be red or blue since it is adjacent to both France and Luxembourg.
Say we color Belgium green. Germany, since it is adjacent to all three, cannot be any of these three colors, so it must be given a fourth color. Thus, we cannot get by with only three colors.
More generally, if the graph representing a map has a piece consisting of 4 vertices, with each vertex connected to each other, then the map will require at least 4 colors. The following graph looks different from the previous 4 vertex graph, but it has the same information.
We have seen that a 3 vertex graph where each vertex is connected to all others needs 3 colors, and a 4 vertex graph where each vertex is connected to all others needs 4 colors. Does this pattern hold in general? Here is the 5 vertex version:
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What is the fewest colors you need in order to color vertices so that adjacent vertices are different colors?
The graph needs 5 colors. If we used 4 or fewer colors, since there are 5 vertices, then two vertices would have to have the same color. This is not possible, since all vertices are connected.
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Actually no, since it turns out that this is not the graph of any map. We will say more about this later.
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Can you recolor it and use fewer colors?
The map can be colored with 4 colors, as the following picture shows.
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Appel and Haken published an article in Scientific American in 1977 which showed that the answer to the problem is yes: you can color any map with at most four colors and not need to color any adjacent regions with the same color.
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This was the first widely known mathematical proof which used a computer to check many cases. At the time this use of machines was very controversial, since nobody could check all the details, only the commands in the program used to do the checking.
Next Time

Why isn’t the example of the pentagon graph we considered earlier not a counterexample to the Four Color Theorem?
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We’ll look at this next time. We’ll also look at how certain shapes called surfaces can be understood through graph-theoretic ideas.
Every map can be colored with at most how many colors, so that adjacent regions do not have the same color?

A  1
B  2
C  3
D  4
E  5